Arch. Rational Mech. Anal.
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-019-01359-8

On Convergence of Arbitrary D-Solution of Steady Navier–Stokes System in 2D Exterior Domains

Mikhail V. Korobkov, Konstantin Pileckas & Remigio Russo

Communicated by P. Constantin

Abstract

We study solutions to stationary Navier–Stokes system in a two dimensional exterior domain. We prove that any such solution with a finite Dirichlet integral converges to a constant vector at infinity uniformly. No additional conditions (on symmetry or smallness, etc.) are assumed. In the proofs we develop the ideas of the classical papers of Gilbarg and Weinberger (Ann Sc Norm Pisa (4) 5:381–404, 1978) and Amick (Acta Math 161:71–130, 1988).

1. Introduction

Let Ω be an exterior domain in $\mathbb{R}^2$ with compact boundary $\partial\Omega = \bigcup\limits_{i=1}^{N} \Gamma_i$, where Γ_i are disjoint curves, homeomorphic to the circle. In particular, $\Omega \supset \mathbb{R}^2 \setminus B$, where B is the disk of radius R_0 centered at the origin with $\partial\Omega \subset B$.

Let $\mathbf{a}$ and $\mathbf{u}_0$ be a given vector field on $\partial\Omega$ and a constant vector. The boundary value problem associated with the Navier–Stokes equations in an exterior $2D$ domain is given by the system

$$\begin{cases} \nu\Delta\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega, \\ \mathbf{u}(z) \to \mathbf{u}_0 & \text{as } |z| \to \infty, \end{cases} \tag{1.1}$$

where $\mathbf{u}$ and p are the unknown velocity field and the pressure, ν denotes the kinematical viscosity coefficient. For $\mathbf{a} = \mathbf{0}$, (1.1) expresses the important problem of a flow around an obstacle.

The first existence theorem of a solution to $(1.1)_{1,2,3}$ (i.e., without the condition at infinity $(1.1)_4$) for sufficiently regular domains and boundary data was proved

Published online: 01 February 2019

by J. Leray in his thesis [13] under the sole hypothesis that the fluxes of $\mathbf{a}$ over all connected components of the boundary are zero, i.e.,

$$\int_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}s = 0, \quad i = 1, 2, \ldots, N. \tag{1.2}$$

Denoting by $\mathbf{u}_k$ the solution to the problem

$$\begin{cases} \nu\Delta\mathbf{u}_k - (\mathbf{u}_k\cdot\nabla)\mathbf{u}_k - \nabla p_k = \mathbf{0} & \text{in } \Omega_k,\\ \operatorname{div}\mathbf{u}_k = 0 & \text{in } \Omega_k,\\ \mathbf{u}_k = \mathbf{a} & \text{on } \partial\Omega,\\ \mathbf{u}_k = \mathbf{u}_0 & \text{for } |z| = k \end{cases} \tag{1.3}$$

on the intersection Ω_k of Ω with the disk B_k of radius $k(\gg R_0)$, whose existence he proved before, Leray showed that the sequence $\mathbf{u}_k$ satisfies the estimate $\int_\Omega |\nabla\mathbf{u}_k|^2 \le c$ for some positive constant c independent of k. Hence, he observed that it is possible to extract a subsequence $\mathbf{u}_{k_n}$ which weakly converges to a solution $\mathbf{u}_L$ of problem $(1.1)_{1,2,3}$ with $\int_\Omega |\nabla\mathbf{u}_L|^2 < +\infty$. This approach of Leray was called an *invading domain method* and the solution obtained by Leray was later called by Amick [1], *Leray's solution*. The arbitrary solution $\mathbf{u}$ to the Navier–Stokes equations

$$\begin{cases} \nu\Delta\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p = \mathbf{0} & \text{in } \Omega,\\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

having the finite Dirichlet integral

$$\int_\Omega |\nabla\mathbf{u}|^2 < +\infty, \tag{1.5}$$

is today called the *D-solution* [6]. As is well known (e.g., [12]), such solutions are real–analytic in $\Omega\backslash B$. The existence of solutions to boundary value problems corresponding to $(1.1)_{1,2,3}$ was also studied in [10,15,16]. In particular, a similar result was proved in our recent paper, [11], under the less restrictive condition that the total flux is equal to zero:

$$\int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n} = 0.$$

As far as condition $(1.4)_4$ is concerned, Leray limited himself to observe that, while in three dimensions (1.5) it is sufficient to guarantee the attainability of the limit $\mathbf{u}_0$ at infinity (at least in a mean square sense) as a consequence of the inequality $\|r^{-1}(\mathbf{u}-\mathbf{u}_0)\|_{L^2(\Omega)} \le 4\|\nabla\mathbf{u}\|_{L^2(\Omega)}$, in the two dimensional case the corresponding inequality $\|(r\log r)^{-1}(\mathbf{u}-\mathbf{u}_0)\|_{L^2(\Omega)} \le c\|\nabla\mathbf{u}\|_{L^2(\Omega)}$ does not imply any type of convergence. Leray concluded that one should not be surprised by this phenomenon, in view of the Stokes paradox, *i.e*, a solution to the system obtained from (1.4) removing the nonlinear term with constant boundary datum (say) does not admit a solution unless $\mathbf{a} = \mathbf{u}_0$. Thirty years later, Fujita [5] and Finn and Smith [4], by means of different techniques, proved the existence of a D-solution to $(1.4)_{1.2.3}$ and (1.4), respectively, under the same hypotheses as in [13] and for

$(\mathbf{a} - \mathbf{u}_0)$ sufficiently small in a suitable norm. Due of the lack of a uniqueness theorem the above solutions are not comparable and so, for instance, while the Finn & Smith solution behaves at infinity as that of the Oseen equations, Leray's solution and Fujita's solution might exhibit another type of behavior. Hence, results holding for every D-solution are of some interest.

The problem of the asymptotic behaviour at infinity of Leray's solution $(\mathbf{u}_L, p_L)$ was tacked by Gilbarg and Weinberger [8]. They proved that $\mathbf{u}_L$ is bounded, and that there is a scalar p_0 and a constant vector $\mathbf{u}_\infty$ such that

$$\lim_{|z|\to+\infty} p_L(z) = p_0 \tag{1.6}$$

(one can choose, say, $p_0 = 0$),

$$\lim_{|z|\to+\infty} \int_0^{2\pi} |\mathbf{u}_L(r,\theta) - \mathbf{u}_\infty|^2 d\theta = 0, \tag{1.7}$$

and

$$\begin{aligned} &\omega_L(z) = o(r^{-3/4}),\\ &\nabla \mathbf{u}_L(z) = o(r^{-3/4}\log r),\\ &\textstyle\int_\Omega r|\nabla\omega_L(z)|^2 < \infty, \end{aligned} \tag{1.8}$$

where $r = |z|$ and

$$\omega = \partial_2 u_1 - \partial_1 u_2$$

is the vorticity. Several years later the same authors [9] showed that any D-solution $(\mathbf{u}, p)$ satisfies (1.6) and

$$\begin{aligned} &\mathbf{u}(z) = o(\log^{1/2} r),\\ &\omega(z) = o(r^{-3/4}\log^{1/8} r),\\ &\nabla\mathbf{u}(z) = o(r^{-3/4}\log^{9/8} r),\\ &\nabla\omega \in L^2(\Omega). \end{aligned}$$

If $\mathbf{u}$, in addition, is bounded, then it satisfies the same properties as the Leray solution, in particular, the relations (1.7)–(1.8) hold true, and if $\mathbf{u}_\infty = \mathbf{0}$, then

$$\mathbf{u}(z) \to 0 \qquad \text{uniformly as } |z| \to \infty. \tag{1.9}$$

Moreover, if $\mathbf{u}_\infty \neq 0$, then there exists a sequence of radii $R_n \in (2^n, 2^{n+1})$, $n \geq n_0$, such that

$$\sup_{\theta\in[0,2\pi]} |\mathbf{u}(R_n,\theta) - \mathbf{u}_\infty| \to 0 \qquad \text{as } n \to \infty. \tag{1.10}$$

In 1988 C.J. Amick [1] proved that a D-solution to the problem of a flow around an obstacle $(\mathbf{a} = \mathbf{0})$ has the following asymptotic properties:

(i) $\mathbf{u}$ is bounded and, as a consequence, it satisfies (1.7)–(1.8);

(ii) the total head pressure $\Phi = p + \frac{1}{2}|\mathbf{u}|^2$ and the absolute value of the velocity $|\mathbf{u}|$ have the uniform limit at infinity, i.e.,

$$|\mathbf{u}(r,\theta)| \to |\mathbf{u}_\infty| \qquad \text{as } r \to \infty, \tag{1.11}$$

where $\mathbf{u}_\infty$ is the constant vector from the condition (1.7);

(iii) if $\partial\Omega$ is symmetric with respect to the x_1–axis, and $\mathbf{u} = (u_1, u_2)$ is also symmetric, i.e., if u_1 is even and u_2 is odd with respect to x_1, then $\mathbf{u}$ converges uniformly at infinity to a constant vector $\mu\mathbf{e}_1$, for some scalar μ. Moreover, the Leray procedure yields a nontrivial solution under these symmetry assumptions.

