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On the steady Navier–Stokes equations in $2D$ exterior domains

Mikhail V. Korobkov [a,b], Konstantin Pileckas [c,*], Remigio Russo [d]

[a] School of Mathematical Sciences, Fudan University, Shanghai 200433, China
[b] Voronezh State University, Universitetskaya pl. 1, Voronezh, 394018, Russia
[c] Institute of Applied Mathematics, Vilnius University, Naugarduko Str., 24, Vilnius, 03225 Lithuania
[d] Dipartimento di Matematica e Fisica, Università degli studi della Campania "Luigi Vanvitelli", viale Lincoln 5, 81100, Caserta, Italy

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Abstract

We study the boundary value problem for the stationary Navier–Stokes system in two dimensional exterior domain. We prove that any solution of this problem with finite Dirichlet integral is uniformly bounded. Also we prove the existence theorem under zero total flux assumption.

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1. Introduction

Let Ω be an exterior domain in $\mathbb{R}^2$, i.e.,

$$\Omega = \mathbb{R}^2 \setminus \bigcup_{i=1}^{N} \overline{\Omega}_i, \tag{1.1}$$

* Corresponding author.
E-mail addresses: korob@math.nsc.ru (M.V. Korobkov), konstantinas.pileckas@mif.vu.lt (K. Pileckas), remigio.russo@unicampania.it (R. Russo).

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where Ω_i are N pairwise disjoint bounded Lipschitz domains, $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$, $i \neq j$. The boundary value problem associated with the Navier–Stokes equations in Ω is to find a solution to the system

$$\begin{aligned} \nu\Delta\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} - \nabla p = \mathbf{0} \qquad & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \qquad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} \qquad & \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

with the condition at infinity

$$\lim_{x\to\infty} \mathbf{u}(x) = \mathbf{u}_0, \tag{1.3}$$

where $\mathbf{a}$ and $\mathbf{u}_0$ are, respectively, an assigned vector field on $\partial\Omega$ and a constant vector. Starting from a pioneering paper by J. Leray [23] it is now customary to look for a solution to (1.2) with finite Dirichlet integral

$$\int_\Omega |\nabla\mathbf{u}|^2 dx < +\infty, \tag{1.4}$$

known also as *D–solution*. As is well known (e.g., [21]), such solution is real–analytic in Ω. Set

$$\mathscr{F}_i = \int_{\partial\Omega_i} \mathbf{a}\cdot\mathbf{n}\, ds. \tag{1.5}$$

The existence of a D–solution to (1.2) has been first established by J. Leray [23] under the assumption

$$\mathscr{F}_i = 0, \quad i = 1\dots, N. \tag{1.6}$$

To show this, Leray introduced an elegant argument, known nowadays as *invading domains method*, which consists in proving first that the Navier–Stokes problem

$$\begin{aligned} -\nu\Delta\mathbf{u}_k + \big(\mathbf{u}_k\cdot\nabla\big)\mathbf{u}_k + \nabla p_k &= 0 \qquad \text{in } \Omega_k, \\ \operatorname{div}\mathbf{u}_k &= 0 \qquad \text{in } \Omega_k, \\ \mathbf{u}_k &= \mathbf{a} \qquad \text{on } \partial\Omega, \\ \mathbf{u}_k &= \mathbf{u}_0 \qquad \text{on } \partial B_k \end{aligned} \tag{1.7}$$

has a weak solution $\mathbf{u}_k$ for every bounded domain $\Omega_k = \Omega\cap B_k$, $B_k = \{x : |x| < k\}$, $k \gg 1$, and then to show that the following estimate holds

$$\int_{\Omega_k} |\nabla\mathbf{u}_k|^2 dx \le c, \tag{1.8}$$

for some positive constant c independent of k. While (1.8) is sufficient to assure the existence of a subsequence $\mathbf{u}_{k_l}$ which converges weakly to a solution $\mathbf{u}$ of (1.2) satisfying (1.4), it does not give any information about the behavior at infinity of the velocity $\mathbf{u}$,[1] i.e., we do not know whether $\mathbf{u}$ satisfies the condition at infinity (1.3). In 1961 H. Fujita [8] recovered, by means of a different method, Leray's result (see also [11, Chapter XII]). Nevertheless, due to the lack of a uniqueness theorem, the solutions constructed by Leray and Fujita are not comparable, even for very large ν.

Pushing a little further the argument of Leray [23], A. Russo [29] showed that the condition (1.6) could be extended to the case of "small" (not zero) fluxes by

$$\sum_{i=1}^{m} |\mathscr{F}_i| < 2\pi\nu. \tag{1.9}$$

The first existence theorem for (1.2)–(1.3) is due to D.R. Smith and R. Finn [7], where it is proved that if $\mathbf{u}_0 \neq \mathbf{0}$ and $|\mathbf{a}-\mathbf{u}_0|$ is sufficiently small, then there is a D–solution to (1.2) which converges uniformly to $\mathbf{u}_0$. This result is particularly meaningful since it rules out (at least for small data) for the non–linear Navier–Stokes system (1.2)–(1.3) the famous *Stokes paradox* which asserts that the equations obtained by linearization of (1.2)–(1.3)

$$\begin{aligned} \nu\Delta\mathbf{u} - \nabla p &= \mathbf{0} && \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{a} && \text{on } \partial\Omega, \\ \lim_{x\to\infty} \mathbf{u}(x) &= \mathbf{u}_0, \end{aligned} \tag{1.10}$$

have a solution if and only if (see, e.g., [30])

$$\int_{\partial\Omega} (\mathbf{a}-\mathbf{u}_0)\cdot\boldsymbol{\psi}\, ds = \mathbf{0} \tag{1.11}$$

for all densities $\boldsymbol{\psi}$ of the simple layer potentials constant on $\partial\Omega$. In particular, since $\int_{\partial\Omega} \boldsymbol{\psi} \neq \mathbf{0}$, if $\mathbf{a}$ vanishes and $\mathbf{u}_0$ is a constant different from zero, then (1.10) is not solvable. Moreover, since for the exterior of a ball, $\boldsymbol{\psi}$ are the constant vectors,[2] a solution to $(1.10)_{1,2,4}$ satisfies

$$\int_0^{2\pi} \mathbf{u}(R,\theta)\, d\theta = 2\pi\mathbf{u}_0. \tag{1.12}$$

Of course, by the linearity of the Stokes equations, it is equivalent to say that a solution to $(1.10)_{1,2}$ constant on the boundary and vanishing at infinity does not exist. The situation is different for the nonlinear problem (1.2). The questions whether it admits a solution constant on $\partial\Omega$ and zero at infinity is not answered yet, also for small data. Nevertheless, for domains symmetric with respect to the coordinate axes, i.e.,

1 Indeed, the unbounded function $\log^\alpha |x|$ ($\alpha \in (0,1/2)$) satisfies (1.4).

2 More in general, for the exterior of an ellipsoid of equation $f(x)=1$, $\boldsymbol{\psi} = \boldsymbol{c}/|\nabla f|$ for every constant vector $\boldsymbol{c}$ [25].

$$(x_1, x_2) \in \Omega \Rightarrow (-x_1, x_2), (x_1, -x_2) \in \Omega,$$

in [27] it is showed that a symmetric D–solution

$$\begin{aligned} u_1(x_1, x_2) &= -u_1(-x_1, x_2) = u_1(x_1, -x_2) \\ u_2(x_1, x_2) &= u_2(-x_1, x_2) = -u_2(x_1, -x_2), \end{aligned} \tag{1.13}$$

to (1.2), uniformly vanishing at infinity, exists under the only natural assumption that **a** satisfies (1.13) and natural regularity conditions. Note that (1.13) meets the mean property (1.12) with $\mathbf{u}_0 = \mathbf{0}$.

The problem of the asymptotic behavior at infinity of an arbitrary D–solution $(\mathbf{u}, p)$ to $(1.2)_{1,2}$ was tackled by D. Gilbarg & H. Weinberger [12,13] and C. Amick [2]. In [13] it is shown that

$$p - p_0 = o(1) \qquad \text{as } r \to \infty, \tag{1.14}$$

i.e., pressure has a limit at infinity (one can choose, say, $p \to 0$), and

$$\begin{aligned} &\mathbf{u}(x) = o(\log^{1/2} r), \\ &\omega = o(r^{-3/4} \log^{1/8} r), \\ &\nabla \mathbf{u}(x) = o(r^{-3/4} \log^{9/8} r), \end{aligned} \tag{1.15}$$

where

$$\omega = \partial_1 u_2 - \partial_2 u_1$$

is the corresponding vorticity. If, in addition, $\mathbf{u}$ is bounded, then there is a constant vector $\mathbf{u}_\infty$ such that

$$\lim_{r \to +\infty} \int_0^{2\pi} |\mathbf{u}(r, \theta) - \mathbf{u}_\infty|^2 d\theta = 0, \tag{1.16}$$

and

$$\begin{aligned} &\omega = o(r^{-3/4}), \\ &\nabla \mathbf{u}(x) = o(r^{-3/4} \log r). \end{aligned} \tag{1.17}$$

Here if $\mathbf{u}_\infty = \mathbf{0}$, then $\mathbf{u} = o(1)$. Moreover, in [28] it is proved that

$$\nabla p = O(r^{\epsilon - 1/2}) \tag{1.18}$$

for every positive ϵ.

In [2] it is proved that if $\mathbf{u}$ vanishes on the boundary, then $\mathbf{u}$ is bounded and, as a consequence, satisfies (1.16), (1.17). However, in this last case the solution could tend to zero at infinity and even be the trivial one. This possibility was excluded by Amick [2] (Section 4.2) for the solution obtained by the Leray method, for symmetric with respect to the x_2–axis (say) domains, i.e., $(x_1, x_2) \in \Omega \Rightarrow (x_1, -x_2) \in \Omega$. This result is remarkable as the first step to exclude the non–linear

Stokes paradox for every ν, at least for axisymmetric domains. For such kind of domains the existence of a D–solution to (1.2) is established in [17] only under the symmetry hypothesis $a_1(x_1,x_2)=a_1(x_1,-x_2)$, $a_2(x_1,x_2)=-a_2(x_1,-x_2)$.

Under similar symmetry assumptions, several results on existence, uniqueness and decay of solutions to (1.2) with external symmetrical force in the right–hand side were established by M. Yamazaki (see, e.g., [33–35]).

Despite the efforts of many researchers (see, $e.g$, the reference in [11]) several relevant problems remain open, among which: existence of a solution to (1.2) for arbitrary fluxes $\mathscr{F}_i$, its uniqueness (for small data); the boundedness of a D–solutions (in the case of non-homogeneous boundary conditions), its uniform convergence to $\mathbf{u}_\infty \neq \mathbf{0}$[3] and the relation between $\mathbf{u}_\infty$ and $\mathbf{u}_0$; more precise asymptotic behavior of ∇p and the derivatives of $\mathbf{u}$.

The present paper is devoted to some of the above issues. The first main result is as follows.

Theorem 1.1. *Let* $\mathbf{u}$ *be a solution to the Navier–Stokes system*

$$\begin{cases} -\nu\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\mathbf{0} & \text{in } \Omega,\\ \operatorname{div}\mathbf{u}=0 & \text{in } \Omega \end{cases} \tag{1.19}$$

in the exterior domain $\Omega\subset\mathbb{R}^2$. *Suppose*

$$\int\limits_{\Omega} |\nabla\mathbf{u}|^2 dx < \infty. \tag{1.20}$$

Then $\mathbf{u}$ *is uniformly bounded in* $\Omega_0=\mathbb{R}^2\setminus B_{R_0}$, *i.e.,*

$$\sup_{x\in\Omega_0} |\mathbf{u}(x)| < \infty, \tag{1.21}$$

where B_{R_0} *is a disk with sufficiently large radius:* $\frac{1}{2}B_{R_0}\Supset\partial\Omega$.

Using the above–mentioned results of D. Gilbarg and H. Weinberger, we obtain immediately

Corollary 1.1. *Let* $\mathbf{u}$ *be a* D*-solution to the Navier–Stokes system (1.19) in a neighborhood of infinity. Then the asymptotic properties (1.14), (1.16)–(1.17) hold.*

Using the results of the above–mentioned paper of Amick [2], we could say something more about asymptotic properties of D-solutions in the case of zero total flux, i.e., when

$$\int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,ds=0, \tag{1.22}$$

[3] By a remarkable result of L.I. Sazonov [31], this ensures that the solution behaves at infinity as that of the linear Oseen equations (see also [10] and [11]).

