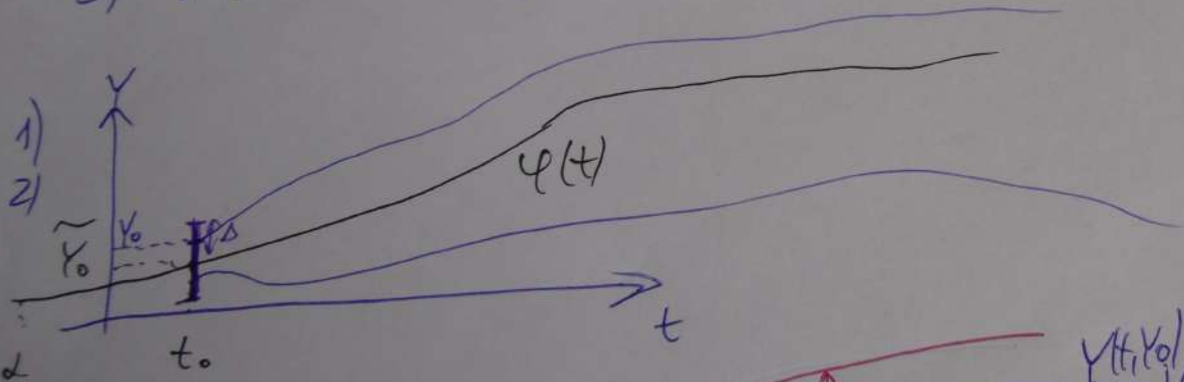
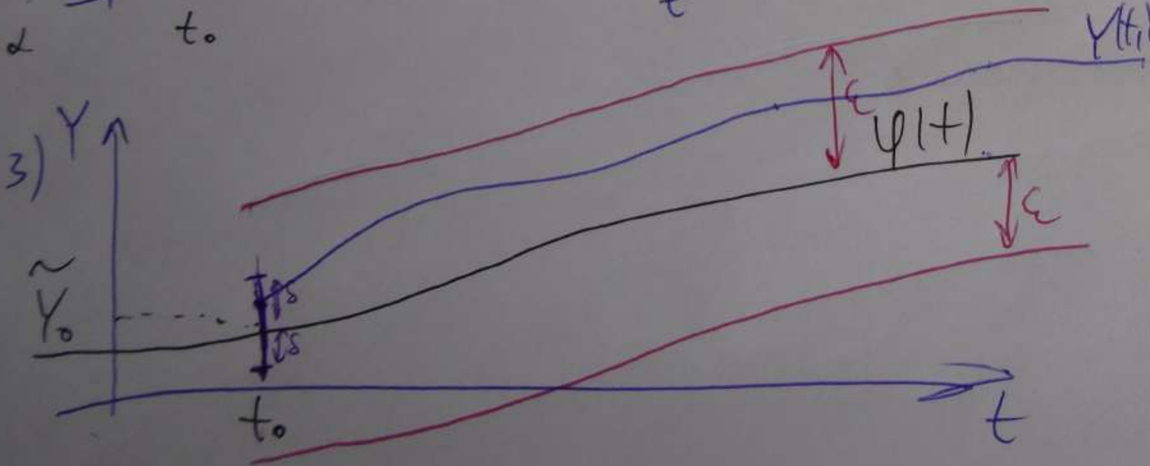
$$\begin{cases} Y' = F(t,Y) \\ Y(t_0) = \tilde{Y}_0 \end{cases} \quad \varphi(t) \text{ — фикс} \quad (\alpha, \omega) \qquad \begin{cases} Y' = F(t,Y) \\ Y(t_0) = Y_0 \end{cases} \quad Y(t, Y_0),\ (\alpha(Y_0), \omega(Y_0))$$

$\varphi(t)$ — устойчиво по Ляпунову, если:

1) $\omega = +\infty$
2) $\exists \Delta: \forall Y_0 \ \|Y_0 - \tilde{Y}_0\| < \Delta$, то $\omega(Y_0) = +\infty$
3) $\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0$: если $\|Y_0 - \tilde{Y}_0\| < \delta$, то $\|Y(t, Y_0) - \varphi(t)\| < \varepsilon$ $\forall t > t_0$