In [2] the same author proved that if $\mu \neq 0$, then the solution in (iii) behaves at infinity as that of the linear Oseen equation. This result was extended by Sazonov [17] to an arbitrary D-solution converging uniformly at large distance to a nonzero constant vector (see also [7] and Ch. XII of [6]).

In [1] Amick has studied the problem of the flow around obstacles, i.e. when

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{1.12}$$

We note that the proofs of the above results (i)–(iii) from [1] pass with tiny (obvious) changes also for the case when the velocity is not equal to zero at the boundary $\partial\Omega$, but the total flux of boundary data is zero, i.e. when instead of (1.12) there holds only the condition

$$\int_{\partial\Omega} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s = 0. \tag{1.13}$$

Recently Korobkov et al. [11] simplified the issue and proved that the first claim (i) holds in the general case of D-solutions without (1.12) or (1.13) assumptions.

Theorem 1.1. ([11]) *Let $\mathbf{u}$ be a D-solution to the Navier–Stokes system (1.4) in the exterior domain $\Omega \subset \mathbb{R}^2$. Then $\mathbf{u}$ is uniformly bounded in $\Omega_0 = \mathbb{R}^2 \setminus B$, i.e.,*

$$\sup_{z\in\Omega_0} |\mathbf{u}(z)| < \infty, \tag{1.14}$$

where $B = B_{R_0}$ is an open disk with sufficiently large radius: $B \supset \partial\Omega$.

Using the above–mentioned results of D. Gilbarg and H. Weinberger, one obtains immediately

Corollary 1.1. *Let $\mathbf{u}$ be a D-solution to the Navier–Stokes system (1.4) in a neighbourhood of infinity. Then the asymptotic properties* (1.6)–(1.8) *hold.*

In the present paper we prove that any D-solution (without no extra assumptions) of the Navier–Stokes system (1.4) uniformly converges at infinity to a constant vector.

Theorem 1.2. *Let* $\mathbf{u}$ *be a D-solution to the Navier–Stokes system* (1.4) *in the exterior domain* $\Omega \subset \mathbb{R}^2$*. Then* $\mathbf{u}$ *converges uniformly at infinity, i.e.,*

$$\mathbf{u}(z) \to \mathbf{u}_\infty \qquad \textit{uniformly as } |z| \to \infty, \tag{1.15}$$

where $\mathbf{u}_\infty \in \mathbb{R}^2$ *is the constant vector from the equality* (1.7).

Although, from Theorem 1.2 we know that there exists a constant vector $\mathbf{u}_\infty \in \mathbb{R}^2$ such that (1.15) holds, the desired equality $\mathbf{u}_\infty = \mathbf{u}_0$ for the solution obtained by the Leray procedure is still an open question. We do not even know whether $\mathbf{u}_\infty$ is nonzero if $\mathbf{u}_0 \neq 0$. Thus, the problem of the flow around an obstacle also remains open; it can occur that the limit $\mathbf{u}$ of the solutions to problems (1.3) with $\mathbf{a} = 0$ and $\mathbf{u}_0 \neq 0$ is identically zero, and it is not known whether the problem (1.1) with $\mathbf{a} = 0$ and $\mathbf{u}_0 \neq 0$ has a solution (in general).

Finally, let us describe briefly the main steps of the proof of Theorem 1.2. We will use two convergence criteria (see Lemma 3.3); namely, the required property (1.15) holds if at least one of the two conditions is fulfilled:

(i) $\omega(z) = o(|z|^{-1})$, where ω is the vorticity;

or

(ii) the absolute value of the velocity has a uniform limit at infinity, i.e., (1.11) is fulfilled.

The first criterion was used by Amick [1] to prove the convergence (1.15) in the symmetric case, but he did not take into account the second one, although it is a simple corollary of some standard lemmas from the Gilbarg and Weinberger paper [9], and Amick has proved the property (1.11) under the condition (1.12).

Accordingly, our proof consists of treating the two cases: case I, when level sets of the vorticity ω separate the origin from infinity (in this case the regular level lines of ω are homeomorphic to the circle surrounding the origin), and case II, when all level sets of ω intersect $\partial\Omega$. In case I we prove (using the coarea formula) that the asymptotic $\omega(z) = o(|z|^{-1})$ is true (see (3.51) and below) and this seems to be the most original part in our arguments. Thus, in case I, the convergence (1.15) follows from the criterion (i). In case II we use the arguments of Amick's paper [1] to prove the uniform convergence at infinity (1.11) of the absolute value of velocity. Thus, to prove (1.15), one can use the criterion (ii). We check the applicability of Amick's arguments in Lemma 3.4. For the reader convenience, we recall briefly the main ideas and steps of the Amick proof of (1.11) in the Appendix.

2. Notations and Preliminaries

By *a domain* we mean an open connected set. We use standard notations for Sobolev spaces $W^{k,q}(\Omega)$, where $k \in \mathbb{N}$, $q \in [1, +\infty]$. In our notation we do not distinguish function spaces for scalar and vector valued functions; it is clear from the context whether we use scalar or vector (or tensor) valued function spaces.

For $q \geq 1$ denote by $D^{k,q}(\Omega)$ the set of functions $f \in W^{k,q}_{\mathrm{loc}}(\Omega)$ such that $\|f\|_{D^{k,q}(\Omega)} = \|\nabla^k f\|_{L^q(\Omega)} < \infty$.

We denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure, i.e., $\mathcal{H}^k(F) = \lim\limits_{t\to 0+} \mathcal{H}^k_t(F)$, where

$$\mathcal{H}^k_t(F) = \big(\frac{\alpha_k}{2}\big)^k \inf\left\{\sum_{i=1}^{\infty}\big(\operatorname{diam} F_i\big)^k : \operatorname{diam} F_i \leqq t,\, F \subset \bigcup_{i=1}^{\infty} F_i\right\},$$

and α_k is a Lebesgue volume of the unit ball in $\mathbb{R}^k$.

In particular, for a curve S the value $\mathcal{H}^1$ coincides with its length, and for sets $E \subset \mathbb{R}^2$ the $\mathcal{H}^2(E)$ coincides with the usual Lebesgue measure in $\mathbb{R}^2$.

Also, for a curve S by $\int_S f\, \mathrm{d}s$ we denote the usual integral with respect to 1-dimensional Hausdorff measure (=length). Further, for a set $E \subset \mathbb{R}^2$ by $\int_E f(x)\, \mathrm{d}\mathcal{H}^2$ or simply $\int_E f(x)$ we denote we integral with respect to the two-dimensional Lebesgue measure.

Below we present some results concerning the behaviour of D-functions.

Lemma 2.1. *Let $f \in D^{1,2}(\Omega)$ and assume that*

$$\int_D |\nabla f|^2\, d\mathcal{H}^2 < \varepsilon^2$$

for some $\varepsilon > 0$ and for some ring $D = \{z \in \mathbb{R}^2 : r_1 < |z - z_0| < r_2\} \subset \Omega$. Then the estimate

$$|\bar{f}(r_2) - \bar{f}(r_1)| \leq \varepsilon\sqrt{\ln\frac{r_2}{r_1}} \tag{2.1}$$

holds, where $\bar{f}$ means the mean value of f over the circle $S(z_0, r)$:

$$\bar{f}(r) := \frac{1}{2\pi r}\int_{|z-z_0|=r} f(z)\, ds.$$

Lemma 2.2. *Fix a number $\beta \in (0, 1)$. Let $f \in D^{1,2}(\Omega)$ and assume that*

$$\int_D |\nabla f|^2\, d\mathcal{H}^2 < \varepsilon^2$$

for some $\varepsilon > 0$ and for some ring $D = \{z \in \mathbb{R}^2 : \beta R < |z - z_0| < R\} \subset \Omega$. Then there exists a number $r \in [\beta R, R]$ such that the estimate

$$\sup_{|z-z_0|=r} |f(z) - \bar{f}(r)| \leq c_\beta \varepsilon \tag{2.2}$$

holds, where the constant c_β depends on β only.