Corollary 1.2. *Let* $\mathbf{u}$ *be a* D*-solution to the Navier–Stokes problem (1.19) in an exterior domain* $\Omega \subset \mathbb{R}^2$ *with zero total flux condition (1.22). Then in addition to the properties of Theorem 1.1 and Corollary 1.1, the total head pressure* $\Phi = p + \frac{1}{2}|\mathbf{u}|^2$ *and the absolute value of the velocity* $|\mathbf{u}|$ *have the uniform limit at infinity, i.e.,*

$$|\mathbf{u}(r,\theta)| \to |\mathbf{u}_\infty| \qquad \text{as } r \to \infty, \tag{1.23}$$

where $\mathbf{u}_\infty$ *is a constant vector from the condition (1.16).*

Let us note that formally Amick [2] established (1.23) under the stronger assumption

$$\mathbf{a} \equiv 0. \tag{1.24}$$

But really his argument for (1.23) cover the more general case (1.22) as well. Indeed, the main tool in [2] was the use of the auxiliary function $\gamma = \Phi - \omega\psi$, where ψ is a stream function: $\nabla\psi = \mathbf{u}^\perp = (u_2, -u_1)$. This auxiliary function γ has remarkable monotonicity properties: it is monotone along level sets of the vorticity $\omega = c$ and vice versa – the vorticity is monotone along level sets $\gamma = c$. But, of course, the stream function ψ (and, consequently, the corresponding auxiliary function γ) could be well defined in the neighborhood of infinity under the more general case (1.22) instead of (1.24). Furthermore, Amick also proved that under the conditions of Corollary 1.2, the convergence

$$\gamma(r,\theta) \to \frac{1}{2}|\mathbf{u}_\infty|^2 \qquad \text{as } r \to \infty \tag{1.25}$$

holds uniformly with respect to θ.

The second result of the paper concerns the existence of solutions to the non-homogeneous boundary value problem (1.2).

Theorem 1.2. *Let* $\Omega \subset \mathbb{R}^2$ *be an exterior domain with* C^2*-smooth boundary. Suppose that* $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ *and the equality (1.22) holds, i.e., the total flux is zero. Then there exists a* D*-solution* $\mathbf{u}$ *to the Navier–Stokes boundary value problem* (1.2)*.*

This theorem shows also that the asymptotic results of Corollaries 1.1, 1.2 and (1.25) have meaning and are not just a figment of the imagination.

Note that the existence theorem for the steady Navier–Stokes problem in *three–dimensional* exterior axially symmetric domains (with axially symmetric data) was proved in the recent paper [19] without any conditions on fluxes $\mathscr{F}_i$.

2. Notations and preliminaries

By *a domain* we mean an open connected set. We use standard notations for function spaces: $W^{k,q}(\Omega)$, $W^{\alpha,q}(\partial\Omega)$, where $\alpha \in (0,1), k \in \mathbb{N}_0, q \in [1,+\infty]$. In our notation we do not distinguish function spaces for scalar and vector valued functions; it is clear from the context whether we use scalar or vector (or tensor) valued function spaces.

For $q \geq 1$ denote by $D^{k,q}(\Omega)$ the set of functions $f \in W^{k,q}_{\text{loc}}(\Omega)$ such that $\|f\|_{D^{k,q}(\Omega)} = \|\nabla^k f\|_{L^q(\Omega)} < \infty$. Further, $D_0^{1,2}(\Omega)$ is the closure of the set of all smooth functions having compact supports in Ω with respect to the norm $\|\cdot\|_{D^{1,2}(\Omega)}$, and $H(\Omega) = \{\mathbf{v} \in D_0^{1,2}(\Omega) : \operatorname{div}\mathbf{v} = 0\}$; $D_\sigma^{1,2}(\Omega) := \{\mathbf{v} \in D^{1,2}(\Omega) : \operatorname{div}\mathbf{v} = 0\}$.

3. Boundedness of general D-solutions: proof of Theorem 1.1

Suppose the assumptions of Theorem 1.1 are fulfilled. By classical regularity results for D-solutions to Navier–Stokes system, the function $\mathbf{u}$ is uniformly bounded on each bounded subset of the set $\Omega_0 = \mathbb{R}^2 \setminus B_{R_0}$; moreover, $\mathbf{u}$ is real analytic in Ω_0. By results of [13], pressure is uniformly bounded in Ω_0:

$$\sup_{x\in\Omega_0} |p(x)| \leq C < +\infty. \tag{3.1}$$

Suppose that the assertion (1.21) of the Theorem is false. Then there exists a sequence of points $x_k \in \Omega_0$ such that

$$|x_k| \to +\infty \qquad \text{and} \qquad |\mathbf{u}(x_k)| \to +\infty. \tag{3.2}$$

This means, by virtue of (3.1), that

$$\Phi(x_k) \to +\infty, \tag{3.3}$$

where $\Phi = p + \frac{1}{2}|\mathbf{u}|^2$ is the total head pressure.

Since $\mathbf{u}$ is a D-solution, $\int\limits_{\Omega_0} |\nabla\mathbf{u}|^2 dx < \infty$, by standard arguments there exists an increasing sequence on numbers $R_m < R_{m+1}$ such that $R_m \to \infty$ and

$$\int\limits_{C_{R_m}} |\nabla\mathbf{u}|\, ds \to 0, \tag{3.4}$$

where $C_R := \{x \in \mathbb{R}^2 : |x| = R\}$. It implies that

$$\sup_{x\in C_{R_m}} |\mathbf{u}(x) - \bar{\mathbf{u}}_m| \to 0, \tag{3.5}$$

here $\bar{\mathbf{u}}_m$ is the mean value of $\mathbf{u}$ on the circle C_{R_m}. Indeed, for any component u_j of $\mathbf{u}$, by mean value theorem, there exists a point $\theta_j^* \in [0, 2\pi)$ such that

$$u_j(R_m, \theta_j^*) = (2\pi)^{-1} \int\limits_0^{2\pi} u_j(R_m, \theta)d\theta = \bar{u}_{jm}, \quad j = 1, 2,$$

and

$$|u_j(R_m,\theta)-\bar u_{jm}|=|u_j(R_m,\theta)-u_j(R_m,\theta_j^*)|\le\int\limits_{\theta_j^*}^{\theta}\Big|\frac{\partial u_j}{\partial\theta}\Big|d\theta\le\int\limits_{C_{R_m}}|\nabla\mathbf{u}|\,ds\to 0.$$

Since Φ satisfies the maximum principle (see, e.g., [13]), in particular, for any subdomain $\Omega_{m_1,m_2}=\{x:R_{m_1}<|x|<R_{m_2}\}$, with $\partial\Omega_{m_1,m_2}=C_{R_{m_1}}\cup C_{R_{m_2}}$ we have

$$\sup_{x\in\Omega_{m_1,m_2}}\Phi(x)=\sup_{x\in C_{R_{m_1}}\cup C_{R_{m_2}}}\Phi(x).$$

Relations (3.2), (3.5) imply that $|\bar{\mathbf{u}}_m|\to+\infty$; consequently, by (3.1), (3.3), (3.5),

$$\inf_{x\in C_{R_m}}\Phi(x)\to+\infty.$$

Then we could assume without loss of generality (choosing a subsequence) that

$$\sup_{x\in C_{R_m}}\Phi(x)<\inf_{x\in C_{R_{m+1}}}\Phi(x). \tag{3.6}$$

Recall that by the classical Morse–Sard Theorem (see, e.g., [14]), applied to the real analytic function Φ, for almost all values $t\in\Phi(\Omega_0)$ the level set $\{\Phi=t\}$ contains no critical points, i.e., $\nabla\Phi(x)\ne 0$ if $x\in\Omega_0$ and $\Phi(x)=t$. Further such values are called *regular*. Take arbitrary regular value $t>t_*=\sup\limits_{x\in C_{R_1}\cup C_{R_0}}\Phi(x)$. Then by the implicit function theorem the level set $\{x\in\Omega_0:\Phi(x)=t\}$ consists of a finite family of disjoint smooth curves which are separated (by construction) both from infinity and from the boundary $\partial\Omega_0=C_{R_0}$. Of course, this implies that every connected component of this level set $\{\Phi=t\}$ is homeomorphic to a circle. Let us call these components *quasicircles*. By obvious geometrical arguments, for every regular $t>t_*$ there exists at least one quasicircle S separating C_{R_1} from infinity, i.e., C_{R_1} is contained in the bounded connected component of the open set $\mathbb{R}^2\setminus S$. Because of the maximum principle, such quasicircle is unique, and we will denote it by S_t.

For $t_*<\tau<t$ let $\Omega_{\tau,t}$ be a domain with $\partial\Omega_{\tau,t}=S_\tau\cup S_t$. Integrating the identity

$$\Delta\Phi=\omega^2+\frac{1}{\nu}\operatorname{div}(\Phi\mathbf{u}) \tag{3.7}$$

over $\Omega_{\tau,t}$, we obtain

$$\begin{aligned}&\int\limits_{S_t}|\nabla\Phi|\,ds-\int\limits_{S_\tau}|\nabla\Phi|\,ds=\int\limits_{\Omega_{\tau,t}}\omega^2dx+\tfrac{1}{\nu}\int\limits_{S_t}\Phi\mathbf{u}\cdot\mathbf{n}\,ds-\tfrac{1}{\nu}\int\limits_{S_\tau}\Phi\mathbf{u}\cdot\mathbf{n}\,ds\\&=\int\limits_{\Omega_{\tau,t}}\omega^2dx+\tfrac{1}{\nu}(t-\tau)\mathscr{F},\end{aligned} \tag{3.8}$$

where $\mathscr{F}=\int\limits_{C_{R_0}}\mathbf{u}\cdot\mathbf{n}$ is the total flux. Notice that by construction the unit normal $\mathbf{n}$ to the level set $S_t=\{x:\Phi(x)=t\}$ is equal to $\frac{\nabla\Phi}{|\nabla\Phi|}$, so that $\nabla\Phi\cdot\mathbf{n}=|\nabla\Phi|$ on S_t; analogously, $\nabla\Phi\cdot\mathbf{n}=-|\nabla\Phi|$ on S_τ. The further proof splits into two cases.

CASE I. The total flux is not zero: $\mathscr{F} \neq 0$. First suppose that $\mathscr{F} > 0$. Then from (3.8) (fixing τ and taking a big t) we obtain

$$C_1 t \le \int_{S_t} |\nabla\Phi|\, ds \le C_2 t \tag{3.9}$$

for sufficiently large t and for some positive constants C_1, C_2 (not depending on t). Denote by $\mathcal{R}$ the set of all regular values $t > t_*$, and put

$$E_t := \bigcup_{\tau\in[t,2t]\cap\mathcal{R}} S_\tau.$$

Applying the classical coarea formula

$$\int_{E_t} f\,|\nabla\Phi|\,dx = \int_t^{2t}\Big(\int_{S_\tau} f\,ds\Big)\,d\tau$$

for $f = |\omega|$ and for $f = |\nabla\Phi|$ we obtain

$$\begin{aligned}
&\int_t^{2t}\Big(\int_{S_\tau}|\omega|\,ds\Big)d\tau = \int_{E_t}|\omega|\cdot|\nabla\Phi|dx \le \Big(\int_{E_t}|\nabla\Phi|^2dx\Big)^{\frac12}\Big(\int_{E_t}\omega^2dx\Big)^{\frac12}\\
&= \Big(\int_t^{2t}\Big(\int_{S_\tau}|\nabla\Phi|\,ds\Big)d\tau\Big)^{\frac12}\Big(\int_{E_t}\omega^2dx\Big)^{\frac12} \le \varepsilon t,
\end{aligned} \tag{3.10}$$

where $\varepsilon \to 0$ as $t \to \infty$ (we used here (3.9) and the assumption that the Dirichlet integral is finite). From (3.10) and from the mean value theorem it follows that there exists a value $\tau \in [t, 2t] \cap \mathcal{R}$ such that

$$\int_{S_\tau} |\omega| ds \le 2\varepsilon. \tag{3.11}$$

Since the pressure is uniformly bounded (see (3.1)), we conclude that $|\mathbf{u}| \sim \sqrt{2\tau} < 2\sqrt{\tau}$ on S_τ for large τ, therefore, using the identity

$$\nabla\Phi = -\nu\nabla^\perp\omega + \omega\mathbf{u}^\perp,$$

we obtain

$$\int_{S_\tau}|\nabla\Phi|\,ds = \int_{S_\tau}\omega\mathbf{u}^\perp\cdot\mathbf{n}\,ds \le 2\sqrt{\tau}\int_{S_\tau}|\omega|\,ds \le 4\sqrt{\tau}\varepsilon \tag{3.12}$$

(the integral of $\nabla^\perp\omega\cdot\mathbf{n} = \operatorname{curl}\omega\cdot\mathbf{n}$ over the closed curve S_τ is equal to zero).

