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Leray's plane steady state solutions are nontrivial

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Abstract

We study solutions to the obstacle problem for the stationary Navier–Stokes system in a two dimensional exterior domain (flow past a prescribed body). We prove that the classical Leray solution to this problem is always nontrivial. No additional condition (on symmetry or smallness, etc.) is assumed. This is a complete extension of a classical result of C.J. Amick (Acta Math. 1988) where nontriviality was proved under symmetry assumption.

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1 Introduction

Let Ω be an exterior domain in $\mathbb{R}^2$ with compact boundary $\partial\Omega = \bigcup_{i=1}^{N} \Gamma_i$, where Γ_i are smooth disjoint curves, homeomorphic to the circle. In particular, $\Omega \supset \mathbb{R}^2 \setminus B$, where B is the disk of radius R_0 centered at the origin with $\partial\Omega \subset B$.

One of the most difficult and still open problem in the theory of the stationary Navier–Stokes equations, initiated by J. Leray in the famous paper of 1933 [12], concerns the existence of a solution to the *flow around an obstacle* (see also [5]):

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{u}(z) \to \mathbf{u}_0 & \text{as } \ |z| \to \infty, \end{cases} \tag{1.1}$$

where $\mathbf{u}$ and p are the unknown velocity and pressure fields, ν denotes the kinematical viscosity coefficient, and $\mathbf{u}_0 \in \mathbb{R}^2$ is a nonzero constant vector (prescribed velocity at infinity).

Leray suggested [12] the following elegant approach to this problem which was called method of "invading domains". Denoting by $\mathbf{u}_k$ the solution to the problem

$$\begin{cases} -\nu\Delta\mathbf{u}_k + (\mathbf{u}_k\cdot\nabla)\mathbf{u}_k + \nabla p_k = \mathbf{0} & \text{in } \Omega_k, \\ \operatorname{div}\mathbf{u}_k = 0 & \text{in } \Omega_k, \\ \mathbf{u}_k = \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{u}_k = \mathbf{u}_0 & \text{for } \ |z| = R_k. \end{cases} \tag{1.2}$$

on the intersection Ω_k of Ω with the disk B_{R_k} of radius $R_k \geq k(\gg R_0)$, whose existence he proved before, Leray showed that the sequence $\mathbf{u}_k$ satisfies the estimate $\int_\Omega |\nabla\mathbf{u}_k|^2 \leq c$ for some positive constant c independent of k. Hence, he observed that it is possible to extract a subsequence $\mathbf{u}_{k_n}$ which weakly converges to a solution $\mathbf{u}_L$ of problem $(1.1)_{1,2,3}$ with $\int_\Omega |\nabla\mathbf{u}_L|^2 < +\infty$. This solution was later called *Leray's solution* (see, *e.g.*, [1]).

An arbitrary solution $\mathbf{u}$ to the Navier–Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega. \end{cases} \tag{1.3}$$

having the finite Dirichlet integral

$$\int_\Omega |\nabla\mathbf{u}|^2 < +\infty, \tag{1.4}$$

is called today *D–solution* [5]. As is well known (*e.g.*, [11]), such solutions are real–analytic in Ω.

As far as condition$(1.1)_4$ is concerned, Leray limited himself to observe that, while in three dimensional problem (1.4) it is sufficient to guarantee the attainability of the limit $\mathbf{u}_0$ at infinity (at least in a mean square sense) as a consequence of the inequality $\|r^{-1}(\mathbf{u}-\mathbf{u}_0)\|_{L^2(\Omega)} \le 4\|\nabla\mathbf{u}\|_{L^2(\Omega)}$, in the two dimension case the corresponding inequality $\|(r\log r)^{-1}(\mathbf{u}-\mathbf{u}_0)\|_{L^2(\Omega)} \le c\|\nabla\mathbf{u}\|_{L^2(\Omega)}$ does not imply any type of convergence. Leray concluded that one should not be surprised of this phenomenon, in view of the *Stokes paradox*, *i.e.*, the system obtained from (1.1) removing the nonlinear term, namely,

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{u}(z) \to \mathbf{u}_0 & \text{as } \ |z| \to \infty, \end{cases} \tag{1.5}$$

does not admit a solution (see, *e.g.*, [15]).

The problem of the asymptotic behaviour at infinity of Leray's solution $(\mathbf{u}_L, p_L)$ was tacked by D. Gilbarg & H. Weinberger in 1974 [6]. They proved that $\mathbf{u}_L$ is bounded, there are a scalar p_0 and a constant vector $\mathbf{u}_\infty$ such that

$$\lim_{|z|\to+\infty} p_L(z) = p_0 \tag{1.6}$$

(one can choose, say, $p_0 = 0$),

$$\lim_{|z|\to+\infty} \int_0^{2\pi} |\mathbf{u}_L(r,\theta) - \mathbf{u}_\infty|^2 d\theta = 0, \tag{1.7}$$

and

$$\begin{aligned} &\omega(z) = o(r^{-3/4}), \\ &\int_\Omega r|\nabla\omega(z)|^2 < \infty, \end{aligned} \tag{1.8}$$

where $r = |z|$ and

$$\omega = \partial_2 u_{L1} - \partial_1 u_{L2}$$

is the vorticity.

In 1988 C.J. Amick [1] proved that a D–solution to the problem of a flow around an obstacle $(1.1)_{1.2.3}$ has the following asymptotic properties:

(i) $\mathbf{u}$ is bounded and, as a consequence, it satisfies (1.7)–(1.8);

(ii) the total head pressure $\Phi = p + \frac{1}{2}|\mathbf{u}|^2$ and the absolute value of the velocity $|\mathbf{u}|$ have the uniform limit at infinity, i.e.,

$$|\mathbf{u}(r,\theta)| \to |\mathbf{u}_\infty| \qquad \text{as} \quad r \to \infty, \tag{1.9}$$

where $\mathbf{u}_\infty$ is the constant vector from the condition (1.7);

(iii) if $\partial\Omega$ is symmetric with respect to the x_1–axis, and $\mathbf{u} = (u_1, u_2)$ is also symmetric, i.e., if u_1 is even and u_2 is odd with respect to x_1, then $\mathbf{u}$ converges uniformly at infinity to a constant vector $\mu\mathbf{e}_1$, for some scalar μ. Moreover, the Leray procedure yields a *nontrivial* (i.e., not identically zero) symmetric solution.

