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On the Asymptotic Form of Navier–Stokes Flow Past a Body in the Plane

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1. INTRODUCTION

In this paper we study the asymptotic structure of a solution (w, p) of the steady Navier–Stokes equations

$$-\nu\,\Delta w + (w \cdot \nabla) w = -\nabla p \qquad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot w = 0 \qquad \text{in } \Omega, \tag{1.2}$$

$$w = 0 \qquad \text{on } \partial\Omega, \tag{1.3}$$

$$w(x, y) \to w_\infty \qquad \text{at infinity in } \Omega. \tag{1.4}$$

Here Ω is an exterior domain, $\Omega = \mathbb{R}^2 - K$, where K is a compact set with smooth boundary. We restrict attention to a non-zero constant vector w_∞, and after a suitable rotation and scaling of the variables, *we restrict attention to* $\nu = 1$ *and* $w_\infty = (1, 0)$. We make the standard assumption that $w = (u, v)$ has finite Dirichlet norm:

$$\int_\Omega |\nabla w|^2 \equiv \int_\Omega \{u_x^2 + u_y^2 + v_x^2 + v_y^2\} < \infty. \tag{1.5}$$

We assume without loss of generality that the origin lies in the interior of the body K, and that K is contained in a ball of radius $R_0 > 2$.

In [1], Smith showed that a solution satisfying

$$|w(x, y) - (1, 0)| = O(r^{-1/4-\varepsilon}), \tag{1.6}$$

$r = \sqrt{x^2 + y^2}$ and some $\varepsilon > 0$, has a faster decay

$$|w(x, y) - (1, 0)| = O(r^{-1/2})$$

and that the asymptotic behavior of the solution is governed by the fundamental solution of the Oseen equations. (Solutions with the decay

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rate (1.6) are called *physically reasonable* by Finn [2] and are said to be of class PR.) These equations consist of (1.1)–(1.4) with $(w \cdot \nabla)w$ replaced by $(w_\infty \cdot \nabla)w$ and they have an explicit solution. The solution has the characteristic wake structure; that is, various quantities decay more rapidly outside of the region $|y| > \sqrt{x}$, $x \to \infty$, than within. Other authors [3, 4] have shown that assumption (1.6) yields the precise asymptotic form of the velocity field w, pressure p, and vorticity $\omega = u_y - v_x$ at infinity.

In this paper we show that any solution of (1.1)–(1.5) satisfies (1.6). For the sake of simplifying the complicated estimates, we restrict attention to symmetric flow so that Ω is symmetric about the x-axis, u is an even function of y, v is an odd function etc. Most of the arguments go over to the case of general flow. The analogous problem for three-dimensional flow past a compact body has been solved in a paper by Babenko [5].

Before proceeding to an outline of this paper, we remark upon the existence of solutions to (1.1)–(1.5) in two and three dimensions. For this discussion, we assume $\nu > 0$ and w_∞ are arbitrary and the flow need not be symmetric. In three dimensions, the work of Finn [6] and Ladyzhenskaya [7] show the existence of a solution, and [5] gives the exact asymptotics of the flow. In two dimensions, Finn and Smith [8] proved the existence of a solution when $|w_\infty|/\nu$ is sufficiently small with Ω fixed. For recent results on the general problem in two dimensions, the reader should consult [9, 10, 11]. These results depend on maximim principles and are unlike the potential theoretic techniques used in three dimensions.

If w_∞ and Ω are fixed and ν is sufficiently large, then [8] gives the existence of a unique solution of class PR to (1.1)–(1.5). A natural question is the behavior of this branch of solutions as ν decreases; if it could be shown to persist for all $\nu > 0$ (possibly only as a connected set), then the existence problem for (1.1)–(1.5) would be solved for arbitrary $\nu > 0$. An important step towards proving such a result would be some type of compactness for solutions. This paper shows that the relatively weak pointwise convergence of (1.5) implies the optimal rate of convergence, and it is hoped that this will contribute to the existence problem.

In Section 2 we state some preliminary estimates and show that

$$|w(x, y) - (1, 0)| = O(r^{-1/4+\varepsilon}) \qquad \text{as} \quad r \to \infty \tag{1.7}$$

for any $\varepsilon > 0$. In Section 3, we show that v, the vertical component of velocity, satisfies

$$|v(x, y)| = O(r^{-3/4+\varepsilon}), \qquad r^{1/2-\varepsilon}|\nabla v| \in L_2(\Omega), \tag{1.8}$$

and that the vorticity $\omega = u_y - v_x$ satisfies

$$\sqrt{r}\,\omega_y, \qquad r^{1-\varepsilon}\omega_x \in L_2(\Omega). \tag{1.9}$$

The vorticity satisfies the equation

$$\Delta\omega - \omega_x = (u-1)\,\omega_x + v\omega_y, \tag{1.10}$$

and if the operator on the left is inverted with the right-hand side estimated by (1.7)–(1.9), then a good estimate for ω results. This estimate is then used in the arguments for (1.7) and proves the desired result (1.6).

2. Preliminary Estimates

We begin with various results from [9, 10] for solutions (w, p) of (1.1)–(1.3) and (1.5). Since $|\nabla w| \in L_2(\Omega)$, it is clear that

$$\omega \in L_2(\Omega) \tag{2.1}$$

and this is used in [9, p. 118; 10, p. 400] to show that

$$\sqrt{r}\,|\nabla\omega| \in L_2(\Omega). \tag{2.2}$$

These estimates together with the maximum principle for (1.10) yield [10, p. 400]

$$\omega(x, y) = o(r^{-3/4}). \tag{2.3}$$

The pressure may be normalized to satisfy [9, p. 118; 10, p. 390]

$$p \to 0 \qquad \text{at infinity in } \Omega. \tag{2.4}$$

The total-head pressure Φ is defined by $\Phi = p + \frac{1}{2}|w|^2$ and satisfies

$$\Delta\Phi = \operatorname{div}(w\Phi) + \omega^2 \qquad \text{in } \Omega. \tag{2.5}$$

With the additional assumption (1.4), it is shown in [11, p. 107] that ω and $\Phi - \frac{1}{2}$ tend to zero exponentially outside of rays centered along the positive x-axis; more precisely, for each $\alpha > 0$ there exists $\beta = \beta(\alpha) > 0$ such that

$$\Phi(x, y) - \tfrac{1}{2},\ \omega(x, y) = O(\exp(-\beta\{|y| - \alpha x\})) \qquad \text{in } \Omega \tag{2.6}$$

with analogous estimates holding for derivatives of Φ and ω. The Stokes stream function ψ is defined by a line integral

$$\psi(x, y) = \int_{(x_0, 0)}^{(x, y)} u\,dy - v\,dx, \qquad (x, y) \in \Omega,$$

where $(x_0, 0) \in \partial\Omega$. The velocity field is given by $w = (\psi_y, -\psi_x)$. Note that