The last estimate contradicts the first inequality in (3.9). Thus, if $\mathscr{F} > 0$, then the assumption (3.2) is false and the solution $\mathbf{u}$ is uniformly bounded.

Let $\mathscr{F} < 0$. Writing relation (3.8) in the form

$$\int\limits_{S_t} |\nabla\Phi|\, ds = \int\limits_{S_\tau} |\nabla\Phi|\, ds + \int\limits_{\Omega_{\tau,t}} \omega^2 dx + \frac{1}{\nu}(t-\tau)\mathscr{F}, \tag{3.13}$$

we immediately see that for large t the right-hand side becomes negative, while the left-hand side is positive for all t. We again obtain a contradiction to assumption (3.2). Thus, the proof for the case $\mathscr{F} \neq 0$ is complete.

CASE II. The total flux is zero: $\mathscr{F} = 0$. Then formula (3.8) takes the form

$$\int\limits_{S_t} |\nabla\Phi|\, ds = \int\limits_{S_\tau} |\nabla\Phi|\, ds + \int\limits_{\Omega_{\tau,t}} \omega^2 dx. \tag{3.14}$$

From the last identity it follows that $\int\limits_{S_t} |\nabla\Phi|\, ds$ is a bounded increasing function, i.e., it has a finite positive limit, in particular,

$$C_1 \le \int\limits_{S_t} |\nabla\Phi|\, ds \le C_2 \tag{3.15}$$

for sufficiently large t and for some positive constants C_1, C_2 (independent of t). Applying the Coarea formula, we obtain now

$$\begin{aligned} \int\limits_t^{2t}\Big(\int\limits_{S_\tau} |\omega|\, ds\Big)\, d\tau &= \int\limits_{E_t} |\omega|\cdot|\nabla\Phi| dx \le \Big(\int\limits_{E_t} |\nabla\Phi|^2 dx\Big)^{\frac12}\cdot\Big(\int\limits_{E_t} \omega^2 dx\Big)^{\frac12} \\ &= \Big(\int\limits_t^{2t}\Big(\int\limits_{S_\tau} |\nabla\Phi|\, ds\Big)\, d\tau\Big)^{\frac12}\cdot\Big(\int\limits_{E_t} \omega^2 dx\Big)^{\frac12} \le \varepsilon\sqrt{t}, \end{aligned} \tag{3.16}$$

where $\varepsilon \to 0$ as $t \to \infty$. From (3.16) and from the mean value theorem the existence of a value $\tau \in [t, 2t] \cap \mathcal{R}$ follows such that

$$\int\limits_{S_\tau} |\omega|\, ds \le \varepsilon \frac{2}{\sqrt{\tau}}. \tag{3.17}$$

As in the Case I, we have $|\mathbf{u}| \sim \sqrt{2\tau}$ on S_τ. Therefore, integrating again the identity

$$\nabla\Phi = -\nu\nabla^\perp\omega + \omega\mathbf{u}^\perp,$$

we obtain

$$\int\limits_{S_\tau} |\nabla\Phi|\, ds = \int\limits_{S_\tau} \omega\mathbf{u}^\perp\cdot\mathbf{n}\, ds \le 2\sqrt{\tau}\int\limits_{S_\tau} |\omega| dx \le 4\varepsilon. \tag{3.18}$$

The last estimate is in contradiction with the first inequality in (3.15). Therefore, in the case $\mathscr{F}=0$ assumption (3.2) is again false and the solution $\mathbf{u}$ is uniformly bounded. Theorem 1.1 is proved.

4. The existence theorem: proof of Theorem 1.2

Here we need some preliminary results on real analysis and topology.

4.1. On Morse-Sard and Luzin N-properties of Sobolev functions from $W^{2,1}$

Let us recall some classical differentiability properties of Sobolev functions.

Lemma 4.1 *(see Proposition 1 in [5]). Let $\psi \in W^{2,1}(\mathbb{R}^2)$. Then the function ψ is continuous and there exists a set A_ψ such that $\mathscr{H}^1(A_\psi)=0$, and the function ψ is differentiable (in the classical sense) at each $x \in \mathbb{R}^2 \setminus A_\psi$. Furthermore, the classical derivative at such points x coincides with $\nabla\psi(x)=\lim\limits_{r\to 0} \fint_{B_r(x)} \nabla\psi(z)dz$, and $\lim\limits_{r\to 0} \fint_{B_r(x)} |\nabla\psi(z)-\nabla\psi(x)|^2 dz=0$.*

Here and henceforth we denote by $\mathscr{H}^1$ the one-dimensional Hausdorff measure, i.e., $\mathscr{H}^1(F)=\lim\limits_{t\to 0+}\mathscr{H}^1_t(F)$, where $\mathscr{H}^1_t(F)=\inf\{\sum\limits_{i=1}^{\infty}\operatorname{diam}F_i : \operatorname{diam}F_i \le t,\ F\subset \bigcup\limits_{i=1}^{\infty} F_i\}$. Note, that in this definition the case $t=\infty$ is also allowed (the value $\mathscr{H}^1_\infty(E)$ is called 'the Hausdorff content of E').

The next theorem has been proved recently by J. Bourgain, M. Korobkov and J. Kristensen [3] (see also [4] for a multidimensional case).

Theorem 4.3. *Let $\mathcal{D}\subset\mathbb{R}^2$ be a bounded domain with Lipschitz boundary and $\psi \in W^{2,1}(\mathcal{D})$. Then*

(i) $\mathscr{H}^1(\{\psi(x) : x\in\bar{\mathcal{D}}\setminus A_\psi \ \&\ \nabla\psi(x)=0\})=0$;

(ii) *for every $\varepsilon>0$ there exists $\delta>0$ such that for any set $U\subset\bar{\mathcal{D}}$ with $\mathscr{H}^1_\infty(U)<\delta$ the inequality $\mathscr{H}^1(\psi(U))<\varepsilon$ holds;*

(iii) *for $\mathscr{H}^1$–almost all $y\in\psi(\bar{\mathcal{D}})\subset\mathbb{R}$ the preimage $\psi^{-1}(y)$ is a finite disjoint family of C^1–curves S_j, $j=1,2,\dots,N(y)$. Each S_j is either a cycle in $\mathcal{D}$ (i.e., $S_j\subset\mathcal{D}$ is homeomorphic to the unit circle $\mathbb{S}^1$) or it is a simple arc with endpoints on $\partial\mathcal{D}$ (in this case S_j is transversal to $\partial\mathcal{D}$).*

4.2. Some facts from topology

We shall need some topological definitions and results. By *continuum* we mean a compact connected set. We understand connectedness in the sense of general topology. A subset of a topological space is called *an arc* if it is homeomorphic to the unit interval [0, 1].

Let us shortly present some results from the classical paper of A.S. Kronrod [20] concerning level sets of continuous functions. Let $Q=[0,1]\times[0,1]$ be a square in $\mathbb{R}^2$ and let f be a continuous function on Q. Denote by E_t a level set of the function f, i.e., $E_t=\{x\in Q : f(x)=t\}$. A component K of the level set E_t containing a point x_0 is a maximal connected subset of E_t containing x_0. By T_f denote the family of all connected components of level sets of f. It was established in [20] that T_f equipped by a natural topology[4] is a one-dimensional topological tree.[5] Endpoints of this tree[6] are the components $C \in T_f$ which do not separate Q, i.e., $Q \setminus C$ is a connected set. Branching points of the tree are the components $C \in T_f$ such that $Q \setminus C$ has more than two connected components (see [20, Theorem 5]). By results of [20, Lemma 1], the set of all branching points of T_f is at most countable. The main property of a tree is that any two points could be joined by a unique arc. Therefore, the same is true for T_f.

Lemma 4.2 *(see Lemma 13 in [20]). If $f \in C(Q)$, then for any two different points $A \in T_f$ and $B \in T_f$, there exists a unique arc $J = J(A, B) \subset T_f$ joining A to B. Moreover, for every inner point C of this arc the points A, B lie in different connected components of the set $T_f \setminus \{C\}$.*

We can reformulate the above Lemma in the following equivalent form.

Lemma 4.3. *If $f \in C(Q)$, then for any two different points $A, B \in T_f$, there exists a continuous injective function $\varphi : [0, 1] \to T_f$ with the properties*

(i) $\varphi(0) = A$, $\varphi(1) = B$;

(ii) *for any $t_0 \in [0, 1]$,*

$$\lim_{[0,1]\ni t \to t_0} \sup_{x \in \varphi(t)} \operatorname{dist}(x, \varphi(t_0)) \to 0;$$

(iii) *for any $t \in (0, 1)$ the sets A, B lie in different connected components of the set $Q \setminus \varphi(t)$.*

Remark 4.1. If in Lemma 4.3 $f \in W^{2,1}(Q)$, then by Theorem 4.3 (iii), there exists a dense subset E of $(0, 1)$ such that $\varphi(t)$ is a C^1–curve for every $t \in E$. Moreover, $\varphi(t)$ is either a cycle or a simple arc with endpoints on ∂Q.

Remark 4.2. All results of Lemmas 4.2–4.3 remain valid for level sets of continuous functions $f : \overline{\Omega}_0 \to \mathbb{R}$, where $\overline{\Omega}_0 \subset \mathbb{R}^2$ is a compact set homeomorphic to the unit square $Q = [0, 1]^2$.

4.3. Leray's argument "reductio ad absurdum"

Consider the Navier–Stokes problem (1.2) in the C^2-smooth exterior domain $\Omega \subset \mathbb{R}^2$ defined by (1.1). Let $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ have zero total flux:

$$\int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n}\, ds = 0. \tag{4.1}$$

[4] The convergence in T_f is defined as follows: $T_f \ni C_i \to C$ iff $\sup_{x \in C_i} \operatorname{dist}(x, C) \to 0$.

[5] A locally connected continuum T is called *a topological tree*, if it does not contain a curve homeomorphic to a circle, or, equivalently, if any two different points of T can be joined by a unique arc. This definition implies that T has topological dimension 1.

[6] A point of a continuum K is called an *endpoint of K* (resp., *a branching point of K*) if its topological index equals 1 (more or equal to 3 resp.). For a topological tree T this definition is equivalent to the following: a point $C \in T$ is an endpoint of T (resp., a branching point of T), if the set $T \setminus \{C\}$ is connected (resp., if $T \setminus \{C\}$ has more than two connected components).

Take an extension $\mathbf{A}$ satisfying

$$\begin{aligned} \mathbf{A} &\in W^{1,2}(\Omega), \\ \operatorname{div} \mathbf{A} &= 0 \quad \text{ in } \Omega, \\ \mathbf{A} &= \mathbf{a} \quad \text{ on } \partial\Omega, \\ \mathbf{A}(x) &= \mathbf{0} \quad \text{ if } x \in \mathbb{R}^2 \setminus B_{R_0}, \end{aligned} \tag{4.2}$$

where $B_{R_0} = B(0, R_0)$ is a disk of sufficiently large radius such that

$$\frac{1}{2} B_{R_0} \supset \partial\Omega$$

(such extension exists because of condition (4.1), see, e.g., [22]).

By a *weak solution* (= D-solution) of problem (1.2) we mean a function $\mathbf{u}$ such that $\mathbf{u} = \mathbf{w} + \mathbf{A}$, $\mathbf{w} \in H(\Omega)$, and the integral identity

$$\nu \int\limits_{\Omega} \nabla \mathbf{u} \cdot \nabla \boldsymbol{\theta} \, dx + \int\limits_{\Omega} \big(\mathbf{u} \cdot \nabla\big)\mathbf{u} \cdot \boldsymbol{\theta} \, dx = 0 \tag{4.3}$$

holds for any $\boldsymbol{\theta} \in J_0^\infty(\Omega)$, where $J_0^\infty(\Omega)$ is the set of all infinitely smooth solenoidal vector-fields with compact support in Ω. In particular, by this definition we have

$$\int\limits_{\Omega} |\nabla \mathbf{u}|^2 dx < \infty. \tag{4.4}$$

Moreover, by classical regularity results for the Navier–Stokes system (see, e.g., [21], [11]) every such solution is C^∞–regular inside the domain.