In the present paper we prove that in general case (without any additional symmetry assumptions) the Leray solution to the the problem of the flow around obstacle is *always* nontrivial.

Theorem 1.1. *Let Ω be an exterior domain in $\mathbb{R}^2$ with smooth compact boundary, $\nu > 0$ and $\mathbf{0} \neq \mathbf{u}_0 \in \mathbb{R}^2$. Take a sequence $\mathbf{u}_k$ of solutions to system (1.2), and take further arbitrary weakly convergent subsequence $\mathbf{u}_{k_n} \rightharpoonup \mathbf{u}$. Then the limiting solution $\mathbf{u}$ to $(1.1)_{1,2,3}$ is nontrivial (i.e., $\mathbf{u}$ is not identically zero). In particular the Leray solution is nontrivial.*

Moreover, we proved a kind of complementary result.

Theorem 1.2. *Let Ω be an exterior domain in $\mathbb{R}^2$ with smooth compact boundary, $\nu > 0$ and let $\mathbf{a} \in \mathbb{R}^2$ be a nonzero constant vector. Take a sequence $\mathbf{u}_k$ of solutions to the system*

$$\begin{cases} -\nu\Delta\mathbf{u}_k + (\mathbf{u}_k \cdot \nabla)\mathbf{u}_k + \nabla p_k = \mathbf{0} & \text{in } \Omega_k, \\ \operatorname{div} \mathbf{u}_k = 0 & \text{in } \Omega_k, \\ \mathbf{u}_k = \mathbf{a} & \text{on } \partial\Omega, \\ \mathbf{u}_k = \mathbf{0} & \textit{for } |z| = R_k, \end{cases} \tag{1.10}$$

and take further arbitrary weakly convergent subsequence $\mathbf{u}_{k_n} \rightharpoonup \mathbf{u}$. Then the limiting solution $\mathbf{u}$ is nontrivial. In other words, the solution to the system

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega, \end{cases}$$

obtained by the Leray method is nontrivial, i.e., $\mathbf{u} \neq \mathbf{a}$.

Note, that for linear case (*e.g.*, for Stokes system (1.5)) the assertions of Theorems 1.1–1.2 are evidently equivalent. But of course it does not hold in general for nonlinear systems.

Recently [9], [10] we proved the following result for general D-solutions.

Theorem 1.3 ([10])**.** *Let* $\mathbf{u}$ *be a D-solution to the Navier–Stokes system (1.3) in the exterior domain* $\Omega \subset \mathbb{R}^2$. *Then* $\mathbf{u}$ *converges uniformly at infinity, i.e.,*

$$\mathbf{u}(z) \to \mathbf{u}_\infty \qquad \textit{uniformly as } \ |z| \to \infty, \tag{1.11}$$

where $\mathbf{u}_\infty \in \mathbb{R}^2$ *is some constant vector.*

By virtue of Theorem 1.3, for every Leray solution there is a constant vector $\mathbf{u}_\infty \in \mathbb{R}^2$ such that (1.11) holds. However, the desired equality $\mathbf{u}_\infty = \mathbf{u}_0$ is still an open question. We even do not know whether $\mathbf{u}_\infty$ is nonzero if $\mathbf{u}_0 \neq \mathbf{0}$. So, the problem of the flow around an obstacle remains still open.

The same open problem exists in Theorem 1.2: we know that the corresponding Leray solution converges uniformly to some constant vector $\mathbf{u}_\infty \in \mathbb{R}^2$, but we were not able to prove the expected equality that $\mathbf{u}_\infty = \mathbf{0}$.

Some additional historical remarks. Note, that thirty years after Leray, H. Fujita [4] by means of different techniques proved the existence of a D–solution to $(1.1)_{1,2,3}$. Due of a lack of a uniqueness theorem, the Leray and Fujita solutions are not comparable.

Recall also the amazing discovery of R. Finn and D.R. Smith in 1967 [3] of the existence of a solution to (1.1) for ν sufficiently large (or, equivalently, for $\mathbf{u}_0$ sufficiently small). Their approach is completely different from that of Leray. Nevertheless, their method does not allow to prove the existence of the solution to the problem

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega, \\ \mathbf{u}(z) \to \mathbf{0} & \text{as } \ |z| \to \infty, \end{cases} \tag{1.12}$$

for a constant vector $\mathbf{a} \in \mathbb{R}^2$. This problem is quite open **even for small** vectors $\mathbf{a} \neq \mathbf{0}$ (the existence was proved for some nonconstant $\mathbf{a}$ under the assumption of symmetry with respect to both coordinate axes [13]). Even the reduced problem (1.12_{1-3}) is open for general boundary value $\mathbf{a}$ (see, *e.g.*, [14] for the case of small fluxes). More detailed survey of results concerning boundary value problems for stationary NS-system in plane exterior domains see, *e.g.*, in [5] or in our recent papers [9], [10], [8].