ψ is well defined in Ω due to (1.3), and that ψ is an odd-function of y satisfying

$$\left|\frac{\psi(x, y)}{y} - 1\right| \to 0 \qquad \text{at infinity in } \Omega \tag{2.7}$$

by (1.4). It is shown in [11, p. 105] that

$$\psi\omega \to 0 \qquad \text{at infinity in } \Omega. \tag{2.8}$$

Our pointwise estimates for $|w-(1, 0)|$ will follow from good estimates for $\psi\omega$, and in this section we prove that

$$\psi\omega = O(r^{-1/4+\varepsilon}) \qquad \text{as} \quad r \to \infty \tag{2.9}$$

for any ε. By (2.3) and (2.7),

$$|\psi\omega| \leqslant \text{const.}\, |y|\; o(r^{-3/4}) \tag{2.10a}$$

or in polar coordinates,

$$|\psi(r, \theta)\, \omega(r, \theta)| \leqslant \text{const.}\, |\theta|\; o(r^{1/4}). \tag{2.10b}$$

If $|\theta| \leqslant \text{const.}\, r^{-1/2+\varepsilon}$, then (2.10b) implies (2.9). This is very suggestive since $|\theta| \leqslant \text{const.}\, r^{-1/2}$ defines the wake. If $\alpha > 0$, and $|\theta| \in [\alpha, \pi]$, then the exponential decay of ω given by (2.6) shows that (2.9) holds; indeed,

$$|\psi(r, \theta)\, \omega(r, \theta)| = o(r^{-1}) \qquad \text{as} \quad r \to \infty \tag{2.11}$$

uniformly for $|\theta| \in [\alpha, \pi]$. Hence, in order to prove (2.9), we must estimate $\omega(r, \theta)$ at points with $\theta \in (r^{-1/2+\varepsilon}, \alpha]$, where α is given, and we recall that ω is odd in y or, equivalently, in θ.

With these preliminaries in hand, we can prove our first

LEMMA 1. *There exists a constant such that*

$$R \int_{-\pi}^{\pi} \psi^2(R, \theta)\, \omega^2(R, \theta)\, d\theta \leqslant \text{const.}, \qquad R \geqslant R_0.$$

Remark 1. We present a function, say f, at a point in Ω by $f(x, y)$ or $f(z)$, $z = x + iy$, or $f(r, \theta)$ in polar coordinates. We use $f(r, \theta)$ as notation for $f(r\cos\theta, r\sin\theta)$, and take $\theta \in (-\pi, \pi]$.

Proof of Lemma 1. Equation (1.10) yields

$$\frac{\psi^2}{2}\, \Delta(\omega^2) = \psi^2 |\nabla\omega|^2 + \frac{1}{2}\, \text{div}(w\psi^2\omega^2), \tag{2.12}$$

where we have used (1.2) and the fact that $w \cdot \nabla\psi = (\psi_y, -\psi_x) \cdot (\psi_x, \psi_y) = 0$. If this relation is integrated over $A_R = \{z \in \Omega: |z| < R\}$, $R > R_0$, then

$$\frac{R}{2}\int_{-\pi}^{\pi} \psi^2(R,\theta)\,\omega^2(R,\theta)(w\cdot n)\,d\theta + \int_{A_R} \psi^2 |\nabla\omega|^2$$

$$= R\int_{-\pi}^{\pi} \psi^2(R,\theta)\,\omega(R,\theta)\,\omega_r(R,\theta)\,d\theta$$

$$- R\int_{-\pi}^{\pi} \psi(R,\theta)\,\psi_r(R,\theta)\,\omega^2(R,\theta)\,d\theta + \int_{A_R} \omega^2(\psi\omega + |\nabla\psi|^2),$$

where n denotes the outward normal to A_R. The final integral on the right-hand side is bounded independently of R by (2.1) and (2.8). The use of (1.4) and (2.6) gives

$$\frac{R}{2}\int_{-\pi}^{\pi} \psi^2(R,\theta)\,\omega^2(R,\theta)(w\cdot n)\,d\theta \geqslant \frac{3R}{8}\int_{-\pi}^{\pi} \psi^2(R,\theta)\,\omega^2(R,\theta)\,d\theta + \text{const.},$$

for all R sufficiently large. Estimate (2.7) and others lead to $\psi_r/\psi \to 0$ at infinity in Ω, whence

$$\frac{R}{4}\,\Gamma(R) + \int_{A_R} \psi^2 |\nabla\omega|^2 \leqslant \text{const.} + \frac{R}{2}\,\Gamma'(R),$$

where

$$\Gamma(R) = \int_{-\pi}^{\pi} \psi^2(R,\theta)\,\omega^2(R,\theta)\,d\theta.$$

This implies that

$$\frac{2}{R}\,e^{-R/2}\int_{A_R} \psi^2 |\nabla\omega|^2 \leqslant \frac{\text{const.}}{R}\,e^{-R/2} + \frac{d}{dR}\left(e^{-R/2}\Gamma(R)\right),$$

and integrating from R to infinity yields the desired result. ∎

Remark 2. The final step also gives $\psi|\nabla\omega| \in L_2(\Omega)$, a result obtained for general solutions by a different method in [11, p. 91]. This is a better estimate than $\sqrt{r}\,|\nabla\omega| \in L_2(\Omega)$ (cf. (2.2)) precisely when $|\psi| \sim |y| \geqslant \sqrt{r}$; that is, outside of the wake.

LEMMA 2. *For every non-negative integer m,*

$$R\int_{-\pi}^{\pi} \psi(R,\theta)^{2m+2}\omega^2(R,\theta)\,d\theta \leqslant \text{const. } R^m, \qquad R \geqslant R_0,$$

where the constant depends only on m.

