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A STEADY EULER FLOW WITH COMPACT SUPPORT

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Abstract. A nontrivial smooth steady incompressible Euler flow in three dimensions with compact support is constructed. Another uncommon property of this solution is the dependence between the Bernoulli function and the pressure.

1 Introduction

A steady flow of an ideal fluid in $\mathbb{R}^3$ is a solution of the Euler equation

$$(u \cdot \nabla)u = -\nabla p, \operatorname{div} u = 0.$$

At present, it is not known if smooth nonzero solutions of this equation with compact support $0 \neq u \in C_0^\infty(\mathbb{R}^3)$ exist [Nad14,NV17]. The problem is trivial in two dimensions where there are obvious vortex-like solutions. In three dimensions, only a few results are known, all on the negative side. It is known that such a flow cannot be Beltrami [Nad14,CC15], and it cannot be axisymmetric without swirl [JX09]. Recently, Nadirashvili and Vladut found some other restrictions [NV17].

Apparently,[1] weak solutions with compact support may be constructed using methods of [CS14]. Also, there is a considerable literature about vortex rings which are solutions with compactly supported vorticity (e.g. [AS89]). Opinions have been expressed that in three dimensions there are no smooth solutions with compact support besides $u = 0$. The main goal of this paper is to show that it is not true.

Theorem 1. *There exists a nontrivial smooth steady Euler flow in* $\mathbb{R}^3$ *with support in an arbitrarily small neighbourhood of a circle.*

We give below an explicit description of an axisymmetric flow with compact support. This solution has also other unusual properties discussed in the last section.

2 Some Differential Equations

In this section we find solutions of some differential equations which will be used later.

[1] The author is no expert in this area.

2.1 Ordinary differential equation.

LEMMA 1. *The singular Cauchy problem*

$$3x\psi'' + 6x(\psi')^3 - 4\psi(\psi')^2 - 3\psi' = 0;\ \psi(0) = 1, \psi'(0) = -\frac{3}{4} \tag{1}$$

has a unique analytic solution $\psi(x)$ in a neighbourhood of $x = 0$.

Proof. The equation (1) becomes first order in variables $t = x\psi^{-2}$ and $v = \psi\psi'$,

$$t\frac{dv}{dt} = v\left(\frac{4}{3}v + 1\right) + \frac{tv^2(2v+9)}{3(1-2tv)}.$$

Denoting $w = v + \frac{3}{4}$ we may write this equation as $t\frac{dw}{dt} = -w + f(t,w)$ where the function f is analytic and $f(0,0) = \frac{\partial}{\partial w}f(0,0) = 0$. By [Hil97, Theorem 11.1] there is an unique analytic solution $v(t)$ such that $v(0) = -\frac{3}{4}$. The Cauchy problem $\frac{d\psi}{dx} = \frac{1}{\psi}v\left(\frac{x}{\psi^2}\right)$, $\psi(0) = 1$ clearly has an unique solution. □

From now ψ always means the function defined by (1). The Taylor series of this function is[2]

$$\psi(x) = 1 - \frac{3}{4}x + \frac{9}{128}x^2 - \frac{21}{1024}x^3 + \frac{1035}{131072}x^4 - \frac{1809}{524288}x^5 + O(x^6).$$

Denote

$$F(x,\alpha) = -2x\psi(\alpha) + 2x^3,\ H(\alpha) = 6\alpha\left(\frac{1}{\psi'(\alpha)} + 2\psi(\alpha)\right),$$
$$G(x,\alpha) = 12x^2\alpha - F^2(x,\alpha) - H(\alpha).$$

Note that at $(x,\alpha) = (1,0)$ we have $F = G = 0$ and

$$\frac{\partial F}{\partial x} = 4,\ \frac{\partial F}{\partial \alpha} = \frac{3}{2},\ \frac{\partial G}{\partial x} = 0,\ \frac{\partial G}{\partial \alpha} = 8.$$

We will also need the following fact.