The proofs of above lemmas are standard, see [9], e.g., for the proofs of similar results. Summarising the results of these lemmas, we receive

Lemma 2.3. *Under conditions of Lemma 2.2, there exists $r \in [\beta R, R]$ such that*

$$\sup_{|z-z_0|=r} |f(z) - \bar{f}(R)| \leq \tilde{c}_\beta \varepsilon. \tag{2.3}$$

3. Proof of the Main Theorem 1.2.

Suppose the assumptions of Theorem 1.2 are fulfilled. By classical regularity results for D-solutions to the Navier–Stokes system (e.g., [6]), the functions $\mathbf{u}$ and p are real–analytical on the set $\Omega_0 = \mathbb{R}^2 \setminus B_{R_0}$. Moreover, it follows from results in [9] and Theorem 1.1 that $\mathbf{u}$ and p are uniformly bounded in Ω_0, such that

$$\sup_{z\in\Omega_0} \big(|p(z)| + |\mathbf{u}(z)|\big) \le C < +\infty, \tag{3.1}$$

and the pressure p has a limit at infinity; we could assume without loss of generality that

$$p(z) \to 0 \qquad \text{uniformly as } |z| \to \infty. \tag{3.2}$$

It is also well known (see [6]) that all derivatives of $\mathbf{u}$ uniformly converge to zero:

$$\forall k = 1, 2, \dots \qquad \nabla^k \mathbf{u}(z) \to \mathbf{0} \qquad \text{uniformly as } |z| \to \infty. \tag{3.3}$$

Further, it is proved in [9] that there exists a vector $\mathbf{u}_\infty \in \mathbb{R}^2$ such that

$$\lim_{r\to+\infty} \int_0^{2\pi} |\mathbf{u}(r,\theta) - \mathbf{u}_\infty|^2 \mathrm{d}\theta = 0, \tag{3.4}$$

and moreover, if $\mathbf{u}_\infty = \mathbf{0}$, then

$$\mathbf{u}(z) \to \mathbf{0} \qquad \text{uniformly as } |z| \to \infty. \tag{3.5}$$

Thus if $\mathbf{u}_\infty = \mathbf{0}$, the statement of Theorem 1.2 is known and we need to consider only the case

$$\mathbf{u}_\infty \ne \mathbf{0}. \tag{3.6}$$

Consider the vorticity $\omega = \partial_2 u_1 - \partial_1 u_2$ which will play the key role in our proof. Recall that ω satisfies the elliptic equation

$$\nu\Delta\omega = (\mathbf{u}\cdot\nabla)\omega. \tag{3.7}$$

In particular, ω satisfies two-sided maximum principle in $\mathbb{R}^2$; moreover,

$$\int_{\Omega_0} r|\nabla\omega|^2 < \infty \tag{3.8}$$

(see [9]). Further, since by the definition of a D-solution $\int_\Omega |\nabla \mathbf{u}|^2 < \infty$, and $\omega = \partial_2 u_1 - \partial_1 u_2$, we have, in particular, that

$$\int_\Omega \omega^2 < \infty. \tag{3.9}$$

We will need also the following statement:

Lemma 3.1. *Let* $\mathbf{u}$ *be a D-solution to the Navier–Stokes system (1.4) in the exterior domain* $\Omega \subset \mathbb{R}^2$. *Denote by* $\bar{\mathbf{u}}(z,r)$ *the mean value of* $\mathbf{u}$ *over the circle* $S(z,r)$, *i.e.,*

$$\bar{\mathbf{u}}(z,r) = \frac{1}{2\pi r}\int_{|\xi - z| = r} \mathbf{u}(\xi)\,ds, \tag{3.10}$$

and let $\varphi(z,r)$ *be the argument of the complex number associated to the vector* $\bar{\mathbf{u}}(z,r) = (\bar{u}_1(r), \bar{u}_2(r))$, *i.e.,* $\varphi(z,r) = \arg(\bar{u}_1(r) + i\bar{u}_2(r))$. *Suppose* $|z|$ *is large enough so that the disk* $D_z = \big\{\xi \in \mathbb{R}^2 : |\xi - z| \le \frac{4}{5}|z|\big\}$ *is contained in* Ω. *Assume also that*

$$|\bar{\mathbf{u}}(z,r)| \ge \sigma$$

for some positive constant $\sigma > 0$ *and for all* $r \in \big(0, \frac{4}{5}|z|\big]$. *Then the estimate*

$$\sup_{0<\rho_1\le\rho_2\le\frac{4}{5}|z|} |\varphi(z,\rho_2) - \varphi(z,\rho_1)| \le \frac{1}{4\pi\sigma^2}\int_{D_z}\left(\frac{1}{r}|\nabla\omega| + |\nabla\mathbf{u}|^2\right) d\mathcal{H}^2_\xi \tag{3.11}$$

holds, where $r = |\xi - z|$.

For the proof of the estimate (3.11) see [9, Proof of Theorem 4, page 399].

To apply the last Lemma 3.1, we also need the following simple technical assertion:

Lemma 3.2. *Let* $\mathbf{u}$ *be a D-solution to the Navier–Stokes system (1.4) in the exterior domain* $\Omega \subset \mathbb{R}^2$. *For* $z \in \Omega$ *denote, as above,*

$$D_z = \left\{\xi \in \mathbb{R}^2 : |\xi - z| \le \frac{4}{5}|z|\right\}.$$

Then the uniform convergence

$$\int_{D_z} \frac{1}{r}|\nabla\omega|\,d\mathcal{H}^2_\xi \to 0 \qquad \text{as } |z| \to \infty \tag{3.12}$$

holds, where again $r = |\xi - z|$.

Proof. Take and fix arbitrary $\varepsilon > 0$. Take also numbers $r_2 > r_1 > 0$ large enough so that

$$2\pi < \varepsilon r_1; \tag{3.13}$$

$$\int_{D_z} r|\nabla\omega|^2\,\mathrm{d}\mathcal{H}^2_\xi < \varepsilon \qquad \text{if } |z| > r_2; \tag{3.14}$$

$$2\pi r_1 \max_{|\xi - z| < r_1} |\nabla\omega(\xi)| < \varepsilon \qquad \text{if } |z| > r_2 \tag{3.15}$$

(the existence of such numbers follows from the estimate (3.8) and from the uniform convergence (3.3)).

Now take arbitrary $z \in \mathbb{R}^2$ with $|z| > r_2$. Then the disk D_z is represented as the union of two sets $D_z = D_1 \cup D_2$, where

$$D_1 = \left\{ \xi \in \mathbb{R}^2 : |\xi - z| < r_1 \right\}, \qquad D_2 = \left\{ \xi \in \mathbb{R}^2 : r_1 \le |\xi - z| < \frac{4}{5}|z| \right\}.$$

We have

$$\begin{aligned}\int_{D_1} \frac{1}{r}|\nabla\omega|\, \mathrm{d}\mathcal{H}^2_\xi &< \max_{|\xi - z|<r_1} |\nabla\omega(\xi)| \int_{D_1} \frac{1}{r}\, \mathrm{d}\mathcal{H}^2_\xi \\ &= 2\pi r_1 \max_{|\xi - z|<r_1} |\nabla\omega(\xi)| \overset{(3.15)}{<} \varepsilon.\end{aligned} \tag{3.16}$$

Further, applying the elementary inequality $\frac{1}{r}|\nabla\omega| < \frac{1}{r^3} + r|\nabla\omega|^2$, for the domain D_2 we have

$$\begin{aligned}\int_{D_2} \frac{1}{r}|\nabla\omega|\, \mathrm{d}\mathcal{H}^2_\xi &< \int_{D_2} \frac{1}{r^3}\, \mathrm{d}\mathcal{H}^2_\xi + \int_{D_2} r|\nabla\omega|^2\, \mathrm{d}\mathcal{H}^2_\xi \\ &= 2\pi \int_{r=r_1}^{\frac{4}{5}|z|} \frac{1}{r^2}\, \mathrm{d}r + \int_{D_2} r|\nabla\omega|^2\, \mathrm{d}\mathcal{H}^2_\xi \overset{(3.13)\text{-}(3.14)}{<} 2\varepsilon.\end{aligned} \tag{3.17}$$

From the inequalities (3.16)–(3.17) it follows that

$$\int_{D_z} \frac{1}{r}|\nabla\omega|\, \mathrm{d}\mathcal{H}^2_\xi < 3\varepsilon. \tag{3.18}$$