Proof. We use induction and note that the case $m=0$ holds by Lemma 1. Now

$$\frac{\psi^{2m+2}}{2}\Delta(\omega^2)=\psi^{2m+2}|\nabla\omega|^2+\frac{1}{2}\operatorname{div}(w\psi^{2m+2}\omega^2) \qquad \text{in } \Omega,$$

and integrating over A_R yields

$$\frac{R}{2}\int_{-\pi}^{\pi}\psi^{2m+2}(R,\theta)\,\omega^2(R,\theta)(w\cdot n)\,d\theta+\int_{A_R}\psi^{2m+2}|\nabla\omega|^2$$

$$=R\int_{-\pi}^{\pi}\psi^{2m+2}(R,\theta)\,\omega(R,\theta)\,\omega_r(R,\theta)\,d\theta$$

$$-(m+1)R\int_{-\pi}^{\pi}\psi^{2m+1}(R,\theta)\,\psi_r(R,\theta)\,\omega^2(R,\theta)\,d\theta$$

$$+(m+1)\int_{A_R}\omega^2\psi^{2m}\{\psi\omega+(2m+1)|\nabla\psi|^2\}.$$

The final integral is $O(R^m)$ by the inductive hypothesis, and the proof continues as for Lemma 1. ∎

In order to turn Lemma 2 into a pointwise estimate, we need an integral estimate for $\nabla\omega$.

LEMMA 3. *There is a constant such that*

$$R^{7/4}\int_{-\pi}^{\pi}\psi^2(R,\theta)|\nabla\omega(R,\theta)|^2\,d\theta\leqslant\text{const.}\,(\log R)^{9/8}, \qquad R\geqslant R_0.$$

Proof. Equation (1.10) gives

$$\Delta\omega_x=\operatorname{div}(w\omega_x)+w_x\cdot\nabla\omega$$

whence

$$\frac{r^{3/4}}{2}\psi^2\,\Delta\omega_x^2=r^{3/4}\psi^2|\nabla\omega_x|^2+\frac{r^{3/4}}{2}\operatorname{div}(w\psi^2\omega_x^2)+r^{3/4}\psi^2\omega_x w_x\cdot\nabla\omega. \tag{2.13}$$

Since $\psi\nabla\omega\in L_2(\Omega)$ by Remark 2 (see also [11, p. 91]) and $r^{3/4}|\nabla w|=O((\log r)^{9/8})$ by [10, p. 402] it follows that $(\log r)^{-9/8}r^{3/4}\psi^2\omega_x w_x\cdot\nabla\omega\in L_1(\Omega)$. One integrates (2.13) and the analogous equation derived by differentiating (1.10) with respect to y over A_R, adds the results, and argues as in the proof of Lemma 1. ∎

THEOREM 4. (a) *Let* $\varepsilon \in [0, \frac{1}{2})$ *and let* m *be a non-negative integer. If* $\theta \in (r^{-1/2+\varepsilon}, \pi]$, *then*

$$\omega^2(r, \theta) \leqslant \text{const.}\, r^{-m\varepsilon} r^{-11/8} (\log r)^{9/16}, \qquad r \geqslant R_0,$$

and the constant depends only on m.

(b) *For any* $\varepsilon > 0$,

$$|\psi\omega| = O(r^{-1/4+\varepsilon}) \qquad as \quad r \to \infty.$$

Proof. (a) For $\theta \in (0, \pi)$ there holds

$$\omega^2(r, \theta) \leqslant \omega^2(r, \theta = \pi) + \frac{1}{2} \int_\theta^\pi |\omega(r, s)\, \omega_\theta(r, s)|\, ds$$

whence

$$r^m \theta^m \omega^2(r, \theta) \leqslant \frac{1}{2} \int_\theta^\pi r^m s^m\, |\omega(r, s)\, \omega_\theta(r, s)|\, ds$$

$$\leqslant \text{const.} \int_{-\pi}^{\pi} |\psi(r, s)|^m\, |\omega(r, s)\, \omega_\theta(r, s)|\, ds$$

by (2.7), where the constant depends on m. The right-hand side may be estimated with the aid of Lemmas 2 and 3 and the Schwarz inequality,

$$r^m \theta^m \omega^2(r, \theta) \leqslant \text{const.} \{1 + r^{m/2} r^{-11/8} (\log r)^{9/16}\}.$$

Since $\theta \geqslant r^{-1/2+\varepsilon}$, there holds

$$r^{m/2} r^{\varepsilon m} \omega^2(r, \theta) \leqslant \text{const.} \{1 + r^{m/2} r^{-11/8} (\log r)^{9/16}\}$$

which completes the proof of (a).

(b) If $\varepsilon > 0$ and $\theta \in (0, r^{-1/2+\varepsilon}]$, then (2.10) gives (b). If $\theta \in (r^{-1/2+\varepsilon}, \pi]$, then (b) follows from (a) upon choosing m sufficiently large. ▮

Remark 3. If (a) is combined with standard elliptic estimates for (1.10), then analogous estimates to (a) hold for arbitrary derivatives of ω.

The estimate in Theorem 4(b) will be used in a representation formula for the velocity w in terms of the vorticity ω. For a suitably smooth complex-valued function B, let

$$B_{\bar{z}} = \tfrac{1}{2}(B_x + iB_y).$$

If $A_R = \{z \in \Omega: |z| < R\}$ for $R > R_0$, then [10, p. 388]

$$B(\tilde{z}) = \frac{1}{2\pi i} \oint_{\partial A_R} \frac{B(z)\,dz}{z - \tilde{z}} - \frac{1}{\pi} \int_{A_R} \frac{B_{\bar{z}}(z)\,dx\,dy}{z - \tilde{z}}, \qquad \tilde{z} \in A_R, \tag{2.14}$$

where $z = x + iy$ and $\tilde{z} = \tilde{x} + i\tilde{y}$. For the case $B = u - iv$, where $w = (u, v)$, we have

$$B_{\bar{z}} = \frac{1}{2}(u_x - iv_x - iu_y + v_y) = \frac{i}{2}(u_y - v_x) = \frac{i}{2}\,\omega.$$

Since $w = 0$ on $\partial\Omega$ by (1.3) while $w \to (1, 0)$ at infinity by (1.4), we may let $R \to \infty$ to deduce

$$u(\tilde{x}, \tilde{y}) - 1 - iv(\tilde{x}, \tilde{y}) = -\frac{i}{2\pi} \int_{\Omega} \frac{\omega(x, y)\,dx\,dy}{z - \tilde{z}}, \qquad \tilde{z} \in \Omega. \tag{2.15}$$

(The use of (2.1), (2.6), and Theorem 4 gives $\omega/r \in L_1(\Omega)$, and so there is no difficulty in letting $R \to \infty$ in the integral over A_R.) Taking the real part of the right-hand side gives

$$u(\tilde{x}, \tilde{y}) - 1 = -\frac{1}{2\pi} \int_{\Omega} \frac{(y - \tilde{y})\,\omega(x, y)\,dx\,dy}{|z - \tilde{z}|^2},$$

where $|z - \tilde{z}|^2 = (x - \tilde{x})^2 + (y - \tilde{y})^2$. If we now use the fact that the flow is to be symmetric so that ω is odd in y, then

$$u(\tilde{x}, \tilde{y}) - 1 = -\frac{1}{\pi} \int_{\Omega_+} \frac{y\omega(x, y)\{(x - \tilde{x})^2 + y^2 - \tilde{y}^2\}}{|z - \tilde{z}|^2\,|\bar{z} - \tilde{z}|^2}\,dx\,dy. \tag{2.16}$$