Finally, let us describe shortly the main steps of the proof of Theorem 1.1 (for defineteness, take $\mathbf{u}_0 = (1, 0)$ and $\nu = 1$). The main ideas are rather simple. By Amick criterion [1], the corresponding limiting Leray solution $\mathbf{u}$ is trivial if and only if the convergence

$$\int\limits_{\Omega_k} |\nabla \mathbf{u}_k|^2 \, dx \to 0 \qquad \text{as} \quad k \to \infty \tag{1.13}$$

holds for the sequence of solutions $\mathbf{u}_k$ to (1.2) in bounded domains $\Omega_k = \Omega \cap B_{R_k}$. Suppose (1.13) to be fulfilled. Then the functions $(\mathbf{u}_k, p_k)$ and all its derivatives go to 0 uniformly on every bounded set; moreover, from [6] it follows that $\sup\limits_{z \in \Omega'_k} |p_k| \to 0$, where $\Omega'_k = \Omega \cap B_{\frac{3}{4}R_k}$. It is well known, that the Bernoulli pressure $\Phi_k = p_k + \frac{1}{2}|\mathbf{u}_k|^2$ satisfies the maximum principle. Using these facts, it can be proved that the level lines $\Phi_k = t$ of the Bernoulli pressure are arranged as circles surrounding the origin; furthermore, the Bernoulli pressure approximately equal to zero near $\partial\Omega$, and it increases up to $\frac{1}{2}$ near the large circle of a radius R_k.

The following steps are crucial in our arguments:

1) The *direction* of the velocity vector $\mathbf{u}_k$ is under control of the Dirichlet integral (it was proved in the Gilbarg–Weinberger paper [7], see below lemma 2.4 for the exact formulation of the result);

2) The vorticity $\omega_k(z)$ does not change sign between two level lines of the Bernoulli pressure Φ_k (it is proved using the results of Amick [1]).

Using these important facts, we prove that for the velocity $\mathbf{u}_k$ the following representation formulas hold:

$$\bar{\mathbf{u}}_k(r) = |\bar{\mathbf{u}}_k(r)| \, (\cos \varphi_k(r), \sin \varphi_k(r)), \tag{1.14}$$

where $\bar{\mathbf{u}}_k(r)$ means the mean value of $\mathbf{u}_k$ over the circle of radius r and

$$|\varphi_k(r)| \le \varepsilon_k \qquad \text{for all sufficiently large } r \le R_k \tag{1.15}$$

with $\varepsilon_k \to 0$ as $k \to \infty$. Recall, that the gradient of the Bernoulli pressure satisfies the identity

$$\nabla \Phi_k(z) = -\nabla^{\perp} \omega_k(z) + \omega_k(z) \cdot \mathbf{u}_k^{\perp}(z), \tag{1.16}$$

where $\mathbf{u}_k^{\perp} := (-u_k^2, u_k^1)$. The first term in (1.16) is negligible (since the integral $\int\limits_{\Omega'_k} r|\nabla \omega_k|^2$ is small, see (1.8_2)). Using these facts we obtain the contradiction with the geometrical structure of the level lines of the Bernoulli pressure Φ_k described above and, as a consequence, the nontriviality of the Leray solution.

2 Notations and preliminaries

By *a domain* we mean an open connected set. We use standard notations for Sobolev spaces $W^{k,q}(\Omega)$, where $k \in \mathbb{N}$, $q \in [1, +\infty]$. In our notation we do not distinguish function spaces for scalar and vector valued functions; it is clear from the context whether we use scalar or vector (or tensor) valued function spaces.

For $q \geq 1$ denote by $D^{k,q}(\Omega)$ the set of functions $f \in W^{k,q}_{\rm loc}(\Omega)$ such that $\|f\|_{D^{k,q}(\Omega)} = \|\nabla^k f\|_{L^q(\Omega)} < \infty$.

We denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure, i.e., $\mathcal{H}^k(F) = \lim\limits_{t\to 0+} \mathcal{H}^k_t(F)$, where

$$\mathcal{H}^1_t(F) = \big(\frac{\alpha_k}{2}\big)^k \inf\Big\{ \sum_{i=1}^{\infty} \big(\mathrm{diam} F_i\big)^k : \mathrm{diam} F_i \leq t, F \subset \bigcup_{i=1}^{\infty} F_i \Big\}$$

and α_k is a Lebesgue volume of the unit ball in $\mathbb{R}^k$.

In particular, for a curve S the value $\mathcal{H}^1(S)$ coincides with its length, and for sets $E \subset \mathbb{R}^2$ the $\mathcal{H}^2(E)$ coincides with the usual Lebesgue measure in $\mathbb{R}^2$.

Also, for a curve S by $\int\limits_S f\,ds$ we denote the usual integral with respect to 1-dimensional Hausdorff measure (=length). Further, for a set $E \subset \mathbb{R}^2$ by $\int\limits_E f(\xi)\,d\mathcal{H}^2$ we denote the integral with respect to the two-dimensional Lebesgue measure. For convenience, we will write simply

$$\int\limits_E rf \qquad \text{instead of} \qquad \int\limits_E |\xi|\, f(\xi)\, d\mathcal{H}^2_\xi.$$

Below we present some results concerning the behavior of D-functions.

Lemma 2.1. *Let $f \in D^{1,2}(\Omega)$ and assume that*

$$\int\limits_D |\nabla f|^2 < \varepsilon^2$$

for some $\varepsilon > 0$ and for some ring $D = \{z \in \mathbb{R}^2 : r_1 < |z - z_0| < r_2\} \subset \Omega$. Then the estimate

$$|\bar{f}(r_2) - \bar{f}(r_1)| \leq \varepsilon \sqrt{\ln \frac{r_2}{r_1}} \tag{2.1}$$

holds, where $\bar{f}$ is the mean value of f over the circle $S(z_0, r)$:

$$\bar{f}(r) := \frac{1}{2\pi r} \int\limits_{|z-z_0|=r} f(z)\, ds.$$

Lemma 2.2. *Fix a number $\beta \in (0,1)$. Let $f \in D^{1,2}(\Omega)$ and assume that*

$$\int\limits_D |\nabla f|^2 < \varepsilon^2$$

for some $\varepsilon > 0$ and for some ring $D = \{z \in \mathbb{R}^2 : \beta R < |z - z_0| < R\} \subset \Omega$. Then there exists a number $r \in [\beta R, R]$ such that the estimate

$$\sup_{|z-z_0|=r} |f(z) - \bar{f}(r)| \le c_\beta \varepsilon \tag{2.2}$$

holds, where the constant c_β depends on β only.