LEMMA 2. *The functions F, G satisfy*

$$\frac{\partial G}{\partial x} + F\frac{\partial G}{\partial \alpha} = 2G\frac{\partial F}{\partial \alpha}, \tag{2}$$
$$x\frac{\partial F}{\partial x} - F = 4x^3. \tag{3}$$

Proof. The part (3) is trivial; (2) boils down to the formula

$$H'(\alpha) = 24\alpha\psi'(\alpha) + 4\psi(\alpha),$$

equivalent to (1). □

[2] This series is for a reader's convenience. In the proof we use $\psi'(0) = -\frac{3}{4}$ but not higher derivatives.

2.2 Partial differential equations.

LEMMA 3. *The system*[3]

$$\frac{\partial}{\partial x}\alpha = F(x,\alpha),\ \left(\frac{\partial}{\partial y}\alpha\right)^2 = G(x,\alpha), \tag{4}$$

has a unique analytic solution $\alpha(x,y)$ in a neighbourhood of the point $(x,y)=(1,0)$ such that $\alpha(1,0)=0$ and $\frac{\partial}{\partial y}\alpha \not\equiv 0$.

Proof. It is convenient to introduce an *ad hoc* variable s and to consider x and α functions of (F,s), where $G=s^2$ by definition. (This is possible because $\frac{\partial(F,G)}{\partial(x,\alpha)}(1,0) = 32 \neq 0$.) Consider a differential form

$$\kappa = \frac{\partial x}{\partial s}dF + \left(\frac{\partial \alpha}{\partial s} - F\frac{\partial x}{\partial s}\right)\frac{ds}{s}.$$

It is analytic near the origin of the (F,s) plane (because $s^{-1}\frac{\partial}{\partial s} = 2\frac{\partial}{\partial G}$). We have the relation (2) which in the new variables takes the form

$$F\frac{\partial x}{\partial F} + s\frac{\partial x}{\partial s} = \frac{\partial \alpha}{\partial F}.$$

It follows that

$$d\kappa = \frac{1}{s}\frac{\partial}{\partial s}\left(F\frac{\partial x}{\partial F} + s\frac{\partial x}{\partial s} - \frac{\partial \alpha}{\partial F}\right)ds\wedge dF = 0.$$

By the Poincaré lemma, there is a unique analytic function $\Phi(F,s)$ such that $\Phi(0,0) = 0$ and $\kappa = d\Phi$.

This form is odd with respect to the second variable, in the sense that $\sigma^*\kappa = -\kappa$ where $\sigma : (F,s)\mapsto(F,-s)$. If γ is a path connecting the origin $(0,0)$ to a given point (F,s), then

$$\Phi(F,-s) = \int_{\sigma\gamma}\kappa = \int_\gamma \sigma^*\kappa = -\int_\gamma \kappa = -\Phi(F,s).$$

We have $\Phi^2(F,s) = \Phi^2(F,-s)$, hence the square Φ^2 is a well defined analytic function of F and $G=s^2$. Now we can change the variables back and denote $f(x,\alpha)=\Phi^2$. We have $f(1,0)=0$ (essentially, by assumptions).

Near the origin $\Phi(F,s) = \left(\frac{1}{4}+O(F)\right)s + O(s^3)$, hence

$$\frac{\partial \Phi^2}{\partial F}(0,0) = 0,\ \frac{\partial \Phi^2}{\partial G}(0,0) = \frac{1}{16},$$

[3] Note that $\frac{\partial}{\partial x}$ means $\frac{\partial}{\partial x}|_\alpha$ when applied to F or G but $\frac{\partial}{\partial x}|_y$ when applied to α. To avoid a (very common) inconsistency in notation we write this partial derivative as $\frac{\partial f}{\partial x}$ for the former and $\frac{\partial}{\partial x}f$ for the latter.

and

$$\frac{\partial}{\partial\alpha} f(1,0) = \left(\frac{3}{2}\frac{\partial}{\partial F} + 8\frac{\partial}{\partial G}\right)\Phi^2(0,0) = \frac{1}{2}.$$