Here $\Omega_+ = \Omega \cap \{y \geqslant 0\}$ and $|\bar{z} - \tilde{z}|^2 = (x - \tilde{x})^2 + (y + \tilde{y})^2$. Since the flow is symmetric, we restrict attention to points $\tilde{z} \in \Omega_+$. A similar argument yields

$$v(\tilde{x}, \tilde{y}) = \frac{2}{\pi} \int_{\Omega_+} \frac{y\omega(x, y)\{\tilde{y}(x - \tilde{x})\}}{|z - \tilde{z}|^2\,|\bar{z} - \tilde{z}|^2}\,dx\,dy. \tag{2.17}$$

The formulae (2.16), (2.17) will be used to prove our results. Since $\psi(x, y) \sim y$ as $r \to \infty$ by (2.7), estimates for $\psi\omega$ yield ones for $y\omega$, which may then be used in (2.16), (2.17).

THEOREM 5. (a) *If* $\psi\omega = O(r^{-\mu})$ *as* $r \to \infty$ *for some* $\mu \in (0, \frac{3}{4}]$, *then*

$$|u(x, y) - 1| + |v(x, y)| = O(r^{-\mu} \log r), \qquad r \to \infty.$$

(b) *For any* $\varepsilon > 0$,

$$|u(x, y) - 1| + |v(x, y)| = O(r^{-1/4+\varepsilon}).$$

Proof. Since (b) follows immediately from (a) and Theorem 4, it suffices to prove (a). The estimate (2.7) gives $y\omega(x, y) = O(r^{-\mu})$. Let $\tilde{z} \in \Omega_+$ with $|\tilde{z}|$ large, and define

$$C_1 = \{z \in \Omega_+ : |z| < |\tilde{z}|/2\},$$

$$C_2 = \left\{z \in \Omega_+ : |z| \in \left(\frac{|\tilde{z}|}{2}, 2|\tilde{z}|\right), |z - \tilde{z}| \geqslant 1\right\},$$

$$C_3 = \{z \in \Omega_+ : |z| > 2|\tilde{z}|\}.$$

Note from (2.16) that

$$|u(\tilde{x}, \tilde{y}) - 1| \leqslant \text{const.} \sum_{j=1}^{3} \int_{C_j} |z|^{-\mu} |z - \tilde{z}|^{-2}\, dx\, dy$$
$$+ \text{const.} \int_{|z - \tilde{z}| < 1} |\omega(x, y)|\, |z - \tilde{z}|^{-1}\, dx\, dy.$$

If $z \in C_1$, then $|z - \tilde{z}| \geqslant |\tilde{z}|/2$ whence

$$\int_{C_1} |z|^{-\mu} |z - \tilde{z}|^{-2}\, dx\, dy \leqslant \text{const.} |\tilde{z}|^{-2} \int_0^{|\tilde{z}|/2} r^{-\mu} r\, dr = O(|\tilde{z}|^{-\mu}).$$

If $z \in C_3$, then $|z - \tilde{z}| \geqslant |z|/2$, so

$$\int_{C_3} |z|^{-\mu} |z - \tilde{z}|^{-2}\, dx\, dy \leqslant \text{const.} \int_{2|\tilde{z}|}^{\infty} r^{-\mu} r^{-2} r\, dr = O(|\tilde{z}|^{-\mu}).$$

Since $\omega = O(r^{-3/4})$ by (2.3), there holds

$$\int_{|z - \tilde{z}| \leqslant 1} |\omega(x, y)|\, |z - \tilde{z}|^{-1}\, dx\, dy = O(|\tilde{z}|^{-3/4}).$$

For the integral over C_2, we have

$$\int_{C_2} |z|^{-\mu} |z - \tilde{z}|^{-2}\, dx\, dy \leqslant \text{const.} |\tilde{z}|^{-\mu} \int_1^{2|\tilde{z}|} r^{-1}\, dr$$
$$= \text{const.} |\tilde{z}|^{-\mu} \log |\tilde{z}|.$$

A similar argument holds for v. ∎

3. Improved Estimates for v and Related Quantities

One expects the vertical component of velocity v to decay faster than $u-1$ [1], and to show this we shall need a representation different from (2.17). In the formula (2.14) let $B=\omega-i\Phi+i(u-iv)$, so that a calculation [11, p. 81] yields

$$B_{\bar{z}}=\frac{1}{2}(u-1)\omega+\frac{i}{2}v\omega.$$

This implies that

$$\omega(\tilde{x},\tilde{y})+v(\tilde{x},\tilde{y})+i\left(u(\tilde{x},\tilde{y})-1-\Phi(\tilde{x},\tilde{y})+\frac{1}{2}\right)$$
$$=\frac{1}{2\pi i}\oint_{\partial\Omega}\frac{\omega(z)-i\Phi(z)\,dz}{z-\tilde{z}}-\frac{1}{2\pi}\int_{\Omega}\frac{(u-1)\omega+iv\omega}{z-\tilde{z}}\,dx\,dy. \tag{3.1}$$

The line integral around $\partial\Omega$ is clearly $O(|\tilde{z}|^{-1})$. Taking the real part in (3.1) and using the symmetry yields

$$\omega(\tilde{x},\tilde{y})+v(\tilde{x},\tilde{y})=O(|\tilde{z}|^{-1})-\frac{2}{\pi}\int_{\Omega_+}\frac{y\omega(u-1)\,\tilde{y}(x-\tilde{x})\,dx\,dy}{|z-\tilde{z}|^2\,|\bar{z}-\tilde{z}|^2}$$
$$+\frac{1}{\pi}\int_{\Omega_+}\frac{\tilde{y}\omega v\{(x-\tilde{x})^2+\tilde{y}^2-y^2\}\,dx\,dy}{|z-\tilde{z}|^2\,|\bar{z}-\tilde{z}|^2}. \tag{3.2}$$

Theorem 6. *For any $\varepsilon>0$, $v(x,y)=O(r^{-1/2+\varepsilon})$ as $r\to\infty$.*

Proof. Since $\omega=o(r^{-3/4})$ it suffices to estimate the integrals on the right-hand side of (3.2). Since $y\omega(u-1)=O(r^{-1/2+\varepsilon})$ for any $\varepsilon>0$ by Theorems 4 and 5, the proof of Theorem 5 takes care of the first integral.