The proofs of above lemmas are standard, see, *e.g.*, [7] for the proofs of similar results. Summarizing the results of these lemmas, we receive

Lemma 2.3. *Under conditions of Lemma 2.2, there exists $r \in [\beta R, R]$ such that*

$$\sup_{|z-z_0|=r} |f(z) - \bar{f}(R)| \le \tilde{c}_\beta \varepsilon. \tag{2.3}$$

The following result was proved in [7, Theorem 4, page 399]. It means, roughly speaking, that the *direction* of the velocity vector $\mathbf{u}$ satisfying the Navier–Stokes system is controlled by the Dirichlet integral.

Lemma 2.4 ([7])**.** *Let $\mathbf{u}$ be a D-solution to the Navier–Stokes system (1.3) in the exterior domain $\Omega \subset \mathbb{R}^2$, and let $D = \{z \in \mathbb{R}^2 : R_1 < |z| < R_2\} \subset \Omega$ be some ring in Ω. Denoted by $\bar{\mathbf{u}}(r)$ the mean value of $\mathbf{u}$ over the circle S_r:*

$$\bar{\mathbf{u}}(r) = \frac{1}{2\pi r} \int\limits_{|\xi|=r} \mathbf{u}(\xi)\, ds \tag{2.4}$$

and let $\varphi(r)$ be the argument of the complex number associated to the vector $\bar{\mathbf{u}}(r) = (\bar{u}_1(r), \bar{u}_2(r))$, i.e., $\varphi(r) = \arg(\bar{u}_1(r) + i\bar{u}_2(r))$. Assume also that

$$|\bar{\mathbf{u}}(r)| \ge \sigma$$

for some positive constant $\sigma > 0$ and for all $r \in [R_1, R_2]$. Then the following estimate

$$\sup_{R_1 < \rho_1 \le \rho_2 \le R_2} |\varphi(\rho_2) - \varphi(\rho_1)| \le \frac{1}{4\pi\sigma^2} \int\limits_D \left(\frac{1}{r} |\nabla \omega| + |\nabla \mathbf{u}|^2 \right) \tag{2.5}$$

holds.

The following statement also follows from (2.5).

Corollary 2.1. *Under the conditions of Lemma 2.4, the estimate*

$$\sup_{R_1<\rho_1\le\rho_2\le R_2} |\varphi(\rho_2)-\varphi(\rho_1)| \le \frac{1}{2\sigma^2 R_1} + \frac{1}{4\pi\sigma^2}\int\limits_D \Big(r\,|\nabla\omega|^2 + |\nabla\mathbf{u}|^2\Big) \qquad (2.6)$$

holds.

3 Proof of the main Theorem 1.1.

We prove Theorem 1.1 by getting a contradiction. Suppose the assumptions of Theorem 1.1 are fulfilled, but the statement is false, i.e., there exists an increasing sequence of radii $R_k \to +\infty$ and solutions $\mathbf{u}_k$ to the system (1.2) such that $\mathbf{u}_{k_n} \rightharpoonup \mathbf{u} \equiv \mathbf{0}$. By the result of Amick [1, Theorem 24, page 115], it is equivalent to the global convergence to zero of the Dirichlet integrals:

$$\int\limits_{\Omega_k} |\nabla \mathbf{u}_k|^2 \to 0. \qquad (3.1)$$

By classical regularity results for D-solutions to the Navier–Stokes system (*e.g.*, [5]), the functions $\mathbf{u}_k$ and p_k are C^∞-smooth on the set $\overline{\Omega}_k$ and real analytical inside Ω_k. Moreover, (3.1) implies in particular, that for every compact set $E \subset \overline{\Omega}$

$$\sup_{x\in E} |\nabla^j \mathbf{u}_k(x)| \to 0 \qquad \forall j = 0,1,2,\ldots \qquad (3.2)$$

uniformly as $k\to\infty$, i.e., $\mathbf{u}_k$ and all its derivatives converges to zero as $k \to \infty$ uniformly on every compact set.

Without loss of generality we may assume that

$$\nu = 1 \qquad \text{and} \qquad \mathbf{u}_0 = (1,0). \qquad (3.3)$$

The proof consists of eight steps.

Step 1. Denote $\Omega_k' = \Omega\cap B(0,\frac34 R_k)$. By results of [6]–[7] (see the proofs of Lemmas 2.2 and 3.2 in [6]), the following estimate

$$\int\limits_{\Omega_k\cap B(0,\frac34 R_k)} r|\nabla\omega_k|^2 \le c\int\limits_{\Omega_k} |\omega_k|^2 \le c\int\limits_{\Omega_k} |\nabla \mathbf{u}_k|^2$$

holds with the constant c independent of k. Hence the assumption (3.1) yields

$$\int_{\Omega_k'} r\,|\nabla\omega_k|^2 \to 0. \tag{3.4}$$

Moreover, from [6]–[7] it follows that (see [6, Lemma 2.4])

$$\sup_{x\in\Omega_k'} |\mathbf{u}_k(x)| \le C. \tag{3.5}$$

and (see the proofs of Lemmas 2.3–2.6 in [6] and, in particular, [6, formula (2.20)])

$$\sup_{x\in\Omega_k'} |p_k(x)| = \varepsilon_k. \tag{3.6}$$

Here and everywhere below the equality $a_k = \varepsilon_k$ means that the sequence a_k tends to 0 as $k \to \infty$.