$\forall \varepsilon$ найдётся $\delta(\varepsilon)$

$\forall Y_0$: $\|Y_0 - \tilde{Y}_0\| < \delta$, то

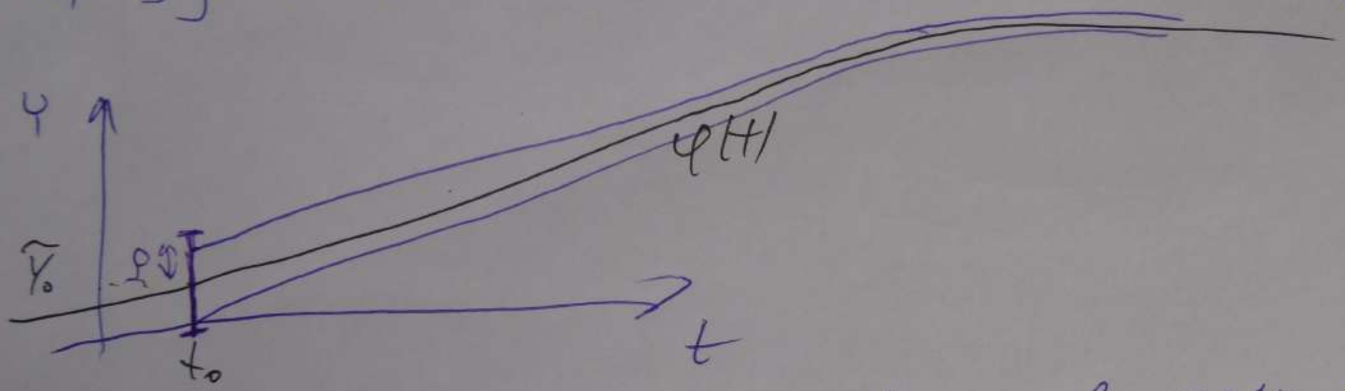
$\|Y(t, Y_0) - \varphi(t)\| < \varepsilon$

$\forall t > t_0$

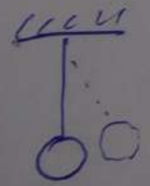
Опр. $\varphi(t)$ - асимптотически уст-вое решение сист $Y' = F(t, Y)$, если

1) $\varphi(t)$ - уст. по Ляпунову

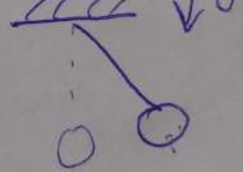
2) $\exists \rho$: $\|Y_0 - \tilde{Y}_0\| < \rho$, то $\|Y(t, Y_0) - \varphi(t)\| \to 0$, $t \to \infty$



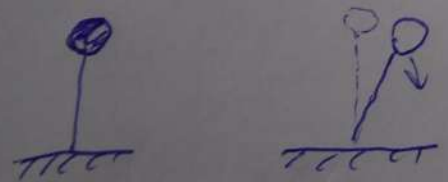
Опр. $\varphi(t)$ - неустойчиво по Ляпунову, если нарушается хотя бы одно условие в определении уст-ти.



$$\begin{cases} \ddot{y} + y = 0 \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \qquad \varphi(t) = 0$$



$$y(t, y_0, y_0') \quad \begin{cases} \ddot{y} + y = 0 \\ y(0) = y_0 \\ y'(0) = y_0' \end{cases}$$



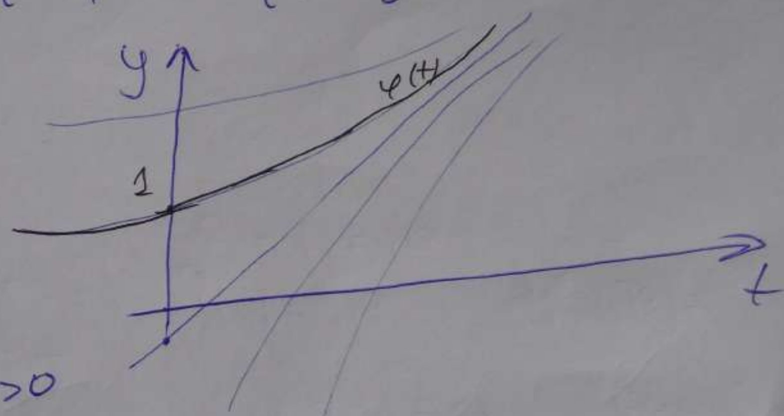
Прим 3. $\begin{cases} y' = -y + t \\ y(0) = 1 \end{cases}$ – иссл-ть реш. на уст-ть.