We proved the last inequality for any $z \in \mathbb{R}^2$ with $|z| > r_2$. Since the number $\varepsilon > 0$ is arbitrary, the required convergence (3.12) is established. □

Further we will use the following two criteria for the uniform convergence of the velocity:

Lemma 3.3. *Let* $\mathbf{u}$ *be a D-solution to the Navier–Stokes system* (1.4) *in the exterior domain* $\Omega \subset \mathbb{R}^2$. *Suppose that at least one of the following two conditions is fulfilled:*

(i) $\omega(z) = o(|z|^{-1})$ *as* $|z| \to \infty$*;*
(ii) *the absolute value of the velocity has a uniform limit at infinity:*

$$|\mathbf{u}(z)| \to |\mathbf{u}_\infty| \qquad \textit{uniformly as } |z| \to \infty, \tag{3.19}$$

where the vector $\mathbf{u}_\infty$ *was specified above.*

Then $\mathbf{u}$ *converges uniformly at infinity as well, i.e., the formula* (1.15) *holds.*

Proof. Part (i) was established by Amick (see [1], Remark 3(i) on p. 103 and the proof of Theorem 19). Recall that his argument is based on the classical Cauchy-type representation formula of complex analysis:

$$w(z) = \frac{1}{2\pi i} \oint\limits_{|\xi - z_0|=r} \frac{w(\xi)\, d\xi}{\xi - z} + \frac{1}{2\pi i} \iint\limits_{|\xi - z_0|<r} \frac{\omega(\xi)}{\xi - z_0}\, \mathrm{d}x\, \mathrm{d}y, \tag{3.20}$$

where $w(\xi) = u_1(\xi) - iu_2(\xi)$ and $\xi = x + iy$.

Let us prove the second part of Lemma 3.3. Suppose that assumption (ii) is fulfilled. If $\mathbf{u}_\infty = \mathbf{0}$, then there is nothing to prove (see the above discussion concerning the results of Gilbarg and Weinberger [8]–[9]), so we assume without loss of generality that

$$|\mathbf{u}_\infty| > 0. \tag{3.21}$$

From assumption (3.19) and Lemmas 2.1–2.3 it follows that

$$\sup_{0<\rho\le\frac{4}{5}|z|} \Big| |\mathbf{u}_\infty| - |\bar{\mathbf{u}}(z,\rho)| \Big| \to 0 \qquad \text{uniformly as } |z| \to \infty, \tag{3.22}$$

where $\bar{\mathbf{u}}(z,r)$ is the mean value of $\mathbf{u}$ over the circle $S(z,r)$. In particular, because of inequality (3.21), there exist numbers $\sigma > 0$ and $R_* > 0$ such that

$$|\bar{\mathbf{u}}(z,r)| \ge \sigma \qquad \text{if } |z| \ge R_* \text{ and } 0 < r \le \frac{4}{5}|z|. \tag{3.23}$$

Then, by Lemma 3.1, the argument $\varphi(z,r)$ of the complex number associated to $\bar{\mathbf{u}}(z,r)$ satisfies the estimate (3.11). From (3.11)–(3.12) it follows immediately that

$$\sup_{0<\rho_1\le\rho_2\le\frac{4}{5}|z|} |\varphi(z,\rho_2) - \varphi(z,\rho_1)| \to 0 \tag{3.24}$$

uniformly as $|z| \to \infty$. In particular,

$$\sup_{0<\rho\le\frac{4}{5}|z|} |\arg \mathbf{u}(z) - \arg \bar{\mathbf{u}}(z,\rho)| \to 0 \tag{3.25}$$

uniformly as $|z| \to \infty$. From the assumptions (3.19) and (3.22) we have

$$\sup_{0<\rho\le\frac{4}{5}|z|} \Big| |\mathbf{u}(z)| - |\bar{\mathbf{u}}(z,\rho)| \Big| \to 0 \qquad \text{uniformly as } |z| \to \infty. \tag{3.26}$$

Summarizing the information from formulas (3.25)–(3.26), we obtain

$$\sup_{0<\rho\le\frac{4}{5}|z|} |\mathbf{u}(z) - \bar{\mathbf{u}}(z,\rho)| \to 0 \qquad \text{uniformly as } |z| \to \infty. \tag{3.27}$$

Consider the sequence of circles $S_{R_n} = \{\xi \in \mathbb{R}^2 : |\xi| = R_n\}$ such that $2^n < R_n < 2^{n+1}$ and

$$\sup_{|\xi|=R_n} |\mathbf{u}(\xi) - \mathbf{u}_\infty| = \varepsilon_n \to 0 \qquad \text{as } n \to \infty \tag{3.28}$$

(the existence of such sequence is guaranteed by above mentioned results of D. Gilbarg and H. Weinberger, see (1.10)).

Now take a point $z \in \mathbb{R}^2$ with sufficiently large $|z|$ and take also the natural number $n = n_z$ such that

$$2^{n+1} \le |z| < 2^{n+2}.$$

Then, by construction and by the triangle inequality, we have

$$S_{R_n} \cap S_{z,\rho} \neq \emptyset \qquad \text{if} \quad \frac{3}{4}|z| < \rho < \frac{4}{5}|z|, \tag{3.29}$$

where $S_{z,\rho} = \{\xi \in \mathbb{R}^2 : |\xi - z| = \rho\}$. From Lemma 2.2 it follows that there exists $\rho_* \in \left(\frac{3}{4}|z|, \frac{4}{5}|z|\right)$ such that

$$\sup_{|\xi - z| = \rho_*} |\mathbf{u}(\xi) - \bar{\mathbf{u}}(z, \rho_*)| = \varepsilon_z, \tag{3.30}$$

where $\varepsilon_z \to 0$ uniformly as $|z| \to \infty$. Summarizing the information from formulas (3.28)–(3.30), we obtain that

$$|\mathbf{u}_\infty - \bar{\mathbf{u}}(z, \rho_*)| = \varepsilon_z' \to 0 \qquad \text{uniformly as } |z| \to \infty. \tag{3.31}$$

Finally, from the last formula and from (3.27), we conclude that

$$|\mathbf{u}_\infty - \mathbf{u}(z)| \to 0 \qquad \text{uniformly as } |z| \to \infty, \tag{3.32}$$

as required. The Lemma 3.3 is proved completely. □

Proof of Theorem 1.2. For a point $z \in \Omega_0$ denote by $K(z)$ the connected component of the level set of the vorticity ω containing z, i.e., $K(z) \ni z$ is a component of the set $X = \{x \in \Omega_0 : \omega(x) = \omega(z)\}$. Here we understand the notion of connectedness in the sense of general topology.[1]

We consider two possible cases:

Case I. Level sets of ω separate infinity from the origin:

$$\exists z_* \in \Omega_0 : \quad \omega(z_*) \neq 0 \quad \text{and} \quad K(z_*) \cap \partial\Omega_0 = \emptyset. \tag{3.33}$$

Case II. Level sets of ω do not separate infinity from the origin:

$$K(z) \cap \partial\Omega_0 \neq \emptyset \qquad \forall z \in \Omega_0. \tag{3.34}$$

In Case I, we shall show that

$$|z|\omega(z) \to 0 \qquad \text{uniformly as } |z| \to \infty, \tag{3.35}$$

and we obtain the statement of Theorem applying Lemma 3.3(i).

In Case II, we prove that

$$|\mathbf{u}(z)| \to |\mathbf{u}_\infty| \qquad \text{uniformly as } |z| \to \infty, \tag{3.36}$$

where $\mathbf{u}_\infty$ is the vector defined in (3.4). In this case the statement of Theorem will follow from Lemma 3.3(ii).

---

[1] Recall, that *a connected component of a point z in a set $X \subset \mathbb{R}^n$* is the union of all connected sets $E \subset X$ containing z. By well known results of general topology (see, e.g., [3, page 356]), the connected component is a connected set as well, i.e., it is the maximal connected subset (ordered by inclusion) of X containing z.

Consider the case (3.33). Note that then the set $K(z_*)$ is compact. Indeed, the set $K(z_*)$ is closed and connected, and if it is not compact, it should "reach" infinity. Since the vorticity tends to zero at infinity, $\omega(z)$ has to be zero on $K(z_*)$, but this contradicts the assumption (3.33).

Next, by elementary compactness and continuity arguments we have that there exists $\delta_0 > 0$ such that

$$K(z) \text{ is a compact set satisfying } K(z) \cap \partial\Omega_0 = \emptyset \text{ whenever } |z - z_*| < \delta_0. \tag{3.37}$$

Note that since ω is an analytical nonconstant function, we have that $\omega(z) \neq$ const in any open neighborhood of z_*.

