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Asymptotic Behavior of the Solution to the Two-Dimensional Stationary Problem of Flow Past a Body Far From It

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ABSTRACT. In the exterior domain $\Omega \subset \mathbb{R}^2$ we consider the two-dimensional Navier–Stokes system

$$\Delta u - \nabla p = (u, \nabla)u, \qquad \operatorname{div} u = 0$$

whose solution possesses a finite Dirichlet integral and satisfies the condition $\lim_{|x|\to\infty} u(x) = (1, 0)$. For this solution, we establish the estimate $|u(x) - (1, 0)| \le c|x|^{-\alpha}$, where $\alpha > 1/4$. This estimate implies an asymptotic expression for the solution indicating the presence of a track behind the body.

KEY WORDS: Navier–Stokes system, flow past a body, Leray's problem, hydrodynamic potential, MacDonald function, Bernoulli function.

§1. Let Ω be an exterior domain not containing (to be definite) the origin in the two-dimensional plane of variables $x = (x_1, x_2)$ with a sufficiently smooth boundary $\partial\Omega$. The two-dimensional stationary flow problem reduces to the solution of the Navier–Stokes system

$$\nu\Delta w = \nabla p + (w, \nabla)w, \qquad \operatorname{div} w = 0 \tag{1}$$

in the domain Ω satisfying the boundary condition

$$w|_{\partial\Omega} = 0 \tag{2}$$

and the limiting relation

$$\lim_{|x|\to\infty} w(x) = w_\infty \qquad (w_\infty \ne 0). \tag{3}$$

It is well known [1] that there exists a solution (w_L, p_L) of problem (1), (2) that is the uniform limit on any compact domain of some sequence of solutions (w_R, p_R) of system (1) for the truncated domains $\Omega_R = \Omega \cap \{x : |x| < R\}$ with boundary conditions $w_R|_{\partial\Omega} = 0$ and $w_R|_{|x|=R} = w_\infty$. This solution satisfies the condition $|\nabla w_L| \in L_2(\Omega)$. By virtue of regularity theorems (see, for example, [2]), any generalized (in the sense of distributions) solution (w, p) of problem (1), (2) and, in particular, the Leray solution (w_L, p_L) is sufficiently smooth in the closure of the domain Ω. However, it was not proved that the Leray solution is a solution of the flow problem, since it is not known whether w_L takes a prescribed value w_∞ at infinity; moreover, in the general case it is not even known if the limit $\lim_{|x|\to\infty} w_L$ exists. Note that for sufficiently small values of $|w_\infty|/\nu$ the existence of a solution of the flow problem was established in [3].

General studies of the behavior at infinity of the solutions of system (1), (2) were carried out in [4–7]. The present paper is related directly to [7], where the following estimate was established for a symmetric (with respect to the x_1-axis) solution of the flow problem in the symmetric domain Ω satisfying the relation $|\nabla w| \in L_2(\Omega)$:

$$|w - w_\infty| = o(|x|^{-1/4-\varepsilon}), \tag{4}$$

where $\varepsilon > 0$ sufficiently is small.

This result will be established for a general (nonsymmetric) solution of the flow problem. The estimate (4) is important in that it implies an asymptotic expansion of the solution, which is determined

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by the fundamental solution of the Ozeen system [8]; moreover, under this assumption the asymptotic formulas for $\omega = \operatorname{rot} w$ and p were established (see [9]).

Note that, unlike [7], the derivation of relation (4) given below is based on the exact L_p-estimates of the solution w; these estimates are obtained from the integral representation of w via the hydrodynamic potentials.

In what follows, without loss of generality, we assume that $\nu = 1$ and (w, p) is a solution of the flow problem with $|\nabla w| \in L_2(\Omega)$ satisfying the limiting relation (3), where $w_\infty = (1, 0)$. Next, we use the following properties of such solutions, which were proved in [4–6]:

a) $|\nabla p| \in L_2(\Omega)$, the limit $\lim_{|x|\to\infty} p(x)$ exists (in what follows, we assume that it is always equal to 0);

b) the following relations are valid for $\omega = \operatorname{rot} w$:

$$|x|^{1/2}|\nabla\omega|, |\omega| \in L_2(\Omega), \qquad |\omega(x)| = o(|x|^{-3/4}), \quad |\nabla w(x)| = o(|x|^{-3/4}\ln|x|);$$

c) the function ω and the Bernoulli function $\Phi = p + |w|^2/2 - 1/2$ decay exponentially outside any sector containing the positive x_1-semiaxis: $\omega = o(e^{-\alpha|x|})$, $\Phi = o(e^{-\alpha|x|})$, where $\alpha > 0$ depends on the sector's angle.