We look for a solution to (1.2) as a limit of weak solutions to the Navier–Stokes problem in a sequence of bounded domain Ω_{bk} that in the limit exhaust the unbounded domain Ω. The following result concerning the solvability of the Navier-Stokes problem in bounded multi connected domains was proved in [18].

Theorem 4.4. *Let* $\Omega' = \Omega_0 \setminus \big(\bigcup\limits_{j=1}^{N} \overline{\Omega}_j\big)$ *be a bounded domain in* $\mathbb{R}^2$ *with multiply connected* C^2*-smooth boundary* $\partial\Omega'$ *consisting of* $N+1$ *disjoint components* $\Gamma_j = \partial\Omega_j$, $j = 0, \dots, N$. *If* $\mathbf{a} \in W^{1/2,2}(\partial\Omega')$ *satisfies*

$$\int\limits_{\partial\Omega'} \mathbf{a} \cdot \mathbf{n} \, ds = 0,$$

then (1.2) *with* $\Omega = \Omega'$ *admits at least one weak solution* $\mathbf{u} \in W^{1,2}(\Omega')$.

Remark 4.3. Formally in the formulation of the existence theorem in [18] we assumed that the boundary value $\mathbf{a}$ satisfies $\mathbf{a} \in W^{3/2,2}(\Omega)$ in order to have the regularity condition $(\mathbf{u}, p) \in W^{2,2}(\Omega)$. But really we used only *local* variant of such regularity $(\mathbf{u}, p) \in W^{2,2}_{\mathrm{loc}}(\Omega)$ (see [18, page 784, line 8 from below]). Now in our situation every D-solution has much better C^∞ regularity inside the domain Ω, so we could assume less restrictive condition $\mathbf{a} \in W^{1/2,2}(\Omega)$.

Take an increasing sequence $R_k \to +\infty$ and consider the boundary value problems

$$\begin{cases} -\nu\Delta\widehat{\mathbf{u}}_k + (\widehat{\mathbf{u}}_k \cdot \nabla)\widehat{\mathbf{u}}_k + \nabla \widehat{p}_k = \mathbf{0} & \text{in } \Omega_{bk}, \\ \operatorname{div} \widehat{\mathbf{u}}_k = 0 & \text{in } \Omega_{bk}, \\ \widehat{\mathbf{u}}_k = \mathbf{a} & \text{on } \partial\Omega, \\ \widehat{\mathbf{u}}_k = \mathbf{0} & \text{on } C_{R_k} = \partial B_k, \end{cases} \tag{4.5}$$

where $\Omega_{bk} = B_k \cap \Omega$ for $k \geq k_0$, $B_k = \{x : |x| < R_k\}$, $\frac{1}{2}B_{k_0} \supset \bigcup_{i=1}^{N} \overline{\Omega}_i$. By Theorem 4.4, each problem (4.5) has a solution $\widehat{\mathbf{u}}_k \in W^{1,2}(\Omega_{bk})$ satisfying $\operatorname{div} \widehat{\mathbf{u}}_k = 0$ and the corresponding integral identities (of (4.3) type).

Assume that there is a positive constant c independent of k such that

$$\int_\Omega |\nabla \widehat{\mathbf{u}}_k|^2 dx \leq c \tag{4.6}$$

(possibly along a subsequence of $\{\widehat{\mathbf{u}}_k\}_{k\in\mathbb{N}}$). This estimate implies the existence of a solution to problem (1.2). Indeed, from (4.6) and from the boundary conditions $(4.5)_3$ it follows that the sequence $\widehat{\mathbf{u}}_k$ is bounded in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$. Hence, $\widehat{\mathbf{u}}_k$ converges weakly (modulo a subsequence) in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$ and strongly in $L^q_{\mathrm{loc}}(\overline{\Omega})$ $(1 \leq q < \infty)$ to a function $\widehat{\mathbf{u}} \in D^{1,2}_\sigma(\Omega)$. It is easy to check that this limiting function $\widehat{\mathbf{u}}$ is a D-solution to the Navier–Stokes problem (1.2) in the exterior domain Ω.

Thus, to prove the assertion of Theorem 1.2, it is sufficient to establish the uniform estimate (4.6). We shall prove (4.6) following a classical *reductio ad absurdum* argument of J. Leray [23] and O.A. Ladyzhenskaia [21]. If (4.6) is not true, then there exists a sequence $\{\widehat{\mathbf{u}}_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty} J_k^2 = +\infty, \qquad J_k^2 = \int_\Omega |\nabla \widehat{\mathbf{u}}_k|^2 dx.$$

The sequence $\mathbf{u}_k = \widehat{\mathbf{u}}_k / J_k$ is bounded in $D^{1,2}_\sigma(\Omega) \cap L^q_{\mathrm{loc}}(\overline{\Omega})$ and it holds

$$\frac{\nu}{J_k} \int_\Omega \nabla \mathbf{u}_k \cdot \nabla \boldsymbol{\theta}\, dx = -\int_\Omega (\mathbf{u}_k \cdot \nabla)\mathbf{u}_k \cdot \boldsymbol{\theta}\, dx \tag{4.7}$$

for all $\boldsymbol{\theta} \in H(\Omega_{bk})$. Extracting a subsequence (if necessary) we can assume that $\mathbf{u}_k$ converges weakly in $D^{1,2}_\sigma(\Omega)$ and strongly in $L^q_{\mathrm{loc}}(\overline{\Omega})$ $(1 \leq q < \infty)$ to a vector field $\mathbf{v} \in H(\Omega)$ with

$$\int_\Omega |\nabla \mathbf{v}|^2 dx \leq 1. \tag{4.8}$$

Fixing in (4.7) a solenoidal smooth $\boldsymbol{\theta}$ with compact support and letting $k \to +\infty$ we get

$$\int\limits_{\Omega} (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \boldsymbol{\theta}\, dx = 0 \quad \forall \boldsymbol{\theta} \in J_0^\infty(\Omega). \tag{4.9}$$

Hence, $\mathbf{v} \in H(\Omega)$ is a weak solution to the Euler equations, and for some $p \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega})$, $(1 < q < 2)$, the pair $(\mathbf{v}, p)$ satisfies the Euler equations almost everywhere:

$$\begin{cases} \big(\mathbf{v} \cdot \nabla\big)\mathbf{v} + \nabla p & = & 0 & \text{in } \ \Omega, \\ \operatorname{div} \mathbf{v} & = & 0 & \text{in } \ \Omega, \\ \mathbf{v} & = & 0 & \text{on } \ \partial\Omega. \end{cases} \tag{4.10}$$

Put $\nu_k = (J_k)^{-1}\nu$. Then the system (4.5) could be rewritten in the following form

$$\begin{cases} -\nu_k \Delta \mathbf{u}_k + (\mathbf{u}_k \cdot \nabla)\mathbf{u}_k + \nabla p_k = \mathbf{0} & \text{in } \Omega_{bk}, \\ \operatorname{div} \mathbf{u}_k = 0 & \text{in } \Omega_{bk}, \\ \mathbf{u}_k = \frac{\nu_k}{\nu}\mathbf{a} & \text{on } \partial\Omega, \\ \mathbf{u}_k = \mathbf{0} & \text{on } C_{R_k} = \partial B_k, \end{cases} \tag{4.11}$$

where $\mathbf{u}_k,\ p_k \in C^\infty_{\mathrm{loc}}(\Omega_{bk})$. In conclusion, we come to the following assertion.

Lemma 4.4. *Assume that $\Omega \subset \mathbb{R}^2$ is an exterior domain of type (1.1) with C^2-smooth boundary $\partial\Omega$, and $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ satisfies zero total flux condition (1.22). If the assertion of Theorem 1.2 is false, then there exist $\mathbf{v}, p$ with the following properties.*

(E) The functions $\mathbf{v} \in H(\Omega)$, $p \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega})$ $(\forall q \in [1, 2))$ satisfy the Euler system (4.10).

(E-NS) Condition (E) is fulfilled and there exist sequences of functions $\mathbf{u}_k \in W^{1,2}(\Omega_{bk})$, $p_k \in W^{1,q}(\Omega_{bk})$, $\Omega_{bk} = \Omega \cap B_{R_k}$, $R_k \to \infty$ as $k \to \infty$, and numbers $\nu_k \to 0+$, such that the pair $(\mathbf{u}_k, p_k)$ satisfies (4.11), and

$$\|\nabla \mathbf{u}_k\|_{L^2(\Omega_{bk})} = 1, \quad \mathbf{u}_k \rightharpoonup \mathbf{v} \ \text{ in } \ W^{1,2}_{\mathrm{loc}}(\overline{\Omega}), \quad p_k \rightharpoonup p \ \text{ in } \ W^{1,q}_{\mathrm{loc}}(\overline{\Omega}), \tag{4.12}$$

$$\nu = \int\limits_{\Omega} (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \mathbf{A}\, dx. \tag{4.13}$$

Moreover, $\mathbf{u}_k,\ p_k \in C^\infty(\Omega_{bk})$ (this notation means C^∞-regularity inside the domain Ω_{bk}).

Proof. We need to prove only the identity (4.13), all other properties are already established above. By construction $\mathbf{u}_k = \mathbf{w}_k + \frac{1}{J_k}\mathbf{A}$, where $\mathbf{w}_k \in H(\Omega_{bk})$, in particular, $\mathbf{w}_k \equiv 0$ on $\partial\Omega_{bk}$. Choosing $\boldsymbol{\theta} = \mathbf{w}_k$ in (4.7) and integration by parts yields

$$\nu = \int\limits_{\Omega} (\mathbf{w}_k \cdot \nabla)\mathbf{w}_k \cdot \mathbf{A}\, dx + \frac{1}{J_k}\int\limits_{\Omega} \mathbf{A} \cdot \nabla \mathbf{w}_k \cdot \mathbf{A}\, dx + \frac{\nu}{J_k}\int\limits_{\Omega} \nabla \mathbf{A} \cdot \nabla \mathbf{u}_k\, dx. \tag{4.14}$$

Since $\mathbf{A} \in W^{1,2}(\Omega)$ has a compact support, it is easy to check that we can pass to the limit in (4.14) and receive the required assertion (4.13). □

Notice that because of (4.13) the limiting solution $\mathbf{v}$ of the Euler system (4.10) is nontrivial.

Now, to finish the proof of Theorem 1.2, we need to show that conditions (E-NS) lead to a contradiction. The next two subsections are devoted to this purpose.

4.4. Some properties of solutions to Euler system

In this section we assume that the assumptions (E) of Lemma 4.4 are satisfied. In particular,

$$\int\limits_{\Omega} |\nabla \mathbf{v}(x)|^2 \, dx < \infty. \tag{4.15}$$

The next statement was proved in [15, Lemma 4] and in [2, Theorem 2.2].

Theorem 4.5. *Let the conditions* (E) *be fulfilled. Then*

$$\forall j \in \{1, \dots, N\} \; \exists \, \widehat{p}_j \in \mathbb{R} : \quad p(x) \equiv \widehat{p}_j \quad \textit{for } \mathscr{H}^1 - \textit{almost all } x \in \Gamma_j. \tag{4.16}$$

Using the last fact, below we assume without loss of generality that the functions $\mathbf{v}$, p are extended to the whole plane $\mathbb{R}^2$ as follows:

$$\mathbf{v}(x) := 0, \quad x \in \mathbb{R}^2 \setminus \Omega, \tag{4.17}$$

$$p(x) := \widehat{p}_j, \;\; x \in \mathbb{R}^2 \cap \bar{\Omega}_j, \;\; j = 1, \dots, N. \tag{4.18}$$

Obviously, the extended functions inherit the properties of the previous ones. Namely, $\mathbf{v} \in H(\mathbb{R}^2)$, $p \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^2)$, and the Euler equations (4.10) are fulfilled almost everywhere in $\mathbb{R}^2$. That means, the pair $(\mathbf{v}, p)$ is a weak (=Sobolev) solution to Euler system (4.10) *in the whole plane*.