Recall that a real number t is called *a regular value of* ω if the set $\{z \in \Omega_0 : \omega(z) = t\}$ is nonempty and $\nabla\omega(z) \neq 0$ whenever $\omega(z) = t$. By the classical Morse–Sard theorem, almost all values of ω are regular. Now take a point z_1 satisfying $|z_1 - z_*| < \delta_0$ with regular value $t_1 = \omega(z_1)$. Then, by definition and regularity assumptions, the set $K(z_1)$ is a smooth compact curve (="*compact one dimensional manifold without boundary*") which does not intersect the boundary $\partial\Omega_0$. For obvious topological reasons, $K(z_1)$ is a smooth curve homeomorphic to the circle. Since ω satisfies maximum principle, this circle surrounds the origin. Therefore, the curve $K(z_1)$ separates the boundary $\partial\Omega_0$ from infinity.[2]

Denote $R_* = \max\{|z| : z \in K(z_1)\}$ and $\Omega_* = \{z \in \mathbb{R}^2 : |z| > R_*\}$. Then by construction we have

$$K(z) \cap \partial\Omega_0 = \emptyset \qquad \forall z \in \Omega_*. \tag{3.38}$$

Applying again the same Morse–Sard theorem, we obtain that for almost all $t \in \mathbb{R}\backslash\{0\}$, if $z \in \Omega_*$ and $\omega(z) = t$, then $K(z)$ is a smooth curve homeomorphic to the circle. Since ω satisfies maximum principle, we conclude that this circle surrounds the origin, moreover,

$$K(z_1) = K(z_2) \qquad \text{if } z_1, z_2 \in \Omega_* \quad \text{and } \omega(z_1) = \omega(z_2) \neq 0. \tag{3.39}$$

This implies that

$$\omega(z) \text{ does not change sign in } \Omega_*. \tag{3.40}$$

Indeed, let there are points $z_1, z_2 \in \Omega_*$ with regular values $\omega(z_1) < 0$ and $\omega(z_2) > 0$. Taking into account that $\omega(z)$ is vanishing at infinity, by maximum principle, $\omega(z)$ is negative in the exterior of $K(z_1)$ and $\omega(z)$ is positive in the exterior of $K(z_2)$. Since this is impossible, $\omega(z)$ cannot change the sign.

Thus we may suppose without loss of generality that

$$\omega(z) \geq 0 \qquad \text{in } \Omega_*. \tag{3.41}$$

---

[2] It means that infinity and the set $\partial\Omega_0$ lie in the different connected components of the set $\mathbb{R}^2\backslash K(z_1)$.

Then by the maximum principle we have the strict inequality

$$\omega(z) > 0 \qquad \text{in } \Omega_*. \tag{3.42}$$

Moreover, from (3.39) and from the uniform convergence (see 3.3)

$$\omega(z) \to 0 \qquad \text{as } |z| \to \infty, \tag{3.43}$$

and from the Mors–Sard theorem, we conclude that there exists a number $\delta > 0$ such that

$$\begin{aligned}&\text{for almost all } t \in (0, \delta) \text{ the set } K_t := \{z \in \Omega_* : \omega(z) = t\}\\ &\text{coincides with the smooth curve homeomorphic to the circle}\\ &\text{such that } K_t \cap \partial\Omega_* = \emptyset \text{ and } \nabla\omega \neq 0 \text{ on } K_t.\end{aligned} \tag{3.44}$$

Denote by $\mathscr{T}$ the set of full measure in the interval $(0, \delta)$ consisting of values t satisfying (3.44). Denote also by Ω_t the unbounded connected component of the set $\mathbb{R}^2 \backslash K_t$. Since ω satisfies the maximum principle, the sets K_t have the following monotonicity property:

$$\Omega_{t_1} \subset \Omega_{t_2} \qquad \text{if} \quad 0 < t_1 < t_2. \tag{3.45}$$

Moreover, from the uniform convergence (3.43), it follows that

$$\inf\{|z| : z \in \Omega_t\} \to \infty \qquad \text{as} \quad t \to 0+. \tag{3.46}$$

Our task is to show the property (i) of Lemma 3.3, i.e., to show that

$$|z|\omega(z) \to 0 \qquad \text{uniformly as} \quad |z| \to \infty. \tag{3.47}$$

The last condition is equivalent to

$$tg(t) \to 0 \qquad \text{as} \quad t \to 0+, \tag{3.48}$$

where the function $g(t)$ is defined by

$$g(t) := \sup\{|z| : z \in K_t\}. \tag{3.49}$$

Obviously, $g(t) \leq \mathcal{H}^1(K_t)$, where, recall, $\mathcal{H}^1$ is the one-dimensional Hausdorff measure (=length).

For $t \in \mathscr{T}$ and $R > R_*$ denote $\Omega_{t,R} = \Omega_t \cap B_R = \{z \in \Omega_t : |z| < R\}$. Then for sufficiently large R

$$\partial\Omega_{t,R} = K_t \cup S_R,$$

where $S_R = \{z \in \mathbb{R}^2 : |z| = R\}$ is the corresponding circle. Assuming for simplicity that $\nu = 1$ and integrating the equation (3.7) over the domain $\Omega_{t,R}$ and taking into account that $(\mathbf{u} \cdot \nabla)\omega = \operatorname{div}(\mathbf{u}\omega)$, we obtain

$$\int_{K_t} |\nabla\omega| \, \mathrm{d}s + \int_{S_R} \nabla\omega \cdot \mathbf{n} \, \mathrm{d}s = t \int_{K_t} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}s + \int_{S_R} \omega \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}s. \tag{3.50}$$

Here $\mathbf{n}$ is a unit vector of the outward with respect to $\Omega_{t,R}$ normal to $\partial\Omega_{t,R}$. Note also that the unit normal to the level set $K_t = \{z \in \Omega_* : \omega(z) = t\}$ is given by the formula $\mathbf{n} = \dfrac{\nabla\omega}{|\nabla\omega|}$.

Since $\operatorname{div}\mathbf{u} = 0$, we have $\int_{K_t} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s = \int_{\partial\Omega_*} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s = C_*$, i.e., this value does not depend on t. On the other hand, the estimate $\int_{\Omega_0} \big(|\omega|^2 + |\nabla\omega|^2\big)\,\mathrm{d}\mathcal{H}^2 < \infty$ (see 3.8–3.9) implies that there is a sequence $R_k \to +\infty$ such that

$$\int_{S_{R_k}} \big(|\omega| + |\nabla\omega|\big)\,\mathrm{d}s \to 0.$$

Taking $R = R_k$ in the equality (3.50) and having in mind the uniform boundedness of the velocity (see (1.14)), we deduce, passing $R_k \to +\infty$, that

$$\int_{K_t} |\nabla\omega|\,\mathrm{d}s = C_* t. \tag{3.51}$$

Further, for $t \in (0, \frac{1}{2}\delta)$ denote $E_t = \{z \in \Omega_* : \omega(z) \in (t, 2t)\}$. By construction,

$$\partial E_t = K_t \cup K_{2t}.$$

Applying the classical Coarea formula (see, e.g., [14])

$$\int_{E_t} f\,|\nabla\omega|\,\mathrm{d}\mathcal{H}^2 = \int_t^{2t} \left(\int_{K_\tau} f\,\mathrm{d}s\right)\mathrm{d}\tau$$

for $f = |\nabla\omega|$, we obtain

$$\int_{E_t} |\nabla\omega|^2\,\mathrm{d}\mathcal{H}^2 = \int_t^{2t} \left(\int_{K_\tau} |\nabla\omega|\,\mathrm{d}s\right)\mathrm{d}\tau \overset{(3.51)}{=} \int_t^{2t} C_*\tau\,\mathrm{d}\tau = 3C_* t^2. \tag{3.52}$$

Applying now the same Coarea formula for $f = 1$ and using the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\int_t^{2t} \mathcal{H}^1(K_\tau)\,\mathrm{d}\tau &= \int_{E_t} |\nabla\omega|\,\mathrm{d}\mathcal{H}^2 \le \left(\int_{E_t} |\nabla\omega|^2\,\mathrm{d}\mathcal{H}^2\right)^{\frac{1}{2}} \Big(\operatorname{meas} E_t\Big)^{\frac{1}{2}} \\ &\overset{(3.52)}{=} \sqrt{3C_*}\Big(t^2 \operatorname{meas}(E_t)\Big)^{\frac{1}{2}} \le \sqrt{\frac{3}{4}C_*}\left(\int_{E_t} \omega^2\,\mathrm{d}\mathcal{H}^2\right)^{\frac{1}{2}} \\ &\le \varepsilon_t \to 0 \quad \text{as } t \to 0.\end{aligned} \tag{3.53}$$

Here we have used also the fact that $t \le |\omega(z)| \le 2t$ in E_t. By virtue of the mean-value theorem, this implies that for any sufficiently small $t \in \mathscr{T}$ there exists a number $\tau \in [t, 2t]$ such that

$$t\mathcal{H}^1(K_\tau) \le \varepsilon_t.$$

By construction, the closed curve K_τ surrounds K_{2t}. Therefore,

$$\sup\{|z| : z \in K_{2t}\} \leq \mathcal{H}^1(K_\tau) \leq \frac{\varepsilon_t}{t},$$

with $\varepsilon_t \to 0$ as $t \to 0$. From the last inequality we receive the relation (3.48) which is equivalent to (3.47). According to Lemma 3.3(i), this finishes the proof of Theorem 1.2 in the considered Case I.