§2. In what follows, an essential role is played by some estimates of the fundamental solution (u_j^k, q^k), $k, j = 1, 2$, of the linearized Ozeen system

$$\nu\Delta u - \partial_1 u - \nabla q = f, \qquad \operatorname{div} u = 0,$$

given by the formulas

$$q^k(x) = \frac{1}{2\pi}\partial_k \ln\frac{1}{|x|}, \qquad u_1^1(x) = \frac{1}{2}\left(-e^{x_1/2}K_0\left(\frac{|x|}{2}\right) + \partial_1\Omega(x)\right),$$

$$u_2^1(x) = u_1^2(x) = \frac{1}{2\pi}\partial_2\Omega(x), \qquad u_2^2(x) = -\frac{1}{2\pi}\partial_1\Omega(x),$$

where $\Omega(x) = \ln|x| + e^{x_1/2}K_0(|x|/2)$ and $K_0(r)$ is the MacDonald function.

In view of the asymptotics of the MacDonald function $K_0(r) \sim e^{-r}/\sqrt{2\pi r}$ as $r \to \infty$, the following inequalities are valid:

$$|u_j^k(x)| \le \begin{cases} c\left(\dfrac{1}{|x|} + \dfrac{e^{(x_1-|x|)/2}}{\sqrt{|x|}}\right), & (k,j) = (1,1), \\ c\dfrac{1}{|x|}, & (k,j) \ne (1,1), \end{cases} \qquad |x| > 1; \tag{5}$$

these, in particular, imply that

$$u_j^k(x) \in L_p(\mathbb{R}^2) \tag{6}$$

where $p \in (3, \infty)$ for $(k, j) = (1, 1)$ and $p \in (2, \infty)$ for $(k, j) \ne (1, 1)$.

An appropriate analysis of the derivatives of the fundamental solution shows that

$$\partial_s u_j^k \in L_p(\mathbb{R}^2) \tag{7}$$

where $p \in (3/2, 2)$ for $(k, j, s) = (1, 1, 2)$ and $p \in (1, 2)$ for $(k, j, s) \ne (1, 1, 2)$; moreover, outside a neighborhood of zero we have

$$\partial_s u_j^k \in L_p(|x| > 1), \tag{8}$$

where $p \in (3/2, \infty]$ for $(k, j, s) = (1, 1, 2)$ and $p \in (1, \infty]$ for $(k, j, s) \ne (1, 1, 2)$.

Further, the application of convolution theorems and operators of potential type (see, for example, [10]) yields the following estimates:

$$\|u_j^k * f\|_{L_q(\mathbb{R}^2)} \le c_{p,q}\|f\|_{L_p(\mathbb{R}^2)}, \tag{9}$$

where $1 < p \le q < \infty$, $1/q = 1/p + 1/r - 1$; moreover, $r \in [3, \infty)$ for $(k, j) = (1, 1)$ and $r \in [2, \infty)$ for $(k, j) \ne (1, 1)$;

$$\|\partial_s u_j^k * f\|_{L_q(\mathbb{R}^2)} \le c_{p,q}\|f\|_{L_p(\mathbb{R}^2)}, \tag{10}$$

where $1 < p \le q < \infty$, $1/q = 1/p + 1/r - 1$; moreover, $r \in [3/2, 2]$ for $(k, j, s) = (1, 1, 2)$ and $r \in [1, 2]$ for $(k, j, s) \ne (1, 1, 2)$.

§3. The proof of the main result is based on the following lemmas.

Lemma 1. *For the field $v = w - w_\infty$, the following integral representation holds:*

$$v_j(x) = \int_\Omega K_{ji}(x-y, v(y))v_i(y)\,dy + F_j(x), \tag{11}$$

where

$$K_{11} = (\partial_1 u_1^1)v_1 + (\partial_2 u_1^1 + \partial_1 u_2^1)v_2, \quad K_{12} = (\partial_2 u_2^1)v_2, \quad K_{21} = 0,$$
$$K_{22} = (\partial_1 u_2^2 - \partial_2 u_1^2)v_1 + (\partial_2 u_2^2)v_2 + 2u_1^2 \partial_2 v_1,$$
$$F_1(x) = \int_\Omega (n_1 u_1^1 - u_j^1 n_s \partial_s v_j + n_s \partial_s u_1^1 - n_1 u_1^1 + p n_j u_j^1 - n_1 q^1)\,dS_y,$$
$$F_2(x) = \int_\Omega (n_s \partial_s u_1^2 - u_j^2 n_s \partial_s v_j - n_1 u_1^2 + n_j u_j^2 p - q^2 n_1)\,dS_y;$$

moreover, in these relations summation is carried out over repeated subscripts; $x-y$ is the argument of the functions u_j^i, q^j; $v_j = v_j(y)$, $p = p(y)$; the $n_j = n_j(y)$ are the components of the outer normal with respect to Ω at $\partial\Omega$.