$\begin{cases} y' = -y + t \\ y(0) = y_0 \end{cases}$ $\quad y_{oo}: y' = -y \quad y_{oo} = Ce^{-t}$

$y_{част} = At + B: \quad A = -At - B + t \Rightarrow \begin{matrix} A = -B \\ 0 = -A + 1 \end{matrix} \Rightarrow \begin{matrix} A = 1 \\ B - 1 \end{matrix}$

$y_{он} = Ce^{-t} + t - 1; \quad y_0 = C - 1$

$y(t, y_0) = (y_0 + 1)e^{-t} + t - 1 \qquad \varphi(t) = y(t, 1) = 2e^{-t} + t - 1$



1) $\omega = +\infty$

2) $\forall y_0 \ \omega(y_0) = +\infty$

3) $\forall \varepsilon$ найдём δ: $|y_0 - 1| < \delta$, то

$|y(t, y_0) - \varphi(t)| < \varepsilon$

$\forall t > 0$

$|(y_0 - 1)e^{-t}| < \varepsilon \quad \forall t > 0$

Если $\delta(\varepsilon) = \varepsilon$, т.е. если $|y_0 - 1| < \varepsilon$, то

$|y(t, y_0) - \varphi(t)| = |(y_0 - 1)e^{-t}| < |y_0 - 1| < \varepsilon \quad \forall t > 0$

$e^{-t} < 1 \quad \forall t > 0$

$\varphi(t)$ – уст-во по Ляпунову

$|y(t, y_0) - \varphi(t)| = |(y_0 - 1)e^{-t}| \xrightarrow[t \to +\infty]{} 0$ – $\varphi(t)$ ас. уст.

Пример 2 $\begin{cases} y' = t(y-1) \\ y(0)=1 \end{cases}$ $\varphi(t) = 1$ $\begin{cases} y' = t(y-1) \\ y(0) = y_0 \end{cases}$

$$\boxed{y(t,y_0) = (y_0-1)e^{t^2}+1}$$

$y' = t(y-1) \quad y \neq 1$

$$\int \frac{dy}{y-1} = \int t\,dt$$

$$\ln|y-1| = t^2 + C$$

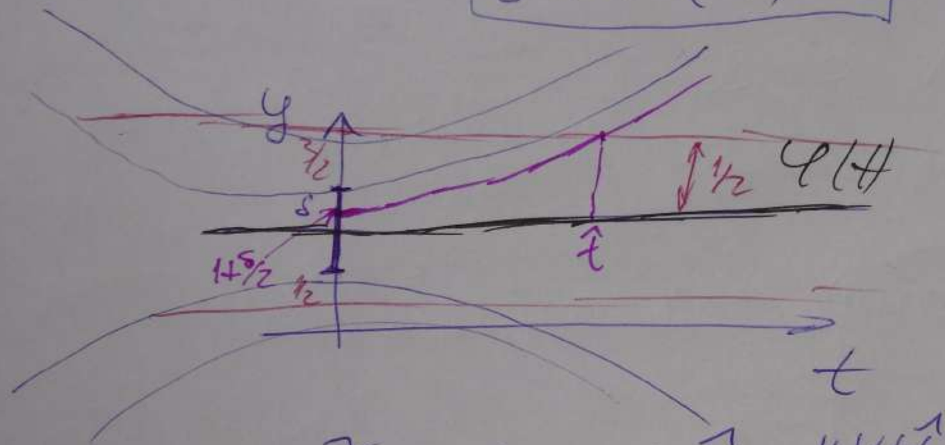
$$y = Ce^{t^2}+1$$

$$y_0 = C + 1 \Rightarrow C = y_0 - 1$$

1) $\omega = +\infty$

2) $\omega(y_0) = +\infty$

3)



$$\exists \varepsilon \quad \forall \delta \quad \exists Y_0 : \|Y_0 - \tilde{Y}_0\| < \delta : \quad \exists \hat{t} \quad \|Y(\hat{t}, Y_0) - \varphi(\hat{t})\| = \varepsilon$$

Возьмём $\varepsilon = 1/2$, тогда $\forall \delta$ можем выбрать y_0: $|y_0 - 1| < \delta$,
например, $y_0 - 1 = \delta/2$, тогда $|y(t,y_0) - \varphi(t)| = 1/2$

$$\left|\frac{\delta}{2}e^{t^2}+1-1\right| = 1/2$$

$$\frac{\delta}{2}e^{t^2} = 1/2 \Rightarrow t^2 = \ln 1/\delta$$

$$\hat{t} = \sqrt{\ln 1/\delta}$$

$\varphi(t)$ – неустойчиво.