First of all, we prove the uniform boundedness and continuity of the pressure.

Theorem 4.6. *Let the conditions* (E) *be fulfilled. Then*

$$p \in D^{2,1}(\mathbb{R}^2) \cap D^{1,2}(\mathbb{R}^2). \tag{4.19}$$

In particular, the function p is continuous and convergent at infinity, i.e.,

$$\exists \lim_{x \to \infty} p(x) \in \mathbb{R}. \tag{4.20}$$

Proof. By well-known fact concerning D-solutions to Euler and Navier–Stokes system (see, e.g., [13, Lemma 4.1]), the averages of the pressure are uniformly bounded:

$$\sup_{r>0} \left| \frac{1}{r} \int\limits_{C_r} p \, ds \right| < \infty, \tag{4.21}$$

where, recall, $C_r = \{x \in \mathbb{R}^2 : |x| = r\}$. Moreover, since $\int\limits_{\mathbb{R}^2} |\nabla \mathbf{v}|^2 dx < \infty$, there exists an increasing sequence $r_i \to +\infty$ such that

$$\int\limits_{C_{r_i}} |\nabla \mathbf{v}|\, ds \le \varepsilon_i \to 0 \quad \text{as } i \to \infty \tag{4.22}$$

and

$$\sup_{x \in C_{r_i}} \big|\mathbf{v}(x)\big| \le \varepsilon_i \sqrt{\ln r_i} \tag{4.23}$$

(see [13, Lemmas 2.1–2.2])). From (4.21)–(4.22) and from the equation $(4.10)_1$ it follows that

$$\sup_{x \in C_{r_i}} \big|p(x)\big| \le C\sqrt{\ln r_i}. \tag{4.24}$$

Indeed,

$$|p(r_i,\theta) - \bar{p}(r_i)| \le \int\limits_{C_{r_i}} |\nabla p|\, ds \le \int\limits_{C_{r_i}} |\mathbf{v}| \cdot |\nabla \mathbf{v}|\, ds \le \varepsilon_i \sqrt{\ln r_i} \int\limits_{C_{r_i}} |\nabla \mathbf{v}|\, ds \le \varepsilon_i^2 \sqrt{\ln r_i},$$

here $\bar{p}(r_i) = \frac{1}{2\pi r_i} \int\limits_{C_{r_i}} p\, ds$. The last inequality and the uniform boundedness of $\bar{p}(r_i)$ (see (4.21)) implies (4.24).

Clearly, $p \in W^{1,q}_{\rm loc}(\mathbb{R}^2)$ is the weak solution to the Poisson equation

$$\Delta p = -\nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top \quad \text{in } \mathbb{R}^2 \tag{4.25}$$

(recall that after our agreement about extension of $\mathbf{v}$ and p, see (4.17)–(4.18), the Euler equations (4.10) are fulfilled in the whole $\mathbb{R}^2$).

Put

$$G(x) = -\frac{1}{2\pi} \int\limits_{\Omega} \log|x-y| (\nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top)(y)\, dy.$$

By the results of [9], $\nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$. Hence by Calderón–Zygmund theorem for Hardy's spaces [32], $G \in D^{2,1}(\mathbb{R}^2) \cap D^{1,2}(\mathbb{R}^2)$. By classical facts from the theory of Sobolev spaces (see, e.g., [26]), the last inclusion implies that G is continuous and convergent at infinity, in particular,

$$\sup_{x \in \mathbb{R}^2} |G(x)| < \infty. \tag{4.26}$$

Consider the function $p_* = p - G$. By construction, $\Delta p_* = 0$ in $\mathbb{R}^2$, i.e., p_* is a harmonic function, and by (4.24), (4.26) we have

$$\sup_{x\in C_{r_i}} |p_*(x)| \le C\sqrt{\ln r_i}. \tag{4.27}$$

From the Liouville type theorems for harmonic functions (see, i.e., [1]) it follows that $p_* \equiv \text{const}$. Consequently, $p \equiv G + \text{const}$, that implies the assertions of the Theorem. □

We say that the function $f \in W^{1,s}_{\text{loc}}(\mathbb{R}^2)$ satisfies a *weak one-side maximum principle*, if

$$\operatorname*{ess\,sup}_{x\in\Omega'} f(x) \le \operatorname*{ess\,sup}_{x\in\partial\Omega'} f(x) \tag{4.28}$$

holds for any bounded subdomain Ω' with the boundary $\partial\Omega'$ not containing singleton connected components. (In (4.28) negligible sets are the sets of 2–dimensional Lebesgue measure zero in the left *esssup*, and the sets of 1–dimensional Hausdorff measure zero in the right *esssup*.)

The total head pressure for the Euler system

$$\Phi := p + \frac{1}{2}|\mathbf{v}|^2$$

plays an important role in the forthcoming considerations. The following two results were proved in [16].

Theorem 4.7. *Suppose that the assumptions $(E - NS)$ from the previous subsection are satisfied. Then the total head pressure Φ satisfies the weak maximum principle in $\mathbb{R}^2$.*

The second equality in (4.10) (which is fulfilled, after the above extension agreement, see (4.17)–(4.18), in the whole plane $\mathbb{R}^2$) implies the existence of a stream function $\psi \in W^{2,2}_{\text{loc}}(\mathbb{R}^2)$ such that

$$\nabla\psi = \mathbf{v}^{\perp}, \tag{4.29}$$

i.e.,

$$\frac{\partial\psi}{\partial x_1} = v_2, \qquad \frac{\partial\psi}{\partial x_2} = -v_1. \tag{4.30}$$

Let us formulate regularity results concerning the considered functions.

Lemma 4.5 *(see, e.g., Theorem 3.1 in [16]). If conditions* (E) *are satisfied, then $\psi \in C(\mathbb{R}^2)$ and there exists a set $A_{\mathbf{v}} \subset \mathbb{R}^2$ such that*

(i) *$\mathscr{H}^1(A_{\mathbf{v}}) = 0$;*

(ii) *for all $x \in \Omega \setminus A_{\mathbf{v}}$*

$$\lim_{r\to 0} \fint_{B_r(x)} |\mathbf{v}(z) - \mathbf{v}(x)|^2 dz = \lim_{r\to 0} \fint_{B_r(x)} |\Phi(z) - \Phi(x)|^2 dz = 0;$$

moreover, the function ψ is differentiable at x and $\nabla\psi(x) = (v_2(x), -v_1(x))$;

(iii) *for every $\varepsilon > 0$ there exists a set $U \subset \mathbb{R}^2$ with $\mathscr{H}^1_\infty(U) < \varepsilon$ such that $A_{\mathbf{v}} \subset U$ and the functions $\mathbf{v}$, Φ are continuous in $\mathbb{R}^2 \setminus U$.*

By virtue of (4.17), we have $\nabla\psi(x) = 0$ for almost all $x \in \Omega_j$. Then

$$\forall j \in \{1, \dots, N\} \ \exists \xi_j \in \mathbb{R}: \quad \psi(x) \equiv \xi_j \qquad \forall x \in \overline{\Omega_j} \cap \mathbb{R}^2. \tag{4.31}$$

By direct calculations one easily gets the identity

$$\nabla\Phi = \omega\nabla\psi, \tag{4.32}$$

here $\omega = \Delta\psi = \partial_1 v_2 - \partial_2 v_1$ means the corresponding vorticity.

The next assertion, obtained in the paper [16], is the another important tool for the proof of Theorem 2.

Theorem 4.8 *(Bernoulli law for Sobolev solutions). Let the conditions* (E) *be valid. Then there exists a set $A_{\mathbf{v}} \subset \mathbb{R}^2$ with $\mathscr{H}^1(A_{\mathbf{v}}) = 0$, such that for any compact connected*[7] *set $K \subset \mathbb{R}^2$ the following property holds: if*

$$\psi\big|_K = \mathrm{const}, \tag{4.33}$$

then

$$\Phi(x_1) = \Phi(x_2) \quad \textit{for all} \ \ x_1, x_2 \in K \setminus A_{\mathbf{v}}. \tag{4.34}$$

Of course, we could assume without loss of generality that the sets $A_{\mathbf{v}}$ from Lemma 4.5 and Theorem 4.8 are the same.

Identities (4.17)–(4.18) mean that

$$\Phi(x) \equiv \widehat{p}_j \qquad \forall x \in \mathbb{R}^2 \cap \overline{\Omega}_j, \ \ j = 1, \dots, N. \tag{4.35}$$

Now consider the behavior of Φ at infinity. By construction, there exists an increasing sequence of numbers $r_i \to +\infty$ such that

$$\sup_{x \in C_{r_i}} |\Phi(x) - \overline{\Phi}_i| \to 0, \tag{4.36}$$

where $\overline{\Phi}_i = \fint_{C_{r_i}} \Phi(x)\,ds$ is the mean value of Φ over the circle C_{r_i}. Indeed, by definition, $|\nabla\Phi| \le |\mathbf{v}| \cdot |\nabla\mathbf{v}|$. By standard estimates (e.g., [11, Lemma 2.1])

$$\int\limits_{C_r} |\mathbf{v}|^2 ds \le Cr \ln r. \tag{4.37}$$

Further, since $\int\limits_{\mathbb{R}^2} |\nabla\mathbf{v}|^2 dx = \int\limits_0^\infty dr \int\limits_{C_r} |\nabla\mathbf{v}|^2 ds < \infty$, there exists an increasing sequence $r_i \to +\infty$ such that

7 We understand the connectedness in the sense of general topology.

$$\int\limits_{C_{r_i}} |\nabla \mathbf{v}|^2 ds \le \frac{\varepsilon_i}{r_i \ln r_i}, \tag{4.38}$$

where $\varepsilon_i \to 0$. Formulas (4.37)–(4.38) and the Hölder inequality imply

$$\int\limits_{C_{r_i}} |\nabla \Phi|\, ds \le \int\limits_{C_{r_i}} |\mathbf{v}| \cdot |\nabla \mathbf{v}|\, ds \le \sqrt{C\varepsilon_i} \to 0, \tag{4.39}$$

thus we obtain (4.36).

From the weak maximum principle (see Theorem 4.7) it follows that there exists a limit $\Phi_\infty = \lim\limits_{i\to\infty} \overline{\Phi_i}$, which does not depend on the choice of circles C_{r_i} (it can be $\Phi_\infty = \infty$). Again the same maximum principle implies that

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^2} \Phi(x) = \max\{\Phi_\infty, \widehat{p}_1, \dots, \widehat{p}_N\}, \tag{4.40}$$

where $\widehat{p}_j$ are the constants form Theorem 4.5. Below we consider separately three possible cases.

(a) The maximum of Φ is attained strictly at infinity,[8] i.e.,

$$\Phi_\infty = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) > \max\{\widehat{p}_1, \dots, \widehat{p}_N\}. \tag{4.41}$$

(b) The maximum of Φ is attained on some boundary component — not at infinity:

$$\max\{\widehat{p}_1, \dots, \widehat{p}_N\} = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) > \Phi_\infty. \tag{4.42}$$

(c) The maximum of Φ is attained both at infinity and on some boundary component:

$$\Phi_\infty = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) = \max\{\widehat{p}_1, \dots, \widehat{p}_N\}. \tag{4.43}$$

4.5. *The case* $\operatorname*{ess\,sup}\limits_{x\in\Omega} \Phi(x) = \Phi_\infty > \max\{\widehat{p}_1, \dots, \widehat{p}_N\}$

Let us consider the first case (4.41). We will adopt the arguments of [18, subsection 2.4.1]. Note that the calculation in the present situations are much easier, since the set where Φ close to the maximum is separated from the boundary components. For the reader convenience, in this subsection we reproduce these arguments in details.

Since the pressure is determined up to an additive constant, without loss of generality we could assume that

$$\Phi_\infty > \delta > 0 > -\delta > \max\{\widehat{p}_1, \dots, \widehat{p}_N\}, \tag{4.44}$$

where δ is sufficiently small positive number.

[8] The case $\operatorname*{ess\,sup}\limits_{x\in\Omega} \Phi(x) = +\infty$ is not excluded.