We prove in a standard way that the integral representation (11) holds. On substitution $w = v + w_\infty$ system (1) is multiplied by the matrix $u_i^j(x-y)$ and is integrated over the truncated domain Ω_R; the integrals over the domain Ω_R are then transformed into integrals along the boundary $\partial\Omega_R$ with the help of Green's formula; after the subsequent passage to the limit as $R \to \infty$ the curvilinear integrals along the circle $\{y : |y| = R\}$ vanish in view of the estimates (5) (for this it suffices to have the conditions $p(y) \to 0$, $v_j(y) \to 0$, $\partial_s v_j(y) \to 0$ as $|y| \to \infty$).

Note that in order to transform the integrals

$$\int_\Omega u_j^k(x-y)(v(y)\nabla)v_j(y)\,dy$$

the following relations are used:

$$(v, \nabla)v_2 = \partial_1(v_1 v_2) + \partial_2(v_2^2), \qquad (v, \nabla)v_1 = \begin{cases} \partial_1(v_1^2) + \partial_2(v_1 v_2), & k = 1, \\ -\partial_2(v_1 v_2) + 2v_2\partial_2 v_1, & k = 2. \end{cases}$$

The application of the estimates (9), (10) leads to the following result.

Lemma 2. *For the integral operators*

$$(A_{ji,\rho} f)(x) = \int_{|y|>\rho} K_{ji}(x-y, v(y)) f(y)\,dy, \qquad j, i = 1, 2, \quad (j, i) \neq (2, 1),$$

the following inequalities are valid: $\|A_{ji,\rho}\|_{L_p(|x|>\rho) \to L_p(|x|>\rho)} \le c_{ji,\rho}$, where $p \in (2, \infty)$ and

$$c_{11,\rho} = c_p\Big(\max_{|x|>\rho} |v(x)| + \|v_2\|_{L_3(|x|>\rho)}\Big), \quad c_{12,\rho} = c_p \max_{|x|>\rho} |v_2(x)|,$$
$$c_{22,\rho} = c_p\Big(\max_{|x|>\rho} |v(x)| + \|\partial_2 v_1\|_{L_2(|x|>\rho)}\Big),$$

where the c_p are constants depending only on p.

Lemma 3. *Let the field $v = w - w_\infty$ belong to some space $L_{p_0}(\Omega)$, $p_0 \in (3, \infty)$. Then the following inclusions are valid: $v_i \in L_p(\Omega)$, where $p \in (3, \infty]$ for $i = 1$ and $p \in (2, \infty]$ for $i = 2$.*

Proof. Let us consider in detail the proof of the inclusion for the component $v_2(x)$. The integral representation (11) for $j = 2$ implies the relation

$$v_2 = A_{22,\rho} v_2 + F_{2,\rho}, \qquad \text{where} \quad F_{2,\rho} = F_2 + \int_{\Omega \cap \{y\,:\,|y|>\rho\}} K_{22} v_2 \, dy. \tag{12}$$

It follows from the estimates of the fundamental solution (6)–(8) that $F_{2,\rho} \in L_p(|x| > \rho)$ for all $p \in (2, \infty)$. Let q be an arbitrary fixed number from the interval $(2, \infty)$. Choosing ρ large enough so that the following inequality is valid:

$$\|A_{22,\rho}\|_{L_p(|x|>\rho) \to L_p(|x|>\rho)} < 1, \tag{13}$$

where $p = q$ and $p = p_0$, and treating relation (12) as an integral equation with respect to $v_2(x)$ in the space $L_{p_0}(|x| > \rho)$, we find that for $|x| > \rho$

$$v_2(x) = \sum_{n=0}^{\infty} (A_{22\rho}^n F_{2,\rho})(x).$$

Therefore, in view of inequalities (13) for $p = q$, the function $v_2(x)$ belongs to the space $L_q(|x| > \rho)$. Since $q \in (2, \infty)$ is arbitrary, the assertion of the lemma holds for the component $v_2(x)$.