The remaining integral may be written as the sum of integrals over C_1, C_2, C_3, and $\{|z-\tilde{z}|<1\}\cap\Omega_+$, the latter yielding a term of order $r^{-3/4}$ as before. Let

$$\Omega_\varepsilon=\{(r,\theta)\in\Omega_+ : \theta\in(r^{-1/2+\varepsilon},\pi]\}.$$

Upon choosing m sufficiently large in Theorem 4, we can make $\omega v=O(r^{-3})$, say, in Ω_ε, whence

$$\int_{C_j\cap\Omega_\varepsilon}\frac{\tilde{y}\,|\omega v\{(x-\tilde{x})^2+\tilde{y}^2-y^2\}|}{|z-\tilde{z}|^2\,|\bar{z}-\tilde{z}|^2}\,dx\,dy=O(|\tilde{z}|^{-1})$$

for $j=1,2,3$. At points in $C_3\backslash\Omega_\varepsilon$, we estimate

$$|\omega v|=o(r^{-3/4}r^{-1/4+\varepsilon})=o(r^{-1+\varepsilon})$$

by (2.3) and Theorem 5. This implies that

$$\left|\int_{C_3\setminus\Omega_\varepsilon}\right| \leqslant \text{const. } \tilde{y}\int_{2|\tilde{z}|}^{\infty} r^{-1+\varepsilon} r^{-2} r \int_0^{r^{-1/2+\varepsilon}} d\theta\, dr = O(|\tilde{z}|^{-1/2+2\varepsilon})$$

with a similar estimate for $C_1\setminus\Omega_\varepsilon$. Finally, in the integral over $C_2\setminus\Omega_\varepsilon$, we write $\tilde{y}\omega v = y\omega v + (\tilde{y}-y)\,\omega v$, whence

$$\left|\int_{C_2\setminus\Omega_\varepsilon}\right| \leqslant \text{const.}\int_{C_2} |y\omega v|\,|z-\tilde{z}|^{-2}\,dx\,dy$$

$$+\text{const.}\int_{C_2\setminus\Omega_\varepsilon} |\omega v|\,(y+\tilde{y})\,|\tilde{z}-\tilde{z}|^{-2}\,dx\,dy. \tag{3.3}$$

Since $|y\omega v| = O(r^{-1/2+\varepsilon})$, the proof of Theorem 5 shows that the first term on the right of (3.3) is $O(|\tilde{z}|^{-1/2+2\varepsilon}\log|\tilde{z}|)$. The second term is dominated by

$$O(|\tilde{z}|^{-3/4}|\tilde{z}|^{-1/4+\varepsilon})\int_{C_2\setminus\Omega_\varepsilon}(y+\tilde{y})\,|\tilde{z}-\tilde{z}|^2\,dx\,dy$$

$$\leqslant \text{const.}\,|\tilde{z}|^{-1+\varepsilon}O(|\tilde{z}|^{1/2+\varepsilon}\log|\tilde{z}|). \tag{3.4}$$

Putting all of the estimates together yields

$$|v(\tilde{x},\tilde{y})| = O(|\tilde{z}|^{-1/2+2\varepsilon}\log|\tilde{z}|),$$

and since $\varepsilon>0$ may be taken as small as we like, the result follows. ∎

Recall from (1.5) and (2.2) that ∇u, ∇v, $\sqrt{r}\,\omega_x$, and $\sqrt{r}\,\omega_y\in L_2(\Omega)$. We now improve these estimates slightly to give $r^{1-\varepsilon}\omega_x$, $r^{1/2-\varepsilon}\nabla v\in L_2(\Omega)$ for any $\varepsilon>0$. These will then give $v=O(r^{-3/4+\varepsilon})$ for any $\varepsilon>0$ in Theorem 9.

Lemma 7. *If* $\varepsilon>0$, *then*

$$\int_\Omega r^{2-\varepsilon}\{\omega_x^2+\omega_{xx}^2+\omega_{xy}^2+\omega_{yy}^2\}<\infty.$$

Proof. Let $\tau\in C_0^\infty((-2,2)\to[0,1])$ be an even function with $\tau(x)=1$ for $|x|\leqslant 1$, and let $\mu(r;n)=\tau(r/n)$ for large positive integers n. From (1.10)

$$\int_\Omega r^{2-\varepsilon}\mu(\omega_{xx}^2+\omega_{yy}^2+\omega_x^2-2\omega_x\omega_{xx}-2\omega_x\omega_{yy}+2\omega_{xx}\omega_{yy})$$

$$\leqslant 2\int_\Omega r^{2-\varepsilon}\mu\{(u-1)^2\omega_x^2+v^2\omega_y^2\},$$

and integrating by parts yields

$$\int_\Omega r^{2-\varepsilon}\mu(\omega_{xx}^2+\omega_{yy}^2+\omega_x^2+2\omega_{xy}^2)$$

$$\leqslant \text{const.} + \text{const.} \int_\Omega |\nabla\omega|^2\, r^{1-\varepsilon} + 2\int_\Omega r^{2-\varepsilon}\mu\{(u-1)^2\,\omega_x^2 + v^2\omega_y^2\},$$

where the constant is independent of n. The first constant arises from contributions on $\partial\Omega$. Theorem 6 ensures that $r^{2-\varepsilon}v^2 \leqslant \text{const.}\, r$, and since $\sqrt{r}\,\nabla\omega \in L_2(\Omega)$, the result follows. ▌

LEMMA 8. *If* $\varepsilon > 0$, *then*

$$\int_\Omega r^{1-\varepsilon}|\nabla v|^2 < \infty.$$

Proof. Write $\Gamma = \omega + v$ and $\tilde{w} = w - (1, 0)$, so that (1.10) yields $\Delta\Gamma = \operatorname{div}(\tilde{w}\omega)$ and

$$\tfrac{1}{2}\Delta(\Gamma^2) = |\nabla\Gamma|^2 + \Gamma \operatorname{div}(\tilde{w}\omega) \qquad \text{in } \Omega.$$

This expression is multiplied by $r^{1-\varepsilon}\mu(r;n)$, μ as in the proof of Lemma 7, and the result integrated over Ω. Now

$$\left|\int_\Omega \mu r^{1-\varepsilon}\,\Delta(\Gamma^2)\right| \leqslant \text{const.} + \int_\Omega \Gamma^2 |\Delta(\mu r^{1-\varepsilon})|,$$

where the constant comes from boundary contributions on $\partial\Omega$. A typical term in the integral is

$$\int_\Omega \Gamma^2 r^{1-\varepsilon}|\mu''| \leqslant \text{const.} \int_0^\infty \frac{r}{n^2}\left|\tau''\left(\frac{r}{n}\right)\right| dr = \text{const.} \int_0^\infty s\,|\tau''(s)|\, ds < \infty,$$

where we have used Theorem 6 and (2.3). It follows that

$$\left|\int_\Omega \mu r^{1-\varepsilon}\,\Delta(\Gamma^2)\right| \leqslant \text{const.}, \tag{3.5}$$

independently of n.