By the implicit function theorem, in a neighbourhood of the origin there is a unique analytic function of two variables $\alpha(x,y)$ such that $\alpha(1,0)=0$ and

$$f(x,\alpha(x,y)) = y^2.$$

From (2) and the definition of κ we have

$$sd\Phi = d\alpha - Fdx.$$

Now, in the variables (x,y) we have $\Phi^2 = y^2$, hence $d\Phi^2 = dy^2$ and[4]

$$(d\alpha - F(x,\alpha)dx)^2 - G(x,\alpha)dy^2 = 0.$$

This equality implies (4) (and is essentially equivalent to it). □

REMARK 1. In Lemma 3, the condition $\alpha(1,0)=0$ is crucial. In this case we cannot take the square root of the second equation, and solving the system is more difficult then for $\alpha(1,0)>0$. Unfortunately, in the latter case the function α would have no extrema, and the Euler flow u in the following section could not be extended to the whole space.

REMARK 2. Note that $\alpha(x,y)=\alpha(x,-y)$ pretty much by definition. An interesting consequence is $G(x,\alpha(x,0))=0$. (Which follows from (4) and $\frac{\partial}{\partial y}\alpha(x,0)=0$.) Because of this, the function $\alpha(x,0)$ is actually another analytic solution of (4).

REMARK 3. In the given proof, the main technical difficulty is the absence of an inverse map to $(F,s)\mapsto(x,\alpha)$. We circumvent this obstacle by artificially constructing a function $\Phi^2(F,s)$ with a well defined "pullback". One of the referees pointed out that there is a more straightforward (although not unrelated) proof using the Cartan–Kähler Theorem. We may consider a form $\omega = d\alpha - pdx - qdy$ on a manifold of dimension 3 defined by equations

$$p = F(x,\alpha), q^2 = G(x,\alpha)$$

in variables x, y, α, p, q. This form satisfies the integrability condition $\omega\wedge d\omega = 0$, so we can use it to construct the function $\alpha(x,y)$. (An important detail is that ω/q is analytic.)

REMARK 4. The method of the proof is constructive and may be used to compute the Taylor series

$$\alpha(x,y) = 2(x-1)^2 + 2y^2 + 3(x-1)^3 + 3(x-1)y^2 + O((|x-1|+|y|)^4).$$

The first two terms are important, so it may be appropriate to include a direct computation of them.

[4] As customary, dy^2 actually means $dy\otimes dy$, a tensor square.

Lemma 4. *The function $\alpha(x,y)$ has a strict local minimum at $(x,y)=(1,0)$.*

Proof. The first derivatives at this point are zero by (4). We have

$$\frac{\partial^2}{\partial x^2}\alpha = \frac{\partial}{\partial x}F = \frac{\partial F}{\partial x} = 4,\ \frac{\partial^2}{\partial x \partial y}\alpha = \frac{\partial}{\partial y}F = \left(\frac{\partial F}{\partial \alpha}\right)\left(\frac{\partial}{\partial y}\alpha\right) = 0.$$

Finally, (at any point) we have the equality

$$\left(\frac{\partial G}{\partial \alpha}\right)\left(\frac{\partial}{\partial y}\alpha\right) = \frac{\partial}{\partial y}G = 2\left(\frac{\partial}{\partial y}\alpha\right)\left(\frac{\partial^2}{\partial y^2}\alpha\right).$$

As the derivative $\frac{\partial}{\partial y}\alpha$ is not identically zero, it implies

$$\frac{\partial^2}{\partial y^2}\alpha = \frac{1}{2}\frac{\partial G}{\partial \alpha}.$$

At the point under consideration we have then $\frac{\partial^2}{\partial y^2}\alpha = 4$. The second differential $d^2\alpha$ is positively definite, so this point is a strict minimum. □