Further, as above, the corresponding result for the function $v_1(x)$ is obtained from the representation

$$v_1 = A_{11,\rho} v_1 + F_{1,\rho}, \qquad \text{where} \quad F_{1,\rho} = F_1 + \int_\Omega K_{12} v_2 \, dy + \int_{\Omega \cap \{y\,:\,|y|>\rho\}} K_{11} v_1 \, dy,$$

since, in view of the estimates of the fundamental solution, the inclusion $v_2 \in L_p(\Omega)$ for all $p \in (2, \infty)$ proved above, and Lemma 2, the function $F_{1,\rho}$ belongs to all the spaces $L_p(|x| > \rho)$ for $p \in (3, \infty)$. □

Thus, in view of Lemma 3, in order to obtain exact L_p-estimates of the field v, we must assume some "initial" estimates. Such estimates are obtained in the following section.

§4. Consider the Bernoulli function $\Phi = p + |w|^2/2 - 1/2$ satisfying the equation $\Delta\Phi = (w, \nabla)\Phi + \omega^2$ and the system of boundary value problems

$$\begin{gathered} \Delta\Phi_R = (w, \nabla)\Phi_R + \omega^2, \\ \Phi_R|_{\partial\Omega} = \Phi|_{\partial\Omega}, \quad \Phi_R|_{|x|=R} = 0. \end{gathered} \tag{14}$$

Since the difference $\Phi - \Phi_R$ satisfies the maximum principle and $\Phi(x) \to 0$ as $|x| \to \infty$, it follows that the sequence of functions Φ_R extended by zero to $\Omega \cap \{x : |x| > R\}$ is uniformly convergent to Φ on Ω and, in particular, $\|\Phi_R\|_{L_\infty(\Omega)}$ are uniformly bounded.

Set $\Phi_R = \zeta_R + b$, where b is a function different from zero only in some neighborhood of $\partial\Omega$ and $b|_{\partial\Omega} = \Phi|_{\partial\Omega}$. Multiplying Eq. (14) by $f\zeta_R$ and integrating over $\Omega_R = \Omega \cap \{x : |x| < R\}$, we obtain the identity

$$\int_{\Omega_R} \Big((\nabla\zeta_R)^2 f - \frac{1}{2}\zeta_R^2 \partial_1 f \Big)\, dx = -\int_{\Omega_R} \Big(\zeta_R(\nabla\zeta_R, \nabla f) - \frac{1}{2}\zeta_R^2 (v, \nabla) f - f\zeta_R(\Delta b - (w, \nabla) b - \omega^2) \Big)\, dx. \tag{15}$$

Choosing

$$f = \begin{cases} \dfrac{1}{\ln a}, & x_1 < a, \\[2mm] \dfrac{1}{\ln x_1}, & x_1 > a, \end{cases}$$

we find from (15) that the following estimate uniform in R is valid for sufficiently large $a > 0$:

$$\int_{\Omega_R} \Phi_R^2 |\partial_1 f| \, dx \le c.$$

(We have used the fact that $\omega \in L_2(\Omega)$, $v \to 0$ as $|x| \to \infty$, and the functions ζ_R are uniformly bounded.) Passing to the limit as $R \to \infty$ and taking into account the fact that Φ decays exponentially outside the semiaxis $x_1 > 0$, we prove that

$$\int_\Omega \Phi^2(x) \frac{dx}{|x| \ln^2 |x|} \le c. \tag{16}$$

Lemma 4. *For the field* v, *the following estimate is valid for any* $\varepsilon > 0$:

$$\int_\Omega v^2 \frac{dx}{|x|^{1+\varepsilon}} < \infty. \tag{17}$$

Proof. To prove (17), we first establish an integral formula relating the field v to the Bernoulli function Φ. Let us introduce the complex velocity $u = w_1 + iw_2$ and the differential operators

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right), \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right).$$

In these terms the Navier–Stokes system can be expressed as

$$4\frac{\partial^2 u}{\partial \overline{z}\,\partial z} - 2\frac{\partial}{\partial \overline{z}}\Phi = \frac{\partial}{\partial z}(u^2 - 1). \tag{18}$$