The remaining integral is

$$\int_\Omega r^{1-\varepsilon}\mu\omega\{(u-1)\,\omega_x + v\omega_y\} + \int_\Omega r^{1-\varepsilon}\mu v\{(u-1)\,\omega_x + v\omega_y\}. \tag{3.6}$$

The first is bounded in magnitude by

$$\text{const.}\ |\omega|_{L_2(\Omega)} \left\{ \int_\Omega (r^{2-\varepsilon}\omega_x^2 + r\omega_y^2 r^{1-2\varepsilon} v^2) \right\}^{1/2} \leqslant \text{const.},$$

by Theorem 6, Lemma 7, and ω, $\sqrt{r}\,\omega_y \in L_2(\Omega)$. The use of Theorems 5 and 6 gives $r^{1-\varepsilon}|v(u-1)\,\omega_x| \leqslant r^{1/4}|\omega_x| \in L_1(\Omega)$ by Lemma 7. Hence

$$\frac{3}{4}\int_\Omega r^{1-\varepsilon}\mu\,|\nabla v|^2 \leqslant \text{const.} + \int_\Omega r^{1-\varepsilon}\mu v^2 \omega_y$$

$$\leqslant \text{const.} + 2\int_\Omega |\omega|\, r^{1-\varepsilon}\mu\,|v|\,|v_y| + \int_\Omega |\omega|\, v^2 |(r^{1-\varepsilon}\mu)_y|. \tag{3.7}$$

The first integral on the right of (3.7) is dominated by

$$\frac{1}{2}\int_\Omega r^{1-\varepsilon}\mu\,|\nabla v|^2 + \text{const.}\int_\Omega \omega^2 |v|^2 r^{1-\varepsilon} \leqslant \frac{1}{2}\int_\Omega r^{1-\varepsilon}\mu\,|\nabla v|^2 + \text{const.} \tag{3.8}$$

by Theorem 6 and $\omega \in L_2(\Omega)$. Because of Theorem 4, the final integral in (3.7) may be bounded by a constant plus an integral over $\Omega \backslash \Omega_\varepsilon$. The latter is dominated by

$$\int_1^\infty r \cdot r^{-3/4} r^{-1+2\varepsilon} r^{-\varepsilon} \int_0^{r^{-1/2+\varepsilon}} d\theta\, dr \leqslant \text{const.}$$

for sufficiently small ε. The use of this with (3.8) in (3.7) completes the proof. ▮

We shall use this lemma to show that $v = O(r^{-3/4+\varepsilon})$ for any $\varepsilon > 0$. Combining this with Theorem 5, Lemma 7, and $\sqrt{r}\,\omega_y \in L_2(\Omega)$ implies that

$$r^{5/4-\varepsilon}\{(u-1)\,\omega_x + v\omega_y\} \in L_2(\Omega), \tag{3.9}$$

and this will be used when the operator on the left of (1.10) is inverted.

To prove the improved estimate for v, we shall need a formula different from (3.1): one which exploits the result from Lemma 8 that $r^{1/2-\varepsilon}\nabla v \in L_2(\Omega)$.

Let

$$B = \omega - i\Phi + i(u - iv) + i(\tfrac{1}{2}(u-1)^2 + iv(u-1))$$

so that

$$B_{\bar z} = -v_x(u-1) + \frac{vv_y}{2} - i\left(\frac{vv_x}{2} + v_y(u-1)\right).$$

The use of (2.14) yields

$$\omega(\tilde{z}) + v(\tilde{z})(2 - u(\tilde{z})) + i\left\{u(\tilde{z}) - 1 - \Phi(\tilde{z}) + \frac{1}{2} + \frac{1}{2}(u(\tilde{z}) - 1)^2\right\}$$

$$= O(|\tilde{z}|^{-1}) - \frac{1}{\pi}\int_\Omega \frac{B_{\bar{z}}(z)\,dx\,dy}{z - \tilde{z}}, \qquad \tilde{z} \in \Omega. \tag{3.10}$$

Theorem 5 and Lemma 8 show that

$$r^{3/4-\varepsilon}|B_{\bar{z}}| \in L_2(\Omega). \tag{3.11}$$

For $|\tilde{z}|$ large, let

$$D_1 = \left\{z \in \Omega: |z| \leqslant \frac{|\tilde{z}|}{2}\right\},$$

$$D_2 = \left\{z \in \Omega: |z| \in \left(\frac{|\tilde{z}|}{2}, 2|\tilde{z}|\right), |z - \tilde{z}| \geqslant 1\right\},$$

$$D_3 = \{z \in \Omega: |z| \geqslant 2|\tilde{z}|\}.$$

The integral in (3.10) is the sum of integrals over these regions plus one over $\{|z - \tilde{z}| \leqslant 1\}$. A result of Gilbarg and Weinberger [10, p. 402] gives $|\nabla w| = O(r^{-3/4}(\log r)^{9/8})$, whence

$$|B_{\bar{z}}| \leqslant \text{const.}\ r^{-1+\varepsilon}$$

for any $\varepsilon > 0$. This yields

$$\left|\int_{|z-\tilde{z}| \leqslant 1} \frac{B_{\bar{z}}(z)\,dx\,dy}{z - \tilde{z}}\right| \leqslant \text{const.}\ |\tilde{z}|^{-1+\varepsilon}. \tag{3.12a}$$

For the remaining integrals we use (3.11)

$$\left|\int_{D_2} \frac{B_{\bar{z}}}{z - \tilde{z}}\right| \leqslant \left(\int_{D_2} |B_{\bar{z}}|^2\right)^{1/2} \left(\int_{D_2} |z - \tilde{z}|^{-2}\right)^{1/2}$$

$$\leqslant \text{const.}\,(|\tilde{z}|^{-3/2+2\varepsilon} \log|\tilde{z}|)^{1/2}, \tag{3.12b}$$

$$\left|\int_{D_3} \frac{B_{\bar{z}}}{z - \tilde{z}}\right| \leqslant \text{const.} \int_{D_3} |z|^{-1} |B_{\bar{z}}|$$

$$\leqslant \text{const.} \left(\int_{D_3} r^{3/2-2\varepsilon} |B_{\bar{z}}|^2\right)^{1/2} \left(\int_{D_3} r^{-3/2+2\varepsilon} r^{-2}\right)^{1/2}$$

$$\leqslant \text{const.}\ |\tilde{z}|^{-3/4+\varepsilon}$$

for ε sufficiently small. A similar argument holds for D_1. Since $\omega = o(r^{-3/4})$ by (2.3), there results

THEOREM 9. *For every* $\varepsilon > 0$,

(a) $v = O(r^{-3/4+\varepsilon})$,

(b) $r^{5/4-\varepsilon}\{(u-1)\,\omega_x + v\omega_y\} \in L_2(\Omega)$.