Consider Case II, i.e., when

$$K(z) \cap \partial\Omega_0 \neq \emptyset \qquad \forall z \in \Omega_0. \tag{3.54}$$

Now we shall prove that the assertion (3.36) is valid.

Let us recall that Ch. Amick [1] has proved the convergence (3.36) under the assumption that

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}. \tag{3.55}$$

The condition (3.36) was used in [1] in order to define the stream function ψ in the neighborhood of infinity:

$$\nabla\psi = \mathbf{u}^\perp = (-v, u), \tag{3.56}$$

where $\mathbf{u} = (u, v)$. Using the stream function ψ, Amick introduced an auxiliary function $\gamma = \Phi - \omega\psi$, where $\Phi := p + \frac{1}{2}|\mathbf{u}|^2$ is the Bernoulli pressure. The gradient of this auxiliary function γ satisfies the identity

$$\nabla\gamma = -\nu\nabla^\perp\omega - \psi\nabla\omega.$$

Then $\nabla\gamma \cdot \nabla^\perp\omega = -\nu|\nabla^\perp\omega|^2$, and therefore γ has the following monotonicity properties:

$$\begin{aligned}&\gamma \text{ is monotone along level sets of the vorticity } \omega = c \text{ and}\\ &\text{vice versa—the vorticity } \omega \text{ is monotone along level sets of } \gamma = c,\end{aligned} \tag{3.57}$$

see [1].

Obviously, the stream function ψ (and, consequently, the corresponding auxiliary function γ) is well defined in the neighborhood of infinity under the more general condition

$$\int_{\partial\Omega_0} \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}s = 0 \tag{3.58}$$

instead of (3.55). However, in the general case the flow-rate of the velocity field is not zero,

$$\int_{\partial\Omega_0} \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}s \neq 0, \tag{3.59}$$

and, therefore, the stream function ψ cannot be defined in the neighborhood of infinity.

We will overcome this difficulty using the assumption (3.54). Take and fix a radius $R_* > R_0$ (R_* could be chosen arbitrary large) and consider the domain $\Omega_* = \{z \in \mathbb{R}^2 : |z| > R_*\}$. Denote by U_i the connected components of the open set $\{z \in \Omega_* : \omega(z) \neq 0\}$. Then there holds the following:

Lemma 3.4. *Under assumption* (3.54) *the following assertions are fulfilled:*

(i) There are only finitely many components U_i, $i = 1, \dots, N$;
(ii) Every U_i is a simply connected open set;
(iii) The vorticity $\omega(z)$ changes sign in every neighborhood of infinity, i.e., there exist two sequences of points z_n^+ and z_n^- such that $\omega(z_n^+) > 0$, $\omega(z_n^-) < 0$ and $\lim_{n\to\infty} |z_n^+| = \lim_{n\to\infty} |z_n^-| = \infty$.

We shall prove Lemma 3.4 below. Let us finish the proof of the theorem using this lemma. The components U_i play also an important role in the arguments of Amick. In particular, he proves in [1] the same properties (i)–(iii) using the boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$. Here we get the properties (i)–(iii) because of the assumption (3.54). Since U_i are simply connected, this allows us to define the stream function ψ in every component U_i. Moreover, since $\omega = 0$ on $\Omega_* \cap \partial U_i$, the auxiliary function $\gamma = \Phi - \omega\psi$ is well defined and continuous on the whole domain Ω_*. After the functions ψ and γ are defined, we can repeat the arguments of the paper [1] and prove the convergence (3.36) of absolute value of the velocity. By Lemma 3.3(ii) this implies the statement of Theorem 1.2. For the reader's convenience we recall the corresponding arguments of Amick [1] in Appendix (we also simplify some of his proofs). $\square$

Proof of Lemma 3.4. Let us prove (iii) first. If we suppose that (iii) is not true, i.e., that there exists $R_1 > 0$ such that $\omega(z)$ does not change sign in $\Omega_1 = \{z : |z| > R_1\}$. Without loss of generality assume that $\omega(z) \geq 0$ in Ω_1. Then by the maximum principle,

$$\omega(z) > 0 \qquad \text{in } \Omega_1. \tag{3.60}$$

Take arbitrary $R_2 > R_1$ and denote

$$\delta := \inf_{z \in S_{R_2}} \omega(z), \tag{3.61}$$

where, we recall, $S_{R_2} = \{z \in \mathbb{R}^2 : |z| = R_2\}$. By (3.60), $\delta > 0$. Now take any z_2 such that $|z_2| > R_2$ and $\omega(z_2) < \delta$. Then by construction, $K(z_2) \cap S_{R_2} = \emptyset$. Therefore, $K(z_2) \cap S_{R_0} = K(z_2) \cap \partial\Omega_0 = \emptyset$, which is a contradiction with (3.54).

(ii). Fix a component U_i. Assume for definiteness that $\omega > 0$ in U_i, and take an arbitrary curve $S \subset U_i$ homeomorphic to the unit circle. By construction, there exists $\delta > 0$ such that

$$\omega(z) > \delta \qquad \forall z \in S.$$

The curve S split the plane $\mathbb{R}^2$ into the two components: $\mathbb{R}^2 \backslash S = \Omega_S \cup \Omega_\infty$, where $\partial\Omega_S = \partial\Omega_\infty = S$, Ω_S is a bounded domain homeomorphic to the disk, and Ω_∞ is a neighborhood of infinity. Now we have to consider two cases:

(α) the curve S surrounds the origin. Then $\Omega_\infty \subset \Omega_*$, and, by maximum principle, $\omega \geq 0$ in Ω_∞. Thus, we receive the contradiction with property (iii) proved just above.

($\alpha\alpha$) the curve S does not surround the origin. Then $\Omega_S \subset \Omega_*$, and, by maximum principle, $\omega > 0$ in Ω_S. Therefore, $\Omega_S \subset U_i$. Since S was arbitrary, it means that U_i is a simply connected set.

Let us prove (i). Since ω is a nonzero analytical function, the set $Z_* = \{z \in S_{R_*} : \omega(z) = 0\}$ is finite (recall that S_{R_*} is a circle of radius R_*). Let S_j, $j = 1, \ldots, M$, be the connected components of the set $S_{R_*} \setminus Z_*$.

Fix arbitrary component U_i. By the maximum principle, $\omega(z)$ is not identically zero on ∂U_i, i.e., there exists a point z_0 such that

$$z_0 \in \partial U_i \qquad \text{and} \qquad \omega(z_0) \neq 0.$$

On the other hand, by definition, U_i is a connected component of the open set

$$\{z \in \Omega_* : \omega(z) \neq 0\},$$

in particular, we have the identity $\omega(z) \equiv 0$ on the set $\Omega_* \cap \partial U_i$. Therefore,

$$z_0 \in \partial \Omega_* = S_{R_*}.$$

This means, using the above notation, that there exists a number $j(i) \in \{1, \ldots, M\}$ such that

$$z_0 \in S_{j(i)}.$$

Then by elementary properties of connected sets and by definitions of S_j and U_i, we have

$$S_{j(i)} \subset \partial U_i$$

and

$$\Big[j(i_1) = j(i_2) \Big] \Rightarrow U_{i_1} = U_{i_2},$$

i.e., the function $i \mapsto j(i)$ is injective. Finally, since the family of components S_j is finite, we conclude that the family U_i is finite as well. This finishes the proof of Lemma 3.4. □

4. Appendix

For reader's convenience we recall here some steps of the corresponding arguments of Amick [1] for the proof of the convergence (3.36).