Step 2. Denote

$$R_{0k} = \min\left\{ r \ge R_0 : |\bar{\mathbf{u}}_k(r)| = \left| \fint_{S_r} \mathbf{u}_k\,ds \right| = \frac{1}{5} \right\}, \tag{3.7}$$

where, as usual, $S_r = \{\xi \in \mathbb{R}^2 : |\xi| = r\}$ is a circle. From (3.2) we have, in particular,

$$R_{0k} \to +\infty \qquad \text{as} \quad k \to \infty. \tag{3.8}$$

Moreover, Lemma 2.1 applied to $\mathbf{u}_k$ and the identity $\bar{\mathbf{u}}_k(R_k) = (1,0)$ imply, by virtue of (3.1), that

$$\frac{R_k}{R_{0k}} \to +\infty \qquad \text{as} \quad k \to \infty. \tag{3.9}$$

Step 3. By construction (see (3.6), (3.7)),

$$\max_{x\in S_{R_{0k}}} \Phi_k(x) > \frac{1}{2}\left(\frac{1}{5}\right)^2 + \varepsilon_k = \frac{1}{50} + \varepsilon_k > \frac{1}{60} > \max_{x\in\partial\Omega} \Phi_k(x), \tag{3.10}$$

for sufficiently large k, where, recall, $\Phi_k = p_k + \frac{1}{2}|\mathbf{u}_k|^2$ is the Bernoulli pressure. It is well known, that Φ_k satisfies the identity

$$\Delta\Phi_k = \omega_k^2 + \nabla\Phi_k \cdot \mathbf{u}_k, \tag{3.11}$$

and thus, it satisfies the classical strong maximum principle (*e.g.*, [6]–[7] or [2]). From this property and (3.10) we obtain

$$\max_{x\in S_R} \Phi_k(x) = \max_{x\in\Omega_R} \Phi_k(x) \qquad \forall R \in [R_{0k}, R_k], \tag{3.12}$$

where, as usual, $S_R = \{x \in \mathbb{R}^2 : |x| = R\}$ is a circle and $\Omega_R = \Omega \cap B_R = \{x \in \Omega : |x| < R\}$. From this maximum principle it follows that

$$\textit{the function} \quad [R_{0k}, R_k] \ni R \mapsto \max_{x \in S_R} \Phi_k(x) \quad \textit{is strictly increasing.} \tag{3.13}$$

In particular, by (3.10),

$$\max_{x \in S_R} \Phi_k(x) > \frac{1}{60} \qquad \forall R \in [R_{0k}, R_k]. \tag{3.14}$$

STEP 4. By results of [6]–[7](see the proof of Lemma 4.1 in [6], in particular, the proof of the estimate (4.5)) there exist a sequence of radii R_{mk}, $m = 1, 2, \dots, M = M(k)$, such that

$$2^{m-1} R_{0k} < R_{mk} < 2^m R_{0k}, \qquad m = 1, 2, \dots, M; \tag{3.15}$$

$$\frac{1}{3} R_k < R_{Mk} < \frac{2}{3} R_k; \tag{3.16}$$

$$\sup_{x \in S_{R_{mk}}} |\mathbf{u}_k(x) - \bar{\mathbf{u}}_k(R_{mk})| = \varepsilon_k. \tag{3.17}$$

Taking k large enough and using estimates (3.14) for the Bernoulli pressure Φ_k and estimates (3.6) for the pressure p_k, we conclude from (3.17) that

$$\big|\bar{\mathbf{u}}_k(R_{mk})\big| = \left| \fint_{S_{R_{mk}}} \mathbf{u}_k \, ds \right| > \sqrt{2 \cdot \frac{1}{60}} + \varepsilon_k > \frac{1}{6} \qquad \forall m = 1, 2, \dots, M. \tag{3.18}$$

Then, applying Lemma 2.1 and using the smallness of the Dirichlet integrals (3.1) and the condition (3.18), (3.15), we obtain

$$\big|\bar{\mathbf{u}}_k(R)\big| = \left| \fint_{S_R} \mathbf{u}_k \, ds \right| > \frac{1}{7} \qquad \forall R \in [R_{0k}, R_k]. \tag{3.19}$$

Denote by $\varphi_k(r)$ the angle direction of the vector $\bar{\mathbf{u}}_k(r)$, i.e.,

$$\bar{\mathbf{u}}_k(r) = |\bar{\mathbf{u}}_k(r)| \big(\cos \varphi_k(r), \sin \varphi_k(r)\big). \tag{3.20}$$

From Corollary 2.1 and relations (3.1), (3.4), (3.16), (3.19) we have

$$\sup_{R_{0k} < \rho_1 \le \rho_2 \le R_{Mk}} |\varphi_k(\rho_2) - \varphi_k(\rho_1)| = \varepsilon_k. \tag{3.21}$$

Further, Lemma 2.1 and the boundary conditions $\mathbf{u}_k(z)\big|_{|z| = R_k} = (1, 0)$ yield

$$|\bar{\mathbf{u}}_k(r) - (1, 0)| = \varepsilon_k \qquad \forall r \in [R_{Mk}, R_k].$$

From the last two formulas we conclude that

$$\sup_{R_{0k}\le r\le R_k} |\varphi_k(r)| = \varepsilon_k. \tag{3.22}$$

Summarizing we can say that the formula (3.20) holds with

$$|\bar{\mathbf{u}}_k(r)| \ge \frac{1}{7}, \qquad |\varphi_k(r)| = \varepsilon_k \qquad \forall r \in [R_{0k}, R_k]. \tag{3.23}$$