Пример 3.

$$\begin{cases} y' = \dfrac{y}{(t^2+1)(\pi/2 + \operatorname{arctg} t)} \\ y(0) = 0 \end{cases} \qquad \varphi(t) = 0$$

$$\begin{cases} y' = \dfrac{y}{(t^2+1)(\pi/2 + \operatorname{arctg} t)} \\ y(0) = y_0 \end{cases}$$

$$y \neq 0: \int \frac{dy}{y} = \int \frac{dt}{(t^2+1)(\pi/2 + \operatorname{arctg} t)} = \int \frac{d(\operatorname{arctg} t)}{\pi/2 + \operatorname{arctg} t} =$$

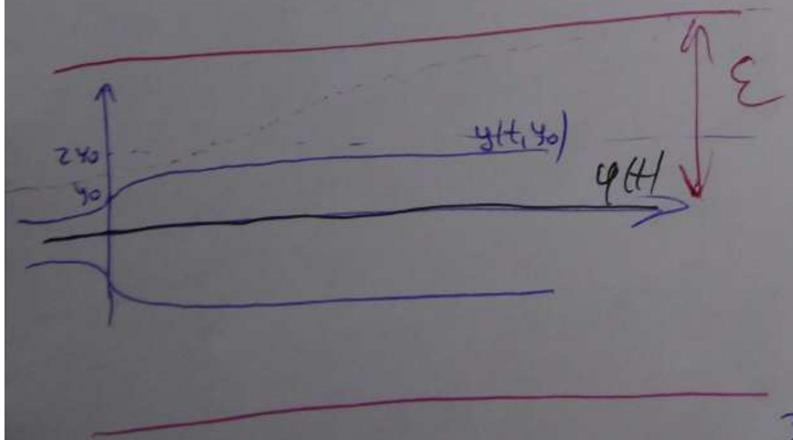
$$\ln|y| \text{ "} \qquad \ln|\pi/2 + \operatorname{arctg} t| + C$$

$$y = C_1(\pi/2 + \operatorname{arctg} t); \quad y_0 = C_1 \pi/2$$

$$y(t, y_0) = \frac{2y_0}{\pi}(\pi/2 + \operatorname{arctg} t)$$

$$-\pi/2 < \operatorname{arctg} t < \pi/2$$

$$0 < y(t, y_0) < 2y_0$$



1, 2): $\omega = +\infty$, $\omega(y_0) = +\infty$.

3) $\forall \varepsilon$ возьмём $\delta = \varepsilon/2 \Rightarrow$ если $|y_0| < \delta$, то $|y(t, y_0)| < 2\delta = \varepsilon$

$\varphi(t)$ - уст-во.

Пример 4.

$$\begin{cases} y' = \dfrac{-y}{(t^2+1)(\pi - \operatorname{arcctg} t)} \\ y(0)=0 \end{cases}$$

$\varphi(t)=0$

$$\begin{cases} y' = \dfrac{-y}{(t^2+1)(\pi - \operatorname{arcctg} t)} \\ y(0)=y_0 \end{cases}$$

$$y \neq 0: \quad \underbrace{\int \frac{dy}{y}}_{\ln|y|} = -\int \frac{dt}{(t^2+1)(\pi - \operatorname{arcctg} t)} = -\int \frac{d \operatorname{arctg} t}{\pi - \operatorname{arctg} t} = \ln|\pi - \operatorname{arctg} t| + c$$

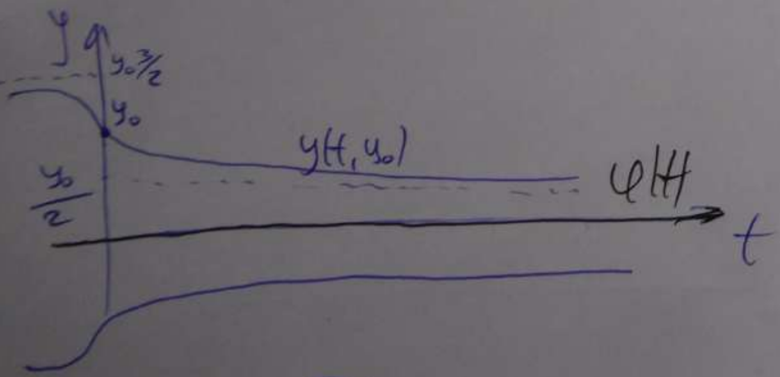
$$y(t) = c(\pi - \operatorname{arctg} t)$$

$$y_0 = c\pi \Rightarrow c = \frac{y_0}{\pi}$$

$$y(t, y_0) = \frac{y_0}{\pi}(\pi - \operatorname{arctg} t)$$

$$-\frac{\pi}{2} < \operatorname{arctg} t < \frac{\pi}{2}$$

$$\frac{y_0}{2} < y(t, y_0) < \frac{3}{2} y_0$$



$\forall \varepsilon \ \exists \delta = \varepsilon \Rightarrow |y_0| < \delta = \varepsilon$, то $\forall t > 0$

$$|y(t, y_0)| < |y_0| = \varepsilon$$

$|y(t, y_0) - \varphi(t)| \xrightarrow[t \to \infty]{} y_0/2$ – ас. уст. нет

$\varphi(t)$ – уст. по Ляпунову

Д/З 881