Our Lemma 3.4 implies, in particular, that there exists at least one unbounded component U_{k_1} where ω is strictly positive and at least one unbounded component U_{k_2} where ω is strictly negative (cf. with [1, Theorem 8, page 84]).

First of all, we mention that by [1, Theorem 15, page 95], if we take the number R_* large enough, then there holds the statement

$$\nabla\omega(z) \neq 0 \quad \text{if } \omega(z) = 0 \ \text{ and } \ |z| \geq R_*. \tag{4.1}$$

This gives the possibility to clarify the geometrical and topological structure of the components U_i. Namely, $\Omega_* \cap \partial U_i$ consists of finitely many smooth (even analytical) curves.

Let U_i, $i = 1, \dots, M$ be a family of *unbounded* components U_i. Then Amick proved the following geometrical and analytical characterization for them:

Theorem 4.1. (see Theorem 11, page 89 in [1]) *For every U_i, $i = 1, \dots, M$, we have that:*

- *(α) The set $\Omega_* \cap \partial U_i$ has precisely two unbounded components which may be parametrised as $\{(x_j(s), y_j(s)) : s \in (0, \infty)\}$, $j = 1, 2$. In addition, $(x_j(0), y_j(0)) \in \{|z| = R_*\}$, s denotes the arc-length measure from these points, and the functions $x_j(\cdot)$ and $y_j(\cdot)$ are real-analytical (if we choose R_* large enough to have (4.1)). The function ω vanishes on these arcs and $|(x_j(s), y_j(s))| \to \infty$ as $s \to \infty$.*
- *($\alpha\alpha$) The maps $s \mapsto \Phi(x_j(s), y_j(s))$ are monotone decreasing and increasing on $(0, \infty)$, respectively, for $j = 1$ and $j = 2$.*

Since the Bernoulli pressure Φ is uniformly bounded, by Weierstrass monotone convergence theorem we have that the functions $s \mapsto \Phi(x_j(s), y_j(s))$ have some limits as $s \to \infty$ for $j = 1, 2$. After the usual agreement that

$$p(z) \to 0 \qquad \text{as } |z| \to \infty, \tag{4.2}$$

and taking into account the convergence on the family of circles (1.10), we obtain

Corollary 4.1. *Functions from item ($\alpha\alpha$) of Theorem 4.1 have the same limit*

$$\Phi(x_j(s), y_j(s)) \to \frac{1}{2}|\mathbf{u}_\infty|^2 \qquad \textit{as } s \to \infty. \tag{4.3}$$

The next step concerns the auxiliary function γ. One of the most important tools in [1] is the following assertion:

Theorem 4.2. (see Theorem 14, page 92 in [1]) *For every U_i, $i = 1, \dots, M$, the convergence*

$$\gamma(z) \to \frac{1}{2}|\mathbf{u}_\infty|^2 \qquad \textit{uniformly as } |z| \to \infty,\ z \in U_i \tag{4.4}$$

holds.

The proof of Theorem 4.2 in [1] is rather short and elegant. Indeed, by construction, we have $\gamma \equiv \Phi$ on $\Omega_* \cap \partial U_i$, therefore, the convergence (4.4) for $z \in \partial U_i$ follows immediately from (4.3). The convergence (4.4) in general case $z \in U_i$ follows from the uniform convergence $\omega(z) \to 0$ as $|z| \to \infty$, from constructive assumption $\omega(z) \neq 0$ in U_i, and from the monotonicity of γ on level sets of ω mentioned in (3.57) (see [1, pages 92–94] for the details).

Since there exist only finitely many components U_i, from Theorem 4.2 we immediately obtain

Corollary 4.2. *The convergence*

$$\gamma(z) \to \frac{1}{2}|\mathbf{u}_\infty|^2 \qquad \textit{uniformly as } |z| \to \infty \tag{4.5}$$

holds.

The function $\gamma = \Phi - \omega\psi$ is closely related to Φ; in particular, $\gamma = \Phi$ if $\omega = 0$ or $\psi = 0$. Having this in mind, it is possible to prove the same convergence as (4.5) for Φ instead of γ.

We assume, without loss of generality, that

$$\mathbf{u}_\infty = (1, 0). \tag{4.6}$$

Recall that, by the results of D. Gilbarg & H. Weinberger [9], the convergence

$$\lim_{r\to+\infty} \int_0^{2\pi} |\mathbf{u}(r,\theta) - \mathbf{u}_\infty|^2 \mathrm{d}\theta = 0 \tag{4.7}$$

holds. In other words, since $\nabla\psi = \mathbf{u}^\perp = (-v, u)$, we have

$$\lim_{r\to+\infty} \frac{1}{r} \int_{|z|=r} |\nabla\psi(z) - (0,1)|^2 \, \mathrm{d}s = 0. \tag{4.8}$$

From this fact and from the finiteness of the Dirichlet integral $\int_\Omega |\nabla\mathbf{u}|^2 < \infty$, we obtain (see [1, pages 99–100] for details) the following asymptotic behaviour of the stream function ψ:

$$\lim_{r\to+\infty} \frac{1}{r} |\psi(x,y) - y| = 0, \tag{4.9}$$

where $r = \sqrt{x^2+y^2}$[3]. For any $\alpha > 0$ denote by Sect_α the sector

$$\mathrm{Sect}_\alpha = \{z = (x,y) \in \Omega_* : \frac{|y|}{|x|} \geq \alpha\}.$$

Since $r \leq c_\alpha |y|$ for $z \in \mathrm{Sect}_\alpha$, from (4.9) it follows that

$$\lim_{(x,y)\in S_\alpha,\ \sqrt{x^2+y^2}\to\infty} \left| \frac{\psi(x,y)}{y} - 1 \right| = 0. \tag{4.10}$$

Let us prove the convergence of Φ in any sector Sect_α.

[3] Stream function ψ is well defined by identity $\nabla\psi = \mathbf{u}^\perp$ in every simply–connected subdomain of Ω_*; in particular, ψ is well-defined in intersection of Ω_* with every of the four half spaces $\{(x,y) \in \mathbb{R}^2 : x > 0\}$, $\{(x,y) \in \mathbb{R}^2 : x \leq 0\}$, $\{(x,y) \in \mathbb{R}^2 : y > 0\}$, $\{(x,y) \in \mathbb{R}^2 : y \leq 0\}$. Since these definitions of ψ differ only by some additive constants, they have no influence on the asymptotic properties discussed here.

Lemma 4.1. (See Theorem 17 and Corollary 18 on page 101 in [1]) *For any* $\alpha > 0$*, the uniform convergences*

$$|z|\omega(z) \to 0 \qquad as\ |z| \to \infty,\ z \in \mathrm{Sect}_\alpha, \tag{4.11}$$

$$\Phi(z) \to \frac{1}{2}|\mathbf{u}_\infty|^2 \qquad as\ |z| \to \infty,\ z \in \mathrm{Sect}_\alpha. \tag{4.12}$$

hold.

Proof. Fix $\alpha > 0$. Then

$$\forall z = (x, y) \in \mathrm{Sect}_{\frac{\alpha}{3}} : |z| \le \tilde{c}_\alpha |y|. \tag{4.13}$$

Take $z_0 = (x_0, y_0) \in \mathrm{Sect}_\alpha$. Without loss of generality assume that $y_0 > 0$. Since

$$\int_{\Omega_*} |\nabla\Phi|^2 < \infty,$$

from Lemma 2.2, from the uniform convergence of the pressure to zero (see (4.2)) and from average convergence of the velocity to $\mathbf{u}_\infty = (1, 0)$ (see (4.7)), we have that

$$\exists r \in [\frac{1}{4}y_0, \frac{1}{2}y_0] : \qquad \sup_{|z-z_0|=r} \left|\Phi(z) - \frac{1}{2}\right| \le \varepsilon_1(r_0), \tag{4.14}$$

where $r_0 = |z_0|$ and $\varepsilon_1(r_0) \to 0$ uniformly as $r_0 \to \infty$ (of course, this function $\varepsilon_1(r_0)$ depends also on the parameter α which was fixed above).