STEP 5. From the choice of the radii R_{1k} and R_{Mk} at Step 4 (see (3.15)–(3.16)), from estimates (3.6), (3.17), and from the conditions $|\bar{\mathbf{u}}_k(R_{0k})| = \frac{1}{5}$ (see (3.7)) and $|\bar{\mathbf{u}}_k(R_k)| = 1$, we have

$$\sup_{x\in S_{R_{1k}}} |\Phi_k(x) - \frac{1}{50}| \to 0, \tag{3.24}$$

$$\sup_{x\in S_{R_{Mk}}} \big|\Phi_k(x) - \frac{1}{2}| \to 0 \tag{3.25}$$

as $k \to \infty$. Without loss of generality we may assume that

$$\Phi_k(x) < \frac{1}{45} \qquad \forall x \in S_{R_{1k}}, \tag{3.26}$$

$$\Phi_k(x) > \frac{1}{3} \qquad \forall x \in S_{R_{Mk}}. \tag{3.27}$$

Denote by I the interval $I = \big[\frac{1}{45}, \frac{1}{3}\big]$. By construction and by the classical Morse–Sard Theorem (which says that the set of critical values of a C^∞ function has zero Lebesgue measure) we conclude that for almost all $t \in I$ the set

$$\{x \in \mathbb{R}^2 : R_{1k} \le |x| \le R_{Mk}, \quad \Phi_k(x) = t\}$$

is a finite disjoint union of smooth closed curves. Moreover, every of these curves is homeomorphic to the circle. (It follows from the fact, that the preimage of a non-critical value is a smooth one dimensional manifold, and, since $\Phi_k(x) \notin I$ for $x \in S_{R_{1k}}$ and $x \in S_{R_{Mk}}$, this manifold has no boundary.) By evident topological reasons, at least one of these curves separate the circles $S_{R_{1k}}$ and $S_{R_{Mk}}$. By maximum principle for the Bernoulli pressure Φ_k this separating curve is unique; denote it by $S_k(t)$. In other words, we have proved that

for almost all $t \in I = \big[\frac{1}{45}, \frac{1}{3}\big]$ there exists exactly one smooth curve $S_k(t)$, homeomorphic to the circle separating $S_{R_{1k}}$ from $S_{R_{Mk}}$, and satisfying the identity $\Phi_k(x) \equiv t \quad \forall x \in S_k(t)$. (3.28)

STEP 6. Take numbers $t_1 \in \left[\frac{1}{45}, \frac{1}{40}\right]$, $t_2 \in \left[\frac{1}{4}, \frac{1}{3}\right]$ which are regular values for all Φ_k, $k = 1, 2, \ldots$. Then denote by Ω_k^s the bounded open subset of Ω_k satisfying

$$\partial\Omega_k^s = S_k(t_1) \cup S_k(t_2). \tag{3.29}$$

We claim that

$$\textit{vorticity } \omega_k(x) \textit{ does not change sign in } \Omega_k^s. \tag{3.30}$$

In order to prove this claim, consider the auxiliary function

$$\gamma_k = \Phi_k - \omega_k \psi_k,$$

where ψ_k is a stream function satisfying $\nabla\psi_k = \mathbf{u}_k^\perp = (-u_k^2, u_k^1)$ (the function γ_k was introduced by Amick in the paper [1]). By direct calculation,

$$\nabla\gamma_k = -\nabla^\perp\omega_k - \psi_k \nabla\omega_k.$$

Then

$$\nabla\gamma_k \cdot \nabla^\perp\omega_k = -|\nabla\omega_k|^2. \tag{3.31}$$

In other words,

$$\frac{\partial\gamma_k}{\partial s} := \nabla\gamma_k \cdot \frac{\nabla^\perp\omega_k}{|\nabla\omega_k|} \equiv -|\nabla\omega_k|, \tag{3.32}$$

where we denote by $\frac{\partial\gamma_k}{\partial s}$ the derivative of γ_k with respect to the direction tangent to the level set $\omega_k = c$. The last identities imply the following monotonicity properties

$$\begin{array}{c}\gamma_k \textit{ is monotone along level sets of the vorticity } \omega_k = c \textit{ and} \\ \textit{vice versa – the vorticity } \omega_k \textit{ is monotone along level sets of } \gamma_k = c\end{array} \tag{3.33}$$

(see [1]). Moreover, there holds the evident identity

$$\gamma_k = \Phi_k \qquad \text{whenever} \quad \omega_k = 0. \tag{3.34}$$

Suppose (3.30) is not true. Let V_k be a connected component of the open set $\{x \in \Omega_k^s : \omega_k(x) > 0\}$. By our assumption, $V_k \neq \Omega_k^s$, therefore, $\Omega_k^s \cap \partial V_k \neq \emptyset$. Take a decreasing sequence of noncritical values $\tau_m > 0$ of the vorticity ω_k satisfying $\tau_m \to 0$ as $m \to \infty$. Since ω_k is a real analytical function in Ω_k, for every τ_m the set

$$\{z \in \partial\Omega_k^s : \omega_k(z) = \tau_m\}$$

is finite. Using this fact and regularity of the values τ_m, it is easy to see that the set

$$\{z \in \overline{\Omega}_k^s : \omega_k(z) = \tau_m\}$$

is a finite disjoint union of smooth curves homeomorphic to the unit interval $[0,1]$ with endpoints on $\partial\Omega_k^s$ (note, that τ_m-level set can not contain curves homeomorphic to the circle because of the monotonicity property (3.33)).

Fix $z_0 \in V_k$ and denote by $V_{k,m}$ the sequence of the connected components of the open set $\{x \in V_k : \omega_k(x) > \tau_m\}$ containing z_0. Evidently,

$$V_{k,m} \subset V_k \tag{3.35}$$

and

$$V_k = \bigcup_{m\in\mathbb{N}} V_{k,m} \tag{3.36}$$

Now we have to consider three possible cases:

(*i*) the equality

$$S_k(t_1) \cap \partial V_{k,m} = \emptyset \tag{3.37}$$

holds for all $m \in \mathbb{N}$;

(*ii*) the equality

$$S_k(t_2) \cap \partial V_{k,m} = \emptyset \tag{3.38}$$

holds for all $m \in \mathbb{N}$;

(*iii*) the relations

$$S_k(t_1) \cap \partial V_{k,m} \neq \emptyset \quad \text{and} \quad S_k(t_2) \cap \partial V_{k,m} \neq \emptyset \tag{3.39}$$

hold for all sufficiently large $m \geq m_0$.