The operator $\Delta\omega - \omega_x$ has an explicit inverse [3], but our calculations will be simplified by using that for the heat operator $\omega_{yy} - \omega_x$. To justify this we must show that ω_{xx} satisfies an estimate similar to Theorem 9(b).

LEMMA 10. *For every* $\varepsilon > 0$, $r^{5/4-\varepsilon}\nabla\omega_x \in L_2(\Omega)$.

Proof. Now

$$\tfrac{1}{2}\Delta\omega_x^2 = |\nabla\omega_x|^2 + \tfrac{1}{2}\operatorname{div}(w\omega_x^2) + u_x\omega_x^2 + v_x\omega_x\omega_y.$$

This expression is multiplied by $r^{5/2-\varepsilon}\mu(r; n)$, μ as in Lemma 7, and the result integrated over Ω. The various terms are bounded as follows:

$$\left|\int_\Omega r^{5/2-\varepsilon}\mu\,\Delta\omega_x^2\right| \leqslant \text{const.} + \left|\int_\Omega \omega_x^2\,\Delta(r^{5/2-\varepsilon}\mu)\right|$$

$$\leqslant \text{const.} + \int_\Omega r^{1/2-\varepsilon}\omega_x^2 \leqslant \text{const.}$$

by (2.2);

$$\left|\int_\Omega r^{5/2-\varepsilon}\mu\,\operatorname{div}(w\omega_x^2)\right| \leqslant \text{const.} + \int_\Omega r^{3/2-\varepsilon}\omega_x^2 \leqslant \text{const.}$$

by Lemma 7. Since $|\nabla w| = O(r^{-3/4}(\log r)^{9/8})$ we have

$$\left|\int_\Omega \mu r^{5/2-\varepsilon}u_x\omega_x^2\right| \leqslant \text{const.}$$

by Lemma 7. The final integral may be integrated by parts to obtain

$$\left|\int_\Omega \mu r^{5/2-\varepsilon}v_x\omega_x\omega_y\right|$$

$$\leqslant \int_\Omega |v|\,\mu r^{5/2-\varepsilon}|\nabla\omega|\,|\nabla\omega_x| + \text{const.}\int_\Omega |v|\,|\nabla\omega|^2\,r^{3/2-\varepsilon}$$

$$\leqslant \frac{1}{2}\int_\Omega \mu r^{5/2-\varepsilon}|\nabla\omega_x|^2 + \text{const.}\int_\Omega |v|^2\,r^{5/2-\varepsilon}|\nabla\omega|^2$$

$$+ \text{const.}\int_\Omega |v|\,|\nabla\omega|^2\,r^{3/2-\varepsilon}$$

$$\leqslant \frac{1}{2}\int_\Omega \mu r^{5/2-\varepsilon}|\nabla\omega_x|^2 + \text{const.}$$

by (2.2) and Theorem 9. It follows that

$$\int_\Omega \mu r^{5/2-\varepsilon}|\nabla\omega_x|^2 \leqslant \frac{1}{2}\int_\Omega \mu r^{5/2-\varepsilon}|\nabla\omega_x|^2 + \text{const.}$$

and the lemma is proved. ∎

4. An Improved Estimate for the Vorticity

We rewrite (1.10) as

$$\omega_{yy} - \omega_x = f \equiv -\omega_{xx} + (u-1)\,\omega_x + v\omega_y \tag{4.1}$$

and note that

$$r^{5/4-\varepsilon} f \in L_2(\Omega) \tag{4.2}$$

for all $\varepsilon > 0$ by Theorem 9 and Lemma 10. Our assumption that $\partial\Omega$ is contained in a ball of radius R_0 ensures that all points (x, y) with $x \geqslant R_0$ belong to Ω. With (2.6) in mind, we restrict our attention to such points. The solution to (4.1) is

$$\omega(x, y) = \omega_1(x, y) + \omega_2(x, y), \qquad x \geqslant R_0,$$

where

$$\omega_1(x, y) = \int_0^\infty \{G(x-R_0, y-t) - G(x-R_0, y+t)\}\, \omega(R_0, t)\, dt$$

and

$$\omega_2(x, y) = \int_{R_0}^x \int_0^\infty \{G(x-s, y-t) - G(x-s, y+t)\}\, f(s, t)\, dt\, ds. \tag{4.3}$$

Here

$$G(x, y) = \frac{1}{\sqrt{4\pi x}} \exp\left(-\frac{y^2}{4x}\right),$$

and we have used the fact that ω and f are odd in y. Since $\omega(R_0, t)$ tends to zero exponentially (cf. (2.6)) as $t \to \infty$, we clearly have

$$|\omega_1(x, y)| \leqslant \frac{\text{const.}\,|y|}{x^{3/2}}, \qquad x > R_0 + 1. \tag{4.4}$$

Because of (2.6), we need to estimate ω_2 only as $x \to \infty$ with $|y| \leqslant x$, say.