From (4.14) and from Corollary 4.2 we have

$$\sup_{|z-z_0|=r} |\omega(z)\psi(z)| \le \varepsilon_2(r_0), \tag{4.15}$$

where again $\varepsilon_2(r_0) \to 0$ uniformly as $r_0 \to \infty$. Denote by B_0 the disk $\{z \in \mathbb{R}^2 : |z - z_0| \le r\}$. By construction,

$$B_0 \subset \mathrm{Sect}_{\frac{\alpha}{3}}.$$

Then, by (4.10),

$$\sup_{(x,y)\in B_0} \left|\frac{\psi(x, y)}{y} - 1\right| \to 0 \text{ as } r_0 \to \infty. \tag{4.16}$$

In particular,

$$\psi(y) \ge d_\alpha r_0 \tag{4.17}$$

if r_0 is sufficiently large; here the constant d_α depends on α only. From (4.17) and (4.15) we obtain immediately that

$$\sup_{|z-z_0|=r} |\omega(z)| \le \frac{1}{r_0}\varepsilon_3(r_0), \tag{4.18}$$

where again $\varepsilon_3(r_0) \to 0$ uniformly as $r_0 \to \infty$. By the maximum principle,

$$|\omega(z_0)| \le \frac{1}{r_0}\varepsilon_3(r_0). \tag{4.19}$$

Thus, we have proved the asymptotic estimate (4.11). Then the convergence (4.12) follows immediately from (4.11) and (4.5). □

The convergence of Φ *outside* of the sectors Sect_α is more delicate and subtle question. Ch. Amick solved this problem [1] using level sets of the stream function ψ.

Define the stream function in the half-domain $\Omega_+ = \{(x, y) : x \ge 0,\ x^2+y^2 \ge R_*^2\}$ and consider the set $C_+ = \{z \in \Omega_+ : \psi(z) = 0\}$.[4] Then $\gamma = \Phi$ on C_+ and from the convergence of γ (4.5) we obtain immediately that $\frac{1}{2}|\nabla\psi(z)|^2 = \frac{1}{2}|\mathbf{u}(z)|^2 \to \frac{1}{2}$ when $|z| \to \infty$, $z \in C_+$. In particular, $\nabla\psi \ne 0$ on C_+ if we choose the parameter R_* sufficiently large. Using similar arguments, Amick proved that the set C_+ has a very simple geometrical structure.

Lemma 4.2. (see Lemma 20 on page 104 in [1]) *If the number R_* is chosen large enough, then the set C_+ is a smooth curve*

$$C_+ = \big\{(p_+(s), q_+(s)) : s \in [0, +\infty)\big\},$$

here p_+ and q_+ are real-analytic functions on $[0, \infty)$, $p_+(s) \to \infty$ *and* $\frac{q_+(s)}{p_+(s)} \to 0$ *as $s \to \infty$. In addition,*

$$|\mathbf{u}(p_+(s), q_+(s))| \to |\mathbf{u}_\infty| \quad \text{as } s \to \infty. \tag{4.20}$$

Of course, a similar assertion holds for another half-domain $\Omega_- = \{(x, y) : x \le 0,\ x^2 + y^2 \ge R_*^2\}$. Using this Lemma and some classical estimates for the Laplace operator (recall, that $\omega = \Delta\psi$), Amick proved the required assertion:

Theorem 4.3. (See Theorem 21 (a) on page 105 in [1]) *The convergence*

$$|\mathbf{u}(z)| \to |\mathbf{u}_\infty| \qquad \text{uniformly as } |z| \to \infty \tag{4.21}$$

holds.

Remark 4.1. The proof of Theorem 4.3 could be essentially simplified in comparison with the original version of [1]. Indeed, from the convergence (4.20) on the curve C_+, using Lemmas 2.1–2.2, it is easy to show that there exists $\sigma > 0$ such that for any $z \in C_+$ with sufficiently large value $|z|$ we have

$$\left|\frac{1}{r}\int_{|\xi-z|=r} \mathbf{u}(\xi)\, \mathrm{d}s\right| > \sigma$$

[4] The asymptotic behavior of $\psi(x, y)$ is similar to that of the linear function $g(x, y) = y$. Since the level set $\{(x, y) \in \Omega_+ : g(x, y) = 0\}$ is a ray $\{(x, y) \in \Omega_+ : y = 0\}$, the set C_+ goes to infinity as well, see also Lemma 4.2 for the precise formulation.

for all $r \in (0, \frac{4}{5}|z|]$. Then the arguments used in the proof of Lemma 3.3 (ii) give us that

$$\mathbf{u}(p_+(s), q_+(s)) \to \mathbf{u}_\infty \quad \text{as } s \to \infty, \tag{4.22}$$

instead of (4.20). This stronger convergence allows us to simplify some technical moments in the proof of [1, Theorem 21 (a)]; see also [1, Theorem 21 (c)].

Acknowledgements. The authors are grateful to the anonymous referee for many useful remarks and suggestions. M. Korobkov was partially supported by the Ministry of Education and Science of the Russian Federation (Grant 14.Z50.31.0037) and by the Russian Federation for Basic Research (Project Numbers 18-01-00649 and 17-01-00875). The research of K. Pileckas was funded by the Grant No. S-MIP-17-68 from the Research Council of Lithuania.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Amick, C.J.: On Leray's problem of steady Navier–Stokes flow past a body in the plane. *Acta Math.* **161**, 71–130, 1988
2. Amick, C.J.: On the asymptotic form of Navier–Stokes flow past a body in the plane. *J. Differ. Equ.* **91**, 149–167, 1991
3. Engelking, R.: *General Topology*, 2nd edn. Sigma Series in Pure Mathematics, Vol. 6. Heldermann Verlag, Berlin, 1989
4. Finn, R., Smith, D.R.: On the stationary solutions of the Navier–Stokes equations in two dimensions. *Arch. Ration. Mech. Anal.* **25**, 26–39, 1967
5. Fujita, H.: On the existence and regularity of the steady-state solutions of the Navier–Stokes equation. *J. Fac. Sci. Univ. Tokyo*(1A) **9**, 59–102, 1961
6. Galdi, G.P.: *An Introduction to the Mathematical Theory of the Navier–Stokes Equations. Steady–State Problems*. Springer, 2011
7. Galdi, G.P., Sohr, H.: On the asymptotic structure of plane steady flow of a viscous fluid in exterior domains. *Arch. Ration. Mech. Anal.* **131**, 101–119, 1995
8. Gilbarg, D., Weinberger, H.F.: Asymptotic properties of Leray's solution of the stationary two–dimensional Navier–Stokes equations. *Russ. Math. Surv.* **29**, 109–123, 1974
9. Gilbarg, D., Weinberger, H.F.: Asymptotic properties of steady plane solutions of the Navier–Stokes equations with bounded Dirichlet integral. *Ann. Sc. Norm. Pisa (4)* **5**, 381–404, 1978
10. Korobkov, M.V., Pileckas, K., Russo, R.: The existence of a solution with finite Dirichlet integral for the steady Navier–Stokes equations in a plane exterior symmetric domain. *J. Math. Pures Appl.* **101**, 257–274, 2014
11. Korobkov, M.V., Pileckas, K., Russo, R.: On the steady Navier–Stokes equations in 2D exterior domains, arXiv:1711.02400
12. Ladyzhenskaia, O.A.: *The Mathematical Theory of Viscous Incompressible Fluid*. Gordon and Breach, 1969

13. Leray, J.: Étude de diverses équations intégrales non linéaire et de quelques problèmes que pose l'hydrodynamique. *J. Math. Pures Appl.* **12**, 1–82, 1933
14. Malý, J., Swanson, D., Ziemer, W.P.: The Coarea formula for Sobolev mappings. *Trans. AMS* **355**(2), 477–492, 2002
15. Pileckas, K., Russo, R.: On the existence of vanishing at infinity symmetric solutions to the plane stationary exterior Navier–Stokes problem. *Math. Ann.* **352**, 643–658, 2012
16. Russo, A.: A note on the two–dimensional steady-state Navier–Stokes problem. *J. Math. Fluid Mech.*, **11**, 407–414, 2009
17. Sazonov, L.I.: On the asymptotic behavior of the solution of the two-dimensional stationary problem of the flow past a body far from it. (Russian) *Mat. Zametki* **65**, 246–253, 1999; translation in *Math. Notes* **65**, 246–253, 1999

Mikhail V. Korobkov
School of Mathematical Sciences,
Fudan University,
Shanghai 200433,
China.
e-mail: korob@math.nsc.ru

and

Voronezh State University,
Universitetskaya pl. 1,
Voronezh, Russia, 394018.

and

Konstantin Pileckas
Faculty of Mathematics and Informatics,
Vilnius University,
Naugarduko Str., 24,
03225 Vilnius,
Lithuania.
e-mail: konstantinas.pileckas@mif.vu.lt

and

Remigio Russo
Dipartimento di Matematica e Fisica
Università degli studi della Campania "Luigi Vanvitelli",
viale Lincoln 5,
81100,
Caserta,
Italy.
e-mail: remigio.russo@unicampania.it

(*Received March 27, 2018 / Accepted January 18, 2019*)