Consider the case (i). First of all, we claim that for this case the relation

$$S_k(t_2) \cap \partial V_{k,m} \neq \emptyset \tag{3.40}$$

holds for all m. Indeed, if $S_k(t_2) \cap \partial V_{k,m} = \emptyset$, then from (3.37) we have $\partial V_{k,m} \cap \big(\partial\Omega_k^s\big) = \emptyset$, but this contradicts the strong maximum principle for the vorticity ω_k (see, *e.g.*, [7]).

By (3.37), (3.40) there exists evidently a smooth arc $L_m \subset V_k \cap \partial V_{k,m}$ such that

$$\omega_k \equiv \tau_m \qquad \textit{on } L_m, \tag{3.41}$$

$$L_m \ \textit{separates the point } z_0 \textit{ from the cycle } S_k(t_1), \tag{3.42}$$

$$\textit{the endpoints } A_m,\, B_m \textit{ of the arc } L_m \textit{ belong to the cycle } S_k(t_2). \tag{3.43}$$

(By *cycle* we mean the plane curve which is homeomorphic to the unit circle.) In particular, from the assertion (3.42) it follows, that

$$\textit{the arc } L_m \textit{ does not degenerate to a point when } m \to \infty. \tag{3.44}$$

In other words,

$$\operatorname{diam} L_m \nrightarrow 0 \quad \text{as } m \to \infty. \tag{3.45}$$

By construction we have

$$\gamma(B_m) - \gamma(A_m) = \Phi_k(B_m) - \Phi_k(A_m) - \tau_m(\psi(B_m) - \psi(A_m))$$
$$= -\tau_m(\psi(B_m) - \psi(A_m)) \to 0$$

as $m \to \infty$. From this fact and from the properties (3.45), (3.32)–(3.33) (which could be applied because of (3.41)) it follows that

$$\sup_{z \in L_m} |\nabla \omega_k(z)| \to 0 \qquad \text{as } m \to \infty.$$

But the last assertion, in view of (3.44)–(3.45), contradicts the fact that ω_k is a nonconstant real analytical function (in particular, $\nabla \omega_k$ can not be identically zero on a compact connected set which is not a single point).

The case (ii) can be proved exactly by the same arguments.

Consider the last possible case (iii). In this case evidently there exist two smooth arcs L_m^+ and L_m^- with the following properties

($\circ$) the arcs L_m^+ and L_m^- are homeomorphic to the closed unit interval $[0,1]$, their endpoints A_m^+, B_m^+ and A_m^-, B_m^- satisfy

$$A_m^+, A_m^- \in S_k(t_1); \qquad B_m^+, B_m^- \in S_k(t_2). \tag{3.46}$$

($\circ\circ$) the identity

$$\omega_m(z) \equiv \tau_m \tag{3.47}$$

holds for all $z \in L_m^+$ and L_m^-;

($\circ\circ\circ$) the function γ is increasing along L_m^+ in the direction from A_m^+ to B_m^+; and γ is decreasing along L_m^- in the direction from A_m^- to B_m^-.

From the last property it follows that

$$0 \le \gamma(A_m^-) - \gamma(B_m^-) = \Phi_k(A_m^-) - \Phi_k(B_m^-) - \tau_m(\psi(A_m^-) - \psi(B_m^-))$$
$$= (t_1 - t_2) - \tau_m(\psi(B_m^-) - \psi(A_m^-)).$$

Since by construction $t_2 > t_1$ and $\tau_m \to 0$ as $m \to \infty$, we conclude that the right hand side of the last formula is strictly negative for sufficiently large m, and that is a required contradiction.

Thus, the property (3.30) is proved. Without loss of generality we may assume that

$$\omega_k(z) > 0 \qquad \forall z \in \Omega_k^s. \tag{3.48}$$

Step 7. For an angle $\theta \in (0, 2\pi)$ denote by L_θ the ray starting from the origin:

$$L_\theta = \{s(\cos\theta, \sin\theta) : \; s \in \mathbb{R}_+\}.$$

Because of assumptions on smallness of the integrals (3.1), (3.4) there exists a value $\tilde\theta$ such that

$$|\tilde\theta - \frac{3}{2}\pi| < \frac{1}{9}, \tag{3.49}$$

and

$$\int\limits_{L_{\tilde\theta}\cap\Omega_k} r|\nabla \mathbf{u}_k|^2(r, \tilde\theta)\, dr \le \varepsilon_k, \tag{3.50}$$

$$\int\limits_{L_{\tilde\theta}\cap\Omega_k} r|\nabla \tilde{\mathbf{u}}_k|^2(r, \tilde\theta)\, dr \le \varepsilon_k, \tag{3.51}$$

$$\int\limits_{L_{\tilde\theta}\cap\Omega'_k} r^2|\nabla \omega_k|^2(r, \tilde\theta)\, dr \le \varepsilon_k, \tag{3.52}$$

where we denote $\tilde{\mathbf{u}}_k(r, \theta) := \mathbf{u}_k(r, \theta) - \bar{\mathbf{u}}_k(r)$ and as usual $\varepsilon_k \to 0$. By estimate (3.17),

$$\sup_{m=1,\dots,M} |\tilde{\mathbf{u}}_k(R_{mk}, \tilde\theta)| \le \varepsilon_k.$$

Therefore, the inequality (3.51) and the assumption (3.15) yield

$$\sup_{R_{m1} \le r \le R_{Mk}} |\tilde{\mathbf{u}}_k(r, \tilde\theta)| \le \varepsilon_k. \tag{3.53}$$

Then from the identities (3.20)–(3.23) we conclude, that

$$\mathbf{u}_k(r, \tilde\theta) = f_k(r)(\cos\varphi_k(r), \sin\varphi_k(r)), \tag{3.54}$$

where

$$f_k(r) = |\mathbf{u}_k(r,\tilde{\theta})| > 0, \qquad |\varphi_k(r)| = \varepsilon_k \qquad \forall r \in \big[R_{m1}, R_{Mk}\big]. \tag{3.55}$$

STEP 8. By construction, the considered domain Ω_k^s is contained in the ring

$$R_{1k} \le |z| \le R_{Mk} \qquad \forall z \in \Omega_k^s \tag{3.56}$$

(see (3.28)).