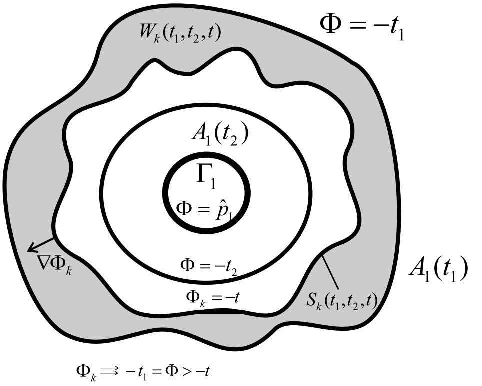

Fig. 1. The surface $S_k(t_1, t_2, t)$ for the case of $N = 1$.

Since $\mathscr{H}^1(A_{\mathbf{v}}) = 0$, the intersection $C_r \cap A_{\mathbf{v}}$ is empty[9] for 1-almost all $r > 0$. Then by definition of Φ_∞ (see, e.g., (4.36)), there exists a radius $r_0 > 0$ such that $B_{\frac{1}{2}r_0} \supset \partial\Omega$ and

$$C_{r_0} \cap A_{\mathbf{v}} = \emptyset; \tag{4.45}$$

$$\inf_{x \in C_{r_0}} \Phi(x) \geq \delta. \tag{4.46}$$

Our first goal is to separate the boundary components Γ_j where $\Phi < 0$ from C_{r_0} by level sets of Φ compactly supported in Ω. More precisely, for any $t \in (0, \delta)$ and $j = 1, \dots, N$ we construct a continuum $A_j(t) \Subset \Omega$ with the following properties:

(i) The set $\Gamma_j = \partial\Omega_j$ lies in a bounded connected component of the open set $\mathbb{R}^2 \setminus A_j(t)$;

(ii) $\psi|_{A_j(t)} \equiv \text{const}$, $\Phi(A_j(t)) = -t$;

(iii) (monotonicity) If $0 < t_1 < t_2 < \delta_p$, then $A_j(t_1)$ lies in the unbounded connected component of the set $\mathbb{R}^2 \setminus A_j(t_2)$ (in other words, the set $A_j(t_2) \cup \Gamma_j$ lies in the bounded connected component of the set $\mathbb{R}^2 \setminus A_j(t_1)$, see Fig. 1).

For this construction, we shall use the results of Subsection 4.2. More precisely, we apply Kronrod's results to the stream function $\psi|_{\bar{B}_{r_0}}$. Accordingly, T_ψ^0 means the corresponding Kronrod tree for the restriction $\psi|_{\bar{B}_{r_0}}$.

For any element $C \in T_\psi^0$ with $C \setminus A_{\mathbf{v}} \neq \emptyset$ we can define the value $\Phi(C)$ as $\Phi(C) = \Phi(x)$, where $x \in C \setminus A_{\mathbf{v}}$. This definition is correct because of the Bernoulli Law. (In particular, $\Phi(C)$ is well defined if $\operatorname{diam} C > 0$.)

Take points $x_0 \in C_{r_0}$ and $x_j \in \Omega_j$, $j = 1, \dots, N$, such that the straight segment L_j with endpoints x_0 and x_j satisfies

$$L_j \cap A_{\mathbf{v}} = \emptyset; \tag{4.47}$$

$$\text{the restriction } \Phi|_{L_j} \text{ is a continuous function} \tag{4.48}$$

(the existence of such points and segments follows from Lemma 4.5 (iii)).

[9] It follows from the fact that the image of a set of 1-measure zero under every smooth transformation has 1-measure zero as well, see, e.g., [6]. Here and in the sequel "1-measure zero" means "$\mathscr{H}^1$-measure zero".

Denote by E_0 and E_j the elements of T_ψ^0 with $x_0 \in E_0$ and $x_j \in E_j$. Note that from $\psi|_{\Omega_j} \equiv$ const it follows that $\overline{\Omega}_j \subset E_j$. Consider the arc $[E_j, E_0] \subset T_\psi^0$. Recall that, by definition, a connected component C of a level set of $\psi|_{\bar{B}_{r_0}}$ belongs to the arc $[E_j, E_0]$ iff $C = E_0$, or $C = E_j$, or C separates E_0 from E_j in $\bar{B}_{r_0}$, i.e., if E_0 and E_j lie in different connected components of $\bar{B}_{r_0} \setminus C$. In particular, since $E_0 \cap L_j \neq \emptyset \neq E_j \cap L_j$, we have

$$C \cap L_j \neq \emptyset \quad \forall C \in [E_j, E_0]. \tag{4.49}$$

Therefore, in view of equality (4.47) the value $\Phi(C)$ is well defined for all $C \in [E_j, E_0]$. Moreover, we have

Lemma 4.6. *The restriction $\Phi|_{[E_j,E_0]}$ is a continuous function.*

Proof. The assertion follows immediately[10] from the assumptions (4.47)–(4.49), from the continuity of $\Phi|_{L_j}$, and from the definition of convergence in T_ψ^0 (see Subsection 4.2). □

Define the natural order[11] on the arc $[E_j, E_0]$. Namely, we say, that $A < C$ for some different elements $A, C \in [E_j, E_0]$ iff C closer to E_0 than A, i.e., if the sets E_0 and C lie in the same connected component of the set $\bar{B}_{r_0} \setminus A$.

Put

$$K_j = \min\{K \in [E_j, E_0] : K \cap C_{r_0} \neq \emptyset\}$$

(this minimum exists since $E_0 \cap C_{r_0} \neq \emptyset$). By elementary and obvious topological arguments we have

$$\forall K \in [E_j, E_0] \quad \big(K \cap C_{r_0} \neq \emptyset \Leftrightarrow K \geq K_j\big). \tag{4.50}$$

From (4.45)–(4.46) and from the Bernoulli Law it follows that

$$\Phi(K) \geq \delta \qquad \forall K \in [K_j, E_0]. \tag{4.51}$$

In particular, since $\Phi(E_j) < -\delta$, we have

$$E_j < K_j \leq E_0. \tag{4.52}$$

By construction,

$$K \cap C_{r_0} = \emptyset \qquad \forall K \in [E_j, K_j), \tag{4.53}$$

where, as usual, $[E_j, K_j) = [E_j, K_j] \setminus \{K_j\}$.

We say that a set $\mathcal{Z} \subset [E_j, E_0]$ has T-measure zero if $\mathscr{H}^1(\{\psi(K) : K \in \mathcal{Z}\}) = 0$.

10 See also the proof of Lemma 3.5 in [18].

11 Recall, that by Lemma 4.2, the set $[E_j, E_0]$ is homeomorphic to the segment of a real line, i.e. it is an arc. So we could define a natural order on this arc and take maxima, minima etc. — as for usual segment. There are two symmetric possibilities to define a usual linear order on the arc; here by our choice $E_j < E_0$.

Lemma 4.7. *For every $j = 1, \dots, N$, T-almost all $K \in [E_j, K_j]$ are C^1-curves homeomorphic to the circle and $K \cap A_{\mathbf{v}} = \emptyset$. Moreover, there exists a subsequence Φ_{k_l} such that $\Phi_{k_l}|_K$ converges to $\Phi|_K$ uniformly $\Phi_{k_l}|_K \rightrightarrows \Phi|_K$ on T-almost all $K \in [E_j, E_0]$.*

Proof. The first assertion of the lemma follows from Theorem 4.3 (iii) and (4.53). The validity of the second one for T-almost all $K \in [E_j, K_j]$ was proved in [16, Lemma 3.3]. □

Below we assume (without loss of generality) that the subsequence Φ_{k_l} coincides with the whole sequence Φ_k. Furthermore, we will call *regular* the cycles K which satisfy the assertion of Lemma 4.7.

Since $\operatorname{diam} C > 0$ for every $C \in [E_j, E_0]$, we obtain, by [18, Lemma 3.6], that the function $\Phi|_{[E_j, E_0]}$ has the following analog of Luzin's N-property.

Lemma 4.8. *For every $j = 1, \dots, N$, if $\mathcal{Z} \subset [E_j, E_0]$ has T-measure zero, then $\mathscr{H}^1(\{\Phi(K) : K \in \mathcal{Z}\}) = 0$.*

Note that Lemma 4.8 is not tautological: in the definition of T-zero measure we have stream function ψ, but Lemma 4.8 deals about another function, total head pressure Φ. It looks like Luzin N-property: $\psi(E)$ has zero measure implies $\Phi(E)$ has zero measure.

From Lemmas 4.7–4.8 and from (4.51) we conclude

Corollary 4.3. *For every $j = 1, \dots, N$ and for almost all $t \in (0, \delta)$ we have*

$$\big(K \in [E_j, E_0] \text{ and } \Phi(K) = -t\big) \Rightarrow K \text{ is a regular cycle.}$$

Below we will say that a value $t \in (0, \delta)$ is *regular* if it satisfies the assertion of Corollary 4.3. Denote by $\mathscr{T}$ the set of all regular values. Then the set $(0, \delta) \setminus \mathscr{T}$ has zero measure.

For $t \in (0, \delta)$ and $j \in \{1, \dots, N\}$ denote

$$A_j(t) = \max\{K \in [E_j, E_0] : \Phi(K) = -t\}.$$

By construction, $A_j(t)$ is nonincreasing and satisfies the properties (i)–(iii) from the beginning of this subsection. Moreover, by definition of regular values we have the following additional property:

(iv) If $t \in \mathscr{T}$, then $A_j(t)$ is a regular cycle.[12]

For $t \in \mathscr{T}$ denote by $V(t)$ the unbounded connected component of the open set $\mathbb{R}^2 \setminus \big(\cup_{j=1}^N A_j(t)\big)$. Since $A_{j_1}(t)$ can not separate $A_{j_2}(t)$ from infinity[13], for $A_{j_1}(t) \neq A_{j_2}(t)$, we have

$$\partial V(t) = A_1(t) \cup \dots \cup A_N(t), \qquad t \in \mathscr{T}. \tag{4.54}$$

By construction, the sequence of domains $V(t)$ is increasing, i.e., $V(t_1) \subset V(t_2)$ for $t_1 < t_2$.

[12] Some of these cycles $A_j(t)$ could coincide, i.e., equalities of type $A_{j_1}(t) = A_{j_2}(t)$ are possible (if Kronrod arcs $[E_{j_1}, E_0)$ and $[E_{j_2}, E_0)$ have nontrivial intersection), but this a priori possibility has no influence on our arguments.

[13] Indeed, if $A_{j_2}(t)$ lies in a bounded component of $\mathbb{R}^2 \setminus A_{j_1}(t)$, then by construction $A_{j_1}(t) \in [E_{j_2}, E_0]$ and $A_{j_1}(t) > A_{j_2}(t)$ with respect to the above defined order on $[E_{j_2}, E_0]$. However, it contradicts the definition of $A_{j_2}(t) = \max\{K \in [E_{j_2}, E_0] : \Phi(K) = -t\}$.

Let $t_1, t_2 \in \mathscr{T}$ and $t_1 < t_2$. The next geometrical objects play an important role in the estimates below: for $t \in (t_1, t_2)$ we define the level set $S_k(t, t_1, t_2) \subset \{x \in \Omega_{bk} : \Phi_k(x) = -t\}$ separating cycles $\cup_{j=1}^N A_j(t_1)$ from $\cup_{j=1}^N A_j(t_2)$ as follows. Namely, take arbitrary $t', t'' \in \mathscr{T}$ such that $t_1 < t' < t'' < t_2$. From Properties (ii), (iv) we have the uniform convergence $\Phi_k|_{A_j(t_1)} \rightrightarrows -t_1$, $\Phi_k|_{A_j(t_2)} \rightrightarrows -t_2$ as $k \to \infty$ for every $j = 1, \dots, N$. Thus there exists $k_\circ = k_\circ(t_1, t_2, t', t'') \in \mathbb{N}$ such that for all $k \ge k_\circ$

$$\Phi_k|_{A_j(t_1)} > -t', \quad \Phi_k|_{A_j(t_2)} < -t'' \quad \forall j = 1, \dots, N. \tag{4.55}$$

In particular,

$$\begin{aligned} &\Phi_k|_{A_j(t_1)} > -t, \quad \Phi_k|_{A_j(t_2)} < -t, \quad \forall t \in [t', t''],\ \forall k \ge k_\circ, \\ &\forall j = 1, \dots, N. \end{aligned} \tag{4.56}$$

For $k \ge k_\circ$, $j = 1, \dots, N$, and $t \in [t', t'']$ denote by $W_k^j(t_1, t_2; t)$ the connected component of the open set $\{x \in V(t_2) \setminus \overline{V}(t_1) : \Phi_k(x) > -t\}$ such that $\partial W_k^j(t_1, t_2; t) \supset A_j(t_1)$ (see Fig. 1) and put

$$W_k(t_1, t_2; t) = \bigcup_{j=1}^N W_k^j(t_1, t_2; t), \qquad S_k(t_1, t_2; t) = (\partial W_k(t_1, t_2; t)) \cap V(t_2) \setminus \overline{V}(t_1).$$

Clearly, $\Phi_k \equiv -t$ on $S_k(t_1, t_2; t)$. By construction (see Fig. 1),

$$\partial W_k(t_1, t_2; t) = S_k(t_1, t_2; t) \cup A_1(t_1) \cup \dots \cup A_N(t_1). \tag{4.57}$$

(Note that $W_k(t_1, t_2; t))$ and $S_k(t_1, t_2; t)$ are well defined for all $t \in [t', t'']$ and $k \ge k_\circ = k_\circ(t_1, t_2, t', t'')$.)