Any easy estimate gives

$$0 \leqslant G(x-s,\ y-t) - G(x-s,\ y+t) \leqslant \text{const. } yt(x-s)^{-3/2} \tag{4.5}$$

for $y,\ t \geqslant 0$ and $s < x$. We break the integral with respect to s in (4.3) into integrals I, II, and III over $(R_0, x/2)$, $(x/2, x-1)$, and $(x-1, x)$, respectively, and begin with the first. With the aid of (4.5) we estimate

$$\begin{aligned} |I| &\leqslant \text{const. } y \int_{R_0}^{x/2} \frac{ds}{(x-s)^{3/2}} \int_0^\infty t\,|f(s,t)|\,dt \\ &\leqslant \text{const. } \frac{y}{x^{3/2}} \int_{R_0}^{x/2} \int_0^\infty t\,|f(s,t)|\,dt\,ds. \end{aligned} \tag{4.6}$$

Let $\varepsilon > 0$ and, as before, $\Omega_\varepsilon = \{(r, \theta) : \theta \in (r^{-1/2+\varepsilon}, \pi]\}$. Remark 3 ensures that $|f(s,t)|$ decays faster than any power of $(s^2+t^2)^{-1}$ in Ω_ε, whence

$$\begin{aligned} |\mathrm{I}| &\leqslant \text{const. } yx^{-3/2} \left\{ 1 + \int_{R_0}^{x/2} \int_0^{s^{1/2+\varepsilon}} t\,|f(s,t)|\,dt\,ds \right\} \\ &\leqslant \text{const. } yx^{-3/2} \left\{ 1 + \left(\int_\Omega r^{5/2-\varepsilon} |f|^2 \right)^{1/2} \right. \\ &\quad \left. \times \left(\int_{R_0}^x r r^{-5/2+\varepsilon} \int_0^{r^{-1/2+\varepsilon}} r^2\theta^2\,d\theta\,dr \right)^{1/2} \right\} \\ &\leqslant \text{const. } yx^{-3/2+2\varepsilon} \end{aligned} \tag{4.7}$$

by (4.2).

To estimate the integral over $(x/2, x-1)$, we use

$$\begin{aligned} 0 &\leqslant G(x-s,\ y-t) - G(x-s,\ y+t) \\ &\leqslant \frac{\text{const.}}{\sqrt{x-s}}, \qquad s < x \qquad \text{and} \qquad y,\ t \geqslant 0. \end{aligned} \tag{4.8}$$

Now

$$\begin{aligned} |\mathrm{II}| &\leqslant \text{const. } \left\{ \int_{x/2}^{x-1} \int_0^{s^{1/2+\varepsilon}} \frac{|f(s,t)|}{\sqrt{x-s}}\,dt\,ds \right. \\ &\quad \left. + \int_{x/2}^{x-1} \int_{s^{1/2+\varepsilon}}^\infty \frac{|f(s,t)|}{\sqrt{x-s}}\,dt\,ds \right\}. \end{aligned} \tag{4.9}$$

At points (s, t) with $t > s^{1/2+\varepsilon}$, we may assume $|f(s,t)| \leqslant \text{const.}\,|t|^{-4}$ by Remark 3 whence

$$
\begin{aligned}
|\mathrm{II}| &\leqslant \text{const.}\left\{x^{-1-3\varepsilon} + \text{const.}\left(\int_\Omega r^{5/2-\varepsilon}|f|^2\right)^{1/2}\right. \\
&\quad \left.\times\left(x^{-5/2+\varepsilon}\int_{x/2}^{x-1}(x-s)^{-1}\int_0^{s^{1/2+\varepsilon}} dt\,ds\right)^{1/2}\right\} \\
&\leqslant \text{const.}\, x^{-1+\varepsilon}(\log x)^{1/2}
\end{aligned}
\tag{4.10}
$$

for sufficiently small ε.

For the final integral, we use (4.8) again,

$$
\begin{aligned}
|\mathrm{III}| &\leqslant \text{const.}\left\{\int_{x-1}^{x}(x-s)^{-1/2}\int_0^{s^{1/2+\varepsilon}}|f(s,t)|\,dt\,ds\right. \\
&\quad \left.+\int_{x-1}^{x}(x-s)^{-1/2}\int_{s^{1/2+\varepsilon}}^{\infty}|f(s,t)|\,dt\,ds\right\}
\end{aligned}
$$

and the second integral is dominated by

$$
\text{const.}\int_{x-1}^{x}(x-s)^{-1/2}s^{-3/2-3\varepsilon}\,ds \leqslant \text{const.}\, x^{-3/2-3\varepsilon}. \tag{4.11}
$$

Let $p>2$ and let q be the conjugate exponent. Then

$$
\begin{aligned}
&\int_{x-1}^{x}(x-s)^{-1/2}\int_0^{s^{1/2+\varepsilon}}|f(s,t)|\,dt\,ds \\
&\quad\leqslant \text{const.}\left(\int_{x-1}^{x}(x-s)^{-q/2}s^{1/2+\varepsilon}\,ds\right)^{1/q}\left(x^{-5/2+\varepsilon}\int_\Omega r^{5/2-\varepsilon}|f|^2|f|^{p-2}\right)^{1/p} \\
&\quad\leqslant \text{const.}\, x^{-1+2\varepsilon}
\end{aligned}
$$

upon setting $p=6/(3-2\varepsilon)$.

Combining all of our estimates yields

THEOREM 11. *For every* $\varepsilon>0$,

$$
|\omega(x,y)| \leqslant \text{const.}\,\{x^{-1+\varepsilon}+x^{-3/2+\varepsilon}|y|\}
$$

as $x\to\infty$ *with* $|y|\leqslant x$. *The constant depends only on* ε.

With this theorem, we can improve the result of Theorem 4(b) that $\psi\omega = O(r^{-1/4+\varepsilon})$ as $r\to\infty$ for every $\varepsilon>0$. Theorem 4(a) ensures that ω (and therefore $\psi\omega$) decays like any inverse power of r in Ω_ε. Hence, it suffices to take $|y|\leqslant x^{1/2+\varepsilon}$ as $x\to\infty$. Since ε is arbitrary, the use of this in Theorem 11 yields

$$
|\omega(x,y)| = O(r^{-1+\varepsilon}) \qquad \text{as } (x,y)\to\infty \text{ in } \Omega \tag{4.12}
$$

for any $\varepsilon > 0$ which improves the estimate $\omega = o(r^{-3/4})$ of (2.3). (The strength of (4.12) is that the optimal estimate for ω [3] under assumption (1.6) is $\omega = O(r^{-1})$ and not $o(r^{-1})$.)

THEOREM 12. *For any* $\varepsilon > 0$,

(a) $\psi\omega = O(r^{-1/2+\varepsilon})$ *as* $r \to \infty$, *and*

(b) $|w(x, y) - (1, 0)| = O(r^{-1/2+\varepsilon})$ *as* $r \to \infty$.

Proof. (a) Theorem 4(a) ensures that we need only take $\theta \in [0, r^{-1/2+\varepsilon}]$ whence

$$|\psi(r, \theta)\, \omega(r, \theta)| \leqslant \text{const. } r\theta\, |\omega(r, \theta)| \leqslant \text{const. } r^{-1/2+2\varepsilon}.$$

(b) This is immediate from (a) and Theorem 5. ∎

The result in (b) shows that assumption (1.6) of Smith [1] is valid, and the full asymptotics given there hold.

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