Consider the intersection of the annulus–shape domain Ω_k^s with the ray $L_{\tilde{\theta}}$. Take a segment $[A, B] \subset L_{\tilde{\theta}}$ with endpoints A, B satisfying

$$|A| = r_1 < r_2 = |B|, \tag{3.57}$$

$$A \in S_k(t_1), \qquad B \in S_k(t_2), \tag{3.58}$$

the interior of the segment $[A, B]$ *is contained in the set* $\Omega_k^s \cap L_{\tilde{\theta}}$ (3.59)

(the existence of such segment is geometrically evident since $\partial\Omega_k^s = S_k(t_1) \cup S_k(t_2)$ and the cycle $S_k(t_2)$ surrounds the cycle $S_k(t_1)$).

Then, by construction and (1.16), we have

$$\begin{aligned} 0 < t_2 - t_1 = \Phi_k(B) - \Phi_k(A) = \int\limits_{[A,B]} \nabla\Phi_k \cdot \mathbf{e}_{\tilde{\theta}}\, dr \\ = -\int\limits_{[A,B]} \nabla^{\perp}\omega_k \cdot \mathbf{e}_{\tilde{\theta}}\, dr + \int\limits_{[A,B]} \omega_k \mathbf{u}_k^{\perp} \cdot \mathbf{e}_{\tilde{\theta}}\, dr = I + II, \end{aligned} \tag{3.60}$$

where $\mathbf{e}_{\tilde{\theta}} = (\cos\tilde{\theta}, \sin\tilde{\theta})$. Estimate the terms I and II separately:

$$I \le \sqrt{\int\limits_{[A,B]} r^2 |\nabla\omega_k|^2\, dr}\ \sqrt{\int\limits_{[A,B]} \frac{1}{r^2}\, dr} \overset{(3.52)}{<} \varepsilon_k. \tag{3.61}$$

By formulas (3.54)–(3.55), we have

$$\mathbf{u}_k^{\perp}(r, \tilde{\theta}) = f_k(r)(\cos\tilde{\varphi}_k(r), \sin\tilde{\varphi}_k(r)), \tag{3.62}$$

where

$$\tilde{\varphi}_k(r) = \frac{\pi}{2} - \varphi_k(r). \tag{3.63}$$

Consequently,

$$\frac{\pi}{2} - \varepsilon_k < \tilde{\varphi}_k(r) < \frac{\pi}{2} + \varepsilon_k. \tag{3.64}$$

By construction (see (3.49)), we also obtain

$$\pi - \frac{1}{9} - \varepsilon_k < \tilde{\theta} - \tilde{\varphi}_k(r) < \pi + \frac{1}{9} + \varepsilon_k. \tag{3.65}$$

Finally, we conclude

$$\mathbf{u}_k^{\perp}(\tilde{\theta}, r) \cdot \mathbf{e}_{\tilde{\theta}} < 0 \qquad \forall r \in \big(R_{m1}, R_{Mk}\big) \supset \big[|A|, |B|\big]. \tag{3.66}$$

Therefore, from (3.48) it follows that

$$\omega_k \mathbf{u}_k^{\perp}(\tilde{\theta}, r) \cdot \mathbf{e}_{\tilde{\theta}} < 0 \qquad \forall r \in \big[|A|, |B|\big]. \tag{3.67}$$

Consequently, the second term II in (3.60) is negative:

$$II < 0,$$

and (3.60)–(3.61) imply the inequality

$$t_2 - t_1 < \varepsilon_k,$$

contradicting the choice of t_2, t_1 (recall, that $t_2 - t_1 \geq \frac{1}{4} - \frac{1}{40} = \frac{9}{40} > \frac{1}{5}$, see Step 5). This contradiction finishes the proof of Theorem 1.1. □

The proof of Theorem 1.2 is analogous. The only differences are the following. The pressure again is "almost zero", i.e., $|p_k| = \varepsilon_k$ in the subdomain $\Omega_k' = B_{\frac{3}{4}R_k}$, but $\mathbf{u}_k \equiv (1, 0)$ on $\partial\Omega$ and $\mathbf{u}_k$ is zero on the big circle S_{R_k}. Denote

$$R_{0k} = \max\bigg\{ r \leq R_k : \bigg| \fint_{S_r} \mathbf{u}_k \, ds \bigg| = \frac{1}{5} \bigg\}.$$

By the same reasons as above,

$$R_{0k} \to +\infty \qquad \text{and} \qquad \frac{R_k}{R_{0k}} \to +\infty \tag{3.68}$$

as $k \to \infty$. Then by the maximum principle for the Bernoulli pressure,

$$\textit{the function} \quad [R_0, R_{0k}] \ni R \mapsto \max_{x \in S_R} \Phi_k(x) \quad \textit{is strictly decreasing}, \tag{3.69}$$

where R_0 is some fixed radius with $B_{R_0} \supset \partial\Omega$ (R_0 does not depend on k). Then Steps 3–8 of the proof repeat exactly the corresponding steps in the proof of Theorem 1.1 with the following obvious change: now the Bernoulli pressure Φ_k is decreasing with respect to R, so that the circle–type level set $S_k(t_1) \subset \{\Phi_k = t_1\}$ surrounds the level set $S_k(t_2) \subset \{\Phi_k = t_2\}$ for $t_1 < t_2$.

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