Since by (E–NS) each Φ_k belongs to $C^\infty(\Omega_{bk})$, by the classical Morse-Sard theorem we have that for almost all $t \in [t', t'']$ the level set $S_k(t_1, t_2; t)$ consists of finitely many C^∞-cycles and Φ_k is differentiable (in classical sense) at every point $x \in S_k(t_1, t_2; t)$ with $\nabla \Phi_k(x) \neq 0$. The values $t \in [t', t'']$ having the above property will be called *k-regular*.

By construction, for every k-regular value $t \in [t', t'']$ the set $S_k(t', t''; t)$ is a finite union of smooth cycles, and

$$\int\limits_{S_k(t_1, t_2; t)} \nabla \Phi_k \cdot \mathbf{n}\, ds = - \int\limits_{S_k(t_1, t_2; t)} |\nabla \Phi_k|\, ds < 0, \tag{4.58}$$

where $\mathbf{n}$ is the unit outward normal vector to $\partial W_k(t_1, t_2; t)$.

The last inequality leads us to the main result of this subsection.

Lemma 4.9. *Assume that $\Omega \subset \mathbb{R}^2$ is a domain of type (1.1) with C^2-smooth boundary $\partial\Omega$, and $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ satisfies zero total flux condition (1.22). Then assumptions (E-NS) and (4.41) lead to a contradiction.*

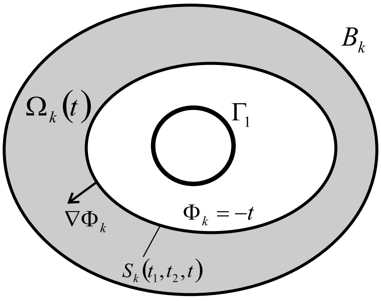

Fig. 2. The domain $\Omega_k(t)$ for the case of $N = 1$.

Proof. Fix $t_1, t_2, t', t'' \in \mathscr{T}$ with $t_1 < t' < t'' < t_2$. Below we always assume that $k \geq k_\circ(t_1, t_2, t', t'')$ (see (4.55)–(4.56)), in particular, the set $S_k(t_1, t_2; t)$ is well defined for all $t \in [t', t'']$.

The main idea of the proof of Lemma 4.9 is quite simple: we will integrate the equation

$$\Delta\Phi_k = \omega_k^2 + \frac{1}{\nu_k}\operatorname{div}(\Phi_k \mathbf{u}_k) \tag{4.59}$$

over a suitable domain $\Omega_k(t)$ with $\partial\Omega_k(t) \supset S_k(t_1, t_2; t)$.

We split the construction of the domain $\Omega_k(t)$ into two steps. Namely, for $t \in \mathscr{T} \cap [t', t'']$ and sufficiently large k denote by $\Omega_{S_k(t_1,t_2;t)}$ the bounded open set in $\mathbb{R}^2$ such that

$$\partial\Omega_{S_k(t_1,t_2;t)} = S_k(t_1, t_2; t).$$

Then put by definition

$$\Omega_k(t) = B_k \setminus \Omega_{S_k(t_1,t_2;t)} \tag{4.60}$$

(see Fig. 2). Here $B_k = \{x \in \mathbb{R}^2 : |x| < R_k\}$ are the balls where the solutions $\mathbf{u}_k \in W^{1,2}(\Omega \cap B_k)$ from (E-NS)-assumptions are defined.

By construction (see Fig. 2), $\partial\Omega_k(t) = S_k(t_1, t_2; t) \cup C_{R_k}$. Integrating the equation (4.59) over the domain $\Omega_k(t)$, we obtain

$$\begin{aligned}\int\limits_{S_k(t_1,t_2;t)} \nabla\Phi_k \cdot \mathbf{n}\, ds + \int\limits_{C_{R_k}} \nabla\Phi_k \cdot \mathbf{n}\, ds &= \int\limits_{\Omega_k(t)} \omega_k^2\, dx \\ + \frac{1}{\nu_k} \int\limits_{S_k(t_1,t_2;t)} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, ds + \frac{1}{\nu_k} \int\limits_{C_{R_k}} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, ds&.\end{aligned} \tag{4.61}$$

By direct calculations, (4.11) implies

$$\nabla\Phi_k = -\nu_k \nabla^\perp \omega_k + \omega_k \mathbf{u}_k^\perp, \tag{4.62}$$

where, recall, for $\mathbf{u} = (u_1, u_2)$ we denote $\mathbf{u}^\perp = (u_2, -u_1)$ and $\nabla^\perp\omega = (\partial_2\omega, -\partial_1\omega)$.

By the Stokes theorem, for any C^1-smooth closed curve $S \subset \Omega$ and $g \in C^1(\Omega)$ we have

$$\int\limits_S \nabla^\perp g \cdot \mathbf{n}\, ds = 0.$$

So, in particular,

$$\int\limits_S \nabla \Phi_k \cdot \mathbf{n}\, ds = \int\limits_S \omega_k \mathbf{u}_k^\perp \cdot \mathbf{n}\, ds. \tag{4.63}$$

Since by construction for every $x \in C_{R_k} = \{y \in \mathbb{R}^2 : |y| = R_k\}$ there holds the equality

$$\mathbf{u}_k(x) = 0, \tag{4.64}$$

we see that

$$\int\limits_{C_{R_k}} \nabla \Phi_k \cdot \mathbf{n}\, ds = 0. \tag{4.65}$$

Furthermore, using (4.64) we get

$$\frac{1}{\nu_k} \int\limits_{C_{R_k}} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, ds = 0. \tag{4.66}$$

Finally, since $\Phi_k(x) \equiv -t$ for all $x \in S_k(t_1, t_2; t)$, we obtain

$$\int\limits_{S_k(t_1,t_2;t)} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, ds = -t \int\limits_{S_k(t_1,t_2;t)} \mathbf{u}_k \cdot \mathbf{n}\, ds = t \int\limits_{C_{R_k}} \mathbf{u}_k \cdot \mathbf{n}\, ds = 0. \tag{4.67}$$

In view of (4.58), (4.61) and (4.65)–(4.67) we get

$$\int\limits_{S_k(t_1,t_2;t)} |\nabla \Phi_k|\, ds = -\int\limits_{\Omega_k(t)} \omega_k^2\, dx, \tag{4.68}$$

a contradiction. The Lemma is proved. □

4.6. The case $\Phi_\infty < \widehat{p}_N = \operatorname*{ess\,sup}_{x \in \bar{\Omega}} \Phi(x)$

Suppose now that (4.42) holds, i.e., the maximum of Φ is attained on the boundary component Γ_N and not at infinity. Then the proof can be reduced to the case with a bounded domain, which was considered in [18]. Let us describe the essential details of this reduction.

Without loss of generality we can assume that $\Phi_\infty < 0$ and $\operatorname{ess\,sup} \Phi(x) = \widehat{p}_N = \Phi(\Gamma_N) = 0$. Repeating the arguments from the first part of Subsection 4.5, we construct a C^1-smooth cycle

$A_N \subset \Omega$ such that $\psi|_{A_N} = \text{const}$, $\Phi_\infty < \Phi(A_N) < 0$ and Γ_N lies in the bounded connected component of the set $\mathbb{R}^2 \setminus A_N$. Denote this component by Ω_b. The cycle A_N separates Γ_N from infinity. Thus, in order to obtain a contradiction, it is enough to consider the bounded domain $\Omega_b \cap \Omega$.

Namely, let

$$\Omega_b \cap \Gamma_j = \emptyset, \qquad j = 1, \dots M_1 - 1,$$

$$\Omega_b \supset \Gamma_j, \qquad j = M_1, \dots, N$$

(the case $M_1 = N$ is not excluded). Making a renumeration (if necessary), we may assume without loss of generality that

$$\Phi(\Gamma_j) < 0, \qquad j = M_1, \dots, M_2,$$

$$\Phi(\Gamma_j) = \widehat{p}_N = 0, \qquad j = M_2 + 1, \dots, N$$

(the case $M_2 = M_1 - 1$, i.e., when Φ attains maximum value at every boundary component inside the domain Ω_b, is not excluded). Now in order to receive the required contradiction, one needs to repeat almost word by word the corresponding arguments of Subsection 2.4.1 in [18]. The only modifications are as follows: now the sets A_N and $\Gamma_{M_1}, \dots, \Gamma_{M_2}$ play the role of the sets $\Gamma_0, \Gamma_1, \dots, \Gamma_M$ from [18, Subsection 2.4.1], and the domain $\Omega_b \cap \Omega$ in the present case plays the role of the domain Ω from [18, Subsection 2.4.1], etc.

4.7. The case $\Phi_\infty = \widehat{p}_N = \operatorname*{ess\,sup}_{x \in \bar{\Omega}} \Phi(x)$

Consider the last possible case, when the maximum of Φ is attained both at infinity and on some boundary component:

$$\Phi_\infty = \operatorname*{ess\,sup}_{x \in \Omega} \Phi(x) = \widehat{p}_N = \max\{\widehat{p}_1, \dots, \widehat{p}_N\} \tag{4.69}$$

(recall, that $\widehat{p}_j = \Phi(\Gamma_j)$).

This case is more delicate: we need to combine the arguments of the previous subsections.

Without loss of generality we may assume that

$$0 = \Phi_\infty = \operatorname*{ess\,sup}_{x \in \Omega} \Phi(x), \tag{4.70}$$

$$\widehat{p}_j < 0, \qquad j = 1, \dots, M, \tag{4.71}$$

$$\widehat{p}_j = 0, \qquad j = M + 1, \dots, N. \tag{4.72}$$

Note that $1 \le M < N$, i.e., the case $\widehat{p}_j \equiv 0$ for all $j = 1, \dots, N$ is impossible. Indeed, from (4.13) and $(4.10)_1$ we have

$$-\nu = \sum_{j=1}^{N} \widehat{p}_j \mathscr{F}_j, \tag{4.73}$$

where, recall,

$$\mathscr{F}_j = \int\limits_{\partial\Omega_j} \mathbf{a}\cdot\mathbf{n}\,ds. \tag{4.74}$$

Let

$$\delta > \max\{-\widehat{p}_j : j = 1,\dots,M\}.$$

Using precisely the same arguments as above in Subsection 4.5, we construct a measurable set $\mathscr{T} \subset [0,\delta]$ of full measure (i.e., meas$\big([0,\delta]\setminus\mathscr{T}\big) = 0$) and smooth cycles $A_j(t) \Subset \Omega$ for all $t \in \mathscr{T}$ and every $j = 1,\dots,M$ with the following properties:

(i) The set $\Gamma_j = \partial\Omega_j$ lies in a bounded connected component of the open set $\mathbb{R}^2 \setminus A_j(t)$;

(ii) $\psi|_{A_j(t)} \equiv \text{const}$, $\Phi(A_j(t)) = -t$;

(iii) (monotonicity) If $0 < t_1 < t_2 < \delta_p$, then $A_j(t_1)$ lies in the unbounded connected component of the set $\mathbb{R}^2 \setminus A_j(t_2)$ (i.e., the set $A_j(t_2)\cup\Gamma_j$ lies in the bounded connected component of the set $\mathbb{R}^2 \setminus A_j(t_1)$);

(iv) $A_j(t)$ is *a regular cycle*, i.e., it is a smooth curve homeomorphic to the unit circle and

$$\Phi_k|_{A_j(t)} \text{ converges to } \Phi|_{A_j(t)} \text{ uniformly for all } t \in \mathscr{T}. \tag{4.75}$$

Further, using also the methods of Subsection 4.5, for any numbers $t_1, t_2, t', t'' \in \mathscr{T}$ with $t_1 < t' < t'' < t_2$ and for all $t \in \mathscr{T} \cap (t', t'')$ and $k \ge k_\circ(t_1, t_2, t', t'')$ we construct[14] a domain $\Omega_k(t)$ with $\partial\Omega_k(t) = C_{R_k} \cup S_k(t_1, t_2; t)$, where $S_k(t_1, t_2; t)$ is a union of smooth cycles satisfying the following conditions:

$$S_k(t_1,t_2;t) \text{ separates } A_j(t_1) \text{ from } A_j(t_2) \text{ for all } j \in 1,\dots,M; \tag{4.76}$$

$$\Phi_k \equiv -t \text{ on } S_k(t_1,t_2;t); \tag{4.77}$$

$$\nabla\Phi \ne 0 \text{ on } S_k(t_1,t_2;t); \tag{4.78}$$

$$\int\limits_{S_k(t_1,t_2;t)} \nabla\Phi_k\cdot\mathbf{n}\,ds = -\int\limits_{S_k(t_1,t_2;t)} |\nabla\Phi_k|\,ds < 0, \tag{4.79}$$

where $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega_k(t)$.