Multiplying (18) by the function $1/(\pi(\overline{z} - \overline{w}))$, integrating over the domain

$$\Omega_{\varepsilon,R} = \big(\Omega \setminus \{z : |z - w| < \varepsilon\}\big) \cap \{z : |z| < R\}$$

and passing to the limit as $\varepsilon \to 0$, $R \to \infty$, we obtain the formula

$$u^2(w) - 1 = (S\Phi)(w) + f(w), \tag{19}$$

where S is the singular operator

$$(S\Phi)(w) = \frac{2}{\pi}\int_\Omega \frac{\Phi(z)}{(\overline{z} - \overline{w})^2}\,dx_1\,dx_2,$$

$$f(w) = 4\frac{\partial u}{\partial \overline{z}}(w) - \frac{1}{\pi i}\int_{\partial\Omega} \frac{\Phi}{\overline{z} - \overline{w}}\,dz + \frac{1}{2\pi i}\int_{\partial\Omega} \frac{u^2 - 1 - 4\partial u/\partial \overline{z}}{\overline{z} - \overline{w}}\,d\overline{z}.$$

Since $\partial u/\partial \overline{z} \in L_2(\Omega)$, the function f belongs to any weight space $L_2(\Omega, |x|^{-\alpha})$ for all $\alpha > 0$. It follows from the estimate (16) that $\Phi \in L_2(\Omega, |x|^{-1-\varepsilon})$ for any arbitrarily small $\varepsilon > 0$. In view of the fact that the singular integral S is bounded in any weight space $L_2(\Omega, |x|^{-\alpha})$, $0 \le \alpha < 2$, we find from (19) that $u^2 - 1 \in L_2(\Omega, |x|^{-1-\varepsilon})$ for all $\varepsilon > 0$. Since $w_1 \to 1$, $w_2 \to 0$ as $|x| \to \infty$, from the relation $u^2 - 1 = w_1^2 - 1 - w_2^2 + 2iw_1w_2$ we obtain $w_2, w_1 - 1 \in L_2(\Omega, |x|^{-1-\varepsilon})$.

Thus Lemma 4 is proved. □

§5. In this section we conclude the proof of the estimate (4).

Theorem 1. *The estimate* (4) *of the solution of the flow problem is valid.*

Proof. For $\omega = \operatorname{rot} w$ and the field $v = w - w_\infty$, the following relation is valid:

$$\Delta v = \nabla^\perp \omega, \tag{20}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$.

Let $x \in \Omega$, and let $B_\rho = \{y : |x - y| < \rho\}$ be the ball centered at x and contained in the domain Ω. Multiplying relation (20) by the function $(2\pi)^{-1}\ln(|x - y|/\rho)$ and integrating over the ball B_ρ, after familiar transformations we obtain

$$v(x) = \frac{1}{2\pi}\int_{\partial B_\rho} \frac{v(y)}{\rho}\,ds + \frac{1}{2\pi}\int_{B_\rho} \frac{\omega(y)(x - y)^\perp}{(x - y)^2}\,dy,$$

where $(x - y)^\perp = (-(x_2 - y_2), x_1 - y_1)$.

Integrating over ρ from $R/2$ to R and estimating the integrals obtained, we find that

$$|v(x)| \le c\left(\int_{B_R} \frac{|v(y)|}{R^2}\, dy + \int_{B_R} \frac{|\omega(y)|}{|x-y|}\, dy\right). \tag{21}$$

In view of the fact that $v \in L_2(\Omega, |x|^{-1-\varepsilon})$, from (21) we obtain the estimate

$$|v(x)| \le cR^{-2}\left(\int_\Omega \frac{|v(y)|^2}{|y|^{1+\varepsilon}}\, dy\right)^{1/2}\left(\int_{B_R} |y|^{1+\varepsilon}\, dy\right)^{1/2} + c\max_{y\in B_R} |\omega(y)| R.$$

Assuming $|x|$ to be sufficiently large and $R < |x|/2$, we prove the inequality

$$v(x) \le c\big(|x|^{(1+\varepsilon)/2} R^{-1} + |x|^{-3/4} R\big).$$

Minimizing this inequality with respect to R, we find that $|v(x)| \le c|x|^{(-1+2\varepsilon)/8}$. Thus $v \in L_p(\Omega)$ for all $p > 16$. This yields the "initial" estimates for the application of Lemma 3. Therefore, $v \in L_p(\Omega)$ for all $p > 3$. In view of this fact, from (21) we obtain the inequality

$$|v(x)| \le c\left(R^{-2/p}\left(\int_{B_R} |v|^p\, dx\right)^{1/p} + R|x|^{-3/4}\right).$$

Setting $R = |x|^\alpha$ for $\alpha = 3/(4+8/p)$ and choosing $p > 3$ arbitrarily close to 3, we obtain $|v(x)| \le c|x|^{-3/10+\varepsilon}$. This concludes the proof. □

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