Now we are ready to prove the key estimate.

Lemma 4.10. *For any $t_1, t_2, t', t'' \in \mathscr{T}$ with $t_1 < t' < t'' < t_2$ there exists $k_* = k_*(t_1, t_2, t', t'')$ such that for every $k \ge k_*$ and for almost all $t \in [t', t'']$ the inequality*

$$\int\limits_{S_k(t_1,t_2;t)} |\nabla\Phi_k|\,ds < \mathscr{F} t, \tag{4.80}$$

holds with the constant $\mathscr{F}$ independent of t, t_1, t_2, t', t'' and k.

[14] See, e.g., (4.55)–(4.56), where now the number M plays the role of N.

Proof. Fix $t_1, t_2, t', t'' \in \mathscr{T}$ with $t_1 < t' < t'' < t_2$. Below we always assume that $k \geq k_\circ = k_\circ(t_1, t_2, t', t'')$, in particular, the set $S_k(t_1, t_2; t)$ is well defined for all $t \in [t', t''] \cap \mathscr{T}$.

Put $\widetilde{\Omega}_k(t) = \Omega \cap \Omega_k(t)$. By construction,

$$\partial \widetilde{\Omega}_k(t) = C_{R_k} \cup S_k(t_1, t_2; t) \cup \Gamma_K \cup \cdots \cup \Gamma_N, \tag{4.81}$$

where $M < K$. This representation follows from the fact that the set $S_k(t_1, t_2; t)$ separates the circle C_{R_k} from the boundary components Γ_j with $j = 1, \dots, M$. However, a priori it does not separate C_{R_k} from other boundary components Γ_i with $i > M$. This is the main difference comparing to the situation of Subsection 4.5, where the boundary of the integration domain consists of only two parts: $C_{R_k} \cup S_k(t_1, t_2; t)$ (see the proof of Lemma 4.9).

It is easy to see that K in the representation (4.81) does not depend on k for sufficiently large k; see, e.g., [18, Subsection 2.4.1] for the detailed explanation of this fact.

Now we have to consider two possible cases:

Case I. $K = N + 1$. It means that no component Γ_j is contained in the domain $\widetilde{\Omega}_k(t)$, i.e.

$$\partial \widetilde{\Omega}_k(t) = C_{R_k} \cup S_k(t_1, t_2; t). \tag{4.82}$$

The contradiction for this case is derived exactly in the same way as in the proof of previous Lemma 4.9.

Case II. $K \leq N$. For $h > 0$ denote $\Gamma_0 = \Gamma_K \cup \cdots \cup \Gamma_N$, $\Gamma_h = \{x \in \Omega : \operatorname{dist}(x, \Gamma_0) = h\}$, $\Omega_k(t, h) = \{x \in \widetilde{\Omega}_k(t) : \operatorname{dist}(x, \Gamma_0) > h\}$. Then

$$\partial \Omega_k(t, h) = C_{R_k} \cup S_k(t_1, t_2; t) \cup \Gamma_h \tag{4.83}$$

for any fixed $t \in \mathscr{T} \cap [t', t'']$, for sufficiently small $h < \delta(t_1)$ and for sufficiently large $k \geq k_\circ$.

It was established in [18] (see the proof of formulas (3.40)–(3.42) on pages 786–788) that for any fixed $\varepsilon > 0$ and for sufficiently large $k \geq k_\varepsilon \geq k_\circ$ there exists a value $\bar{h}_k < \delta(t_1)$ such that

$$\Bigg| \int\limits_{\Gamma_{\bar{h}_k}} \nabla \Phi_k \cdot \mathbf{n}\, ds \Bigg| < \varepsilon, \tag{4.84}$$

$$\frac{1}{\nu_k} \Bigg| \int\limits_{\Gamma_{\bar{h}_k}} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, dS \Bigg| < \varepsilon. \tag{4.85}$$

It was shown before (see formulas (4.65)–(4.66)) that

$$\int\limits_{C_{R_k}} \nabla \Phi_k \cdot \mathbf{n}\, ds = 0, \tag{4.86}$$

$$\int\limits_{C_{R_k}} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, ds = 0. \tag{4.87}$$

Denote $\Omega_{0k}(t) := \Omega_k(t, \bar{h}_k)$. Then

$$\partial\Omega_{0k}(t) = C_{R_k} \cup S_k(t_1, t_2; t) \cup \Gamma_{\bar{h}_k}.$$

Integrating the equation (4.59) over the domain $\Omega_{0k}(t)$ and using (4.86)–(4.87), we get

$$\begin{aligned} &\int\limits_{S_k(t_1,t_2;t)} \nabla\Phi_k \cdot \mathbf{n}\, ds + \int\limits_{\Gamma_{\bar{h}_k}} \nabla\Phi_k \cdot \mathbf{n}\, ds = \int\limits_{\Omega_k(t)} \omega_k^2\, dx \\ &+\frac{1}{\nu_k} \int\limits_{S_k(t_1,t_2;t)} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, ds + \frac{1}{\nu_k} \int\limits_{\Gamma_{\bar{h}_k}} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, ds. \end{aligned} \tag{4.88}$$

Using (4.79), (4.84)–(4.85), we obtain the estimate

$$\int\limits_{S_k(t_1,t_2;t)} |\nabla\Phi_k|\, ds < 2\varepsilon - \frac{1}{\nu_k} \int\limits_{S_k(t_1,t_2;t)} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, ds. \tag{4.89}$$

Finally, since $\Phi_k(x) \equiv -t$ for all $x \in S_k(t_1, t_2; t)$, we derive

$$\begin{aligned} &\int\limits_{S_k(t_1,t_2;t)} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, ds = -t \int\limits_{S_k(t_1,t_2;t)} \mathbf{u}_k \cdot \mathbf{n}\, ds = t \int\limits_{\Gamma_0} \mathbf{u}_k \cdot \mathbf{n}\, ds \\ &= t\, \nu_k \int\limits_{\Gamma_0} \mathbf{a} \cdot \mathbf{n}\, ds = t \nu_k \mathscr{F}_\circ, \end{aligned} \tag{4.90}$$

here $\mathscr{F}_\circ = \frac{1}{\nu} \sum\limits_{j=K}^{N} \mathscr{F}_j$ and we have used the identities $(4.11)_3$, (4.81) and

$$0 = \int\limits_{\partial\widetilde{\Omega}_k(t)} \mathbf{u}_k \cdot \mathbf{n} ds = \int\limits_{C_{R_k} \cup S_k(t_1,t_2;t) \cup \Gamma_0} \mathbf{u}_k \cdot \mathbf{n} ds = \int\limits_{S_k(t_1,t_2;t)} \mathbf{u}_k \cdot \mathbf{n} ds + \int\limits_{\Gamma_0} \mathbf{u}_k \cdot \mathbf{n} ds.$$

Since the parameter $\varepsilon > 0$ could be chosen to be arbitrary small, from (4.89)–(4.90) the inequality

$$\int\limits_{S_k(t_1,t_2;t)} |\nabla\Phi_k|\, ds \le \big(|\mathscr{F}_\circ| + 1\big) t \tag{4.91}$$

follows for sufficiently large k. The Lemma is proved. □

Now we apply the argument from [18, proof of Lemma 3.9] and receive the required contradiction using the Coarea formula.

Lemma 4.11. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain of type* (1.1) *with C^2-smooth boundary $\partial\Omega$, and $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ satisfies zero total flux condition (1.22). Then assumptions (E-NS) and* (4.43) *lead to a contradiction.*

Proof. Take a number $t_0 \in \mathscr{T}$ such that $t_i := 2^{-i} t_0 \in \mathscr{T}$ for all $i \in \mathbb{N}$. Let R_0 be a sufficiently large radius such that $B_{\frac{1}{2}R_0} \supset \partial\Omega$. Denote $S_{ik}(t) := B_{R_0} \cap S_k(t_{i+1}, t_i, t)$ (it is well defined for almost all $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$ and for $k \geq k_* \geq k_\circ$, see paragraph before Lemma 4.10) and put

$$E_i = \bigcup_{t \in [\frac{5}{8}t_i, \frac{7}{8}t_i] \cap \mathscr{T}} S_{ik}(t).$$

By the Coarea formula (see, e.g., [24]), for any integrable function $g : E_i \to \mathbb{R}$ the equality

$$\int_{E_i} g|\nabla\Phi_k|\, dx = \int_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \int_{S_{ik}(t)} g(x)\, ds\, dt \tag{4.92}$$

holds. In particular, taking $g = |\nabla\Phi_k|$ and using (4.80), we obtain

$$\int_{E_i} |\nabla\Phi_k|^2\, dx = \int_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \int_{S_{ik}(t)} |\nabla\Phi_k|(x)\, ds\, dt \leq \int_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \mathscr{F} t\, dt = \mathscr{F}' t_i^2 \tag{4.93}$$

where $\mathscr{F}' = \frac{3}{16}\mathscr{F}$ is independent of i. Now, taking $g = 1$ in (4.92) and using the Hölder inequality we have

$$\begin{aligned} &\int_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \mathscr{H}^1\big(S_{ik}(t)\big)\, dt = \int_{E_i} |\nabla\Phi_k|\, dx \\ &\leq \Big(\int_{E_i} |\nabla\Phi_k|^2\, dx\Big)^{\frac{1}{2}} \big(\text{meas}(E_i)\big)^{\frac{1}{2}} \leq \sqrt{\mathscr{F}'}\, t_i \big(\text{meas}(E_i)\big)^{\frac{1}{2}}. \end{aligned} \tag{4.94}$$

By construction, for almost all $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$ the set $S_{ik}(t)$ is a finite union of smooth curves and $S_{ik}(t)$ separates $A_j(t_{i+1})$ from $A_j(t_i)$ in B_{R_0} for $j = 1, \dots, M$. Thus, each set $S_{ik}(t)$ separates Γ_j from Γ_N. In particular, $\mathscr{H}^1(S_{ik}(t)) \geq \min\big(\text{diam}(\Gamma_j), \text{diam}(\Gamma_N)\big)$. Hence, the left integral in (4.94) is greater than Ct_i, where $C > 0$ does not depend on i. On the other hand, the uniformly bounded sets E_i are pairwise disjoint and, therefore, $\text{meas}(E_i) \to 0$ as $i \to \infty$. The obtained contradiction finishes the proof of Lemma 4.11. □

We can summarize the results of Subsections 4.5–4.7 in the following statement.

Lemma 4.12. *Assume that $\Omega \subset \mathbb{R}^2$ is an exterior plane domain of type (1.1) with C^2-smooth boundary $\partial\Omega$ and $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ satisfies zero total flux condition (1.22). Let (E-NS) be fulfilled. Then every possible assumption (4.41), (4.42) and (4.43) lead to a contradiction.*

Proof of Theorem 1.2. Let the hypotheses of Theorem 1.2 be satisfied. Suppose that its assertion fails. Then, by Lemma 4.4, there exist $\mathbf{v}$, p and a sequence $(\mathbf{u}_k, p_k)$ satisfying (E-NS), and by Lemma 4.12 these assumptions lead to a contradiction. □

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