3 The Flow

We use the standard cylindrical coordinates.[5] For a velocity field with axial symmetry the Euler equation $(u\cdot\nabla)u = -\nabla p$ takes the form

$$\begin{cases} u_\rho \frac{\partial}{\partial \rho}u_\rho + u_z\frac{\partial}{\partial z}u_\rho - \frac{1}{\rho}u_\varphi^2 = -\frac{\partial}{\partial \rho}p, \\ u_\rho \frac{\partial}{\partial \rho}u_\varphi + u_z\frac{\partial}{\partial z}u_\varphi + \frac{1}{\rho}u_\rho u_\varphi = 0, \\ u_\rho \frac{\partial}{\partial \rho}u_z + u_z\frac{\partial}{\partial z}u_z = -\frac{\partial}{\partial z}p. \end{cases} \tag{5}$$

For $R>0$, denote $a = \alpha\left(\frac{\rho}{R},\frac{z}{R}\right)$. For the sake of convenience, we denote by $\mathcal{C}$ the circle $\rho = R, z = 0$ where $a = 0$. Let

$$p = \frac{aR^4}{4},\ b = \frac{R^3}{4}\sqrt{H(a)},\ u = \frac{1}{\rho}\left(\frac{\partial p}{\partial z}e_\rho - \frac{\partial p}{\partial \rho}e_z + be_\varphi\right). \tag{6}$$

(Note that b is not smooth on $\mathcal{C}$.) Obviously, $\operatorname{div} u = 0$ outside $\mathcal{C}$.

Lemma 5. *The fields (u,p) given by (6) satisfy (5) in a neighbourhood of $\mathcal{C}$ (but not on the curve itself).*

[5] Alternatively, one may use toroidal coordinates (which is in a sense more natural). However, they do not seem to offer any real advantage because the surfaces $p = \text{const}$ are not actual tori (with circular section).

Proof. The second equation of (5) is obvious. The last one is equivalent to

$$x\left(\frac{\partial}{\partial x}\alpha\right)\frac{\partial^2}{\partial x\partial y}\alpha+\left(\frac{\partial}{\partial y}\alpha\right)\left(\frac{\partial}{\partial x}\alpha+4x^3-x\frac{\partial^2}{\partial x^2}\alpha\right)=0,$$

where $x=\frac{\rho}{R}$, $y=\frac{z}{R}$ and $\alpha=\alpha(x,y)$. After multiplication by $\frac{\partial}{\partial y}\alpha$, using (4) and

$$\frac{\partial G}{\partial x}+F\frac{\partial G}{\partial\alpha}=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\alpha\right)^2=2\left(\frac{\partial}{\partial y}\alpha\right)\frac{\partial^2}{\partial x\partial y}\alpha$$

we have

$$\frac{1}{2}xF\left(\frac{\partial G}{\partial x}+F\frac{\partial G}{\partial\alpha}\right)+G\left(F+4x^3-x\left(\frac{\partial F}{\partial x}+F\frac{\partial F}{\partial\alpha}\right)\right)=0,$$

which follows from (2), (3).

Finally, the first equation is

$$x\left(\frac{\partial}{\partial x}\alpha\right)\frac{\partial^2}{\partial y^2}\alpha-x\left(\frac{\partial}{\partial y}\alpha\right)\frac{\partial^2}{\partial x\partial y}\alpha+\left(\frac{\partial}{\partial y}\alpha\right)^2-4x^3\frac{\partial}{\partial x}\alpha+H(\alpha)=0,$$

or

$$\frac{1}{2}xF\frac{\partial G}{\partial\alpha}-\frac{1}{2}x\left(\frac{\partial G}{\partial x}+F\frac{\partial G}{\partial\alpha}\right)+G-4x^3F+H(\alpha)=0,$$

which is again a consequence of (2), (3). □

As introduced, this Euler flow is only defined in a vicinity of the circle $\mathcal{C}$. However, this flow satisfies an additional condition $u\cdot\nabla p=0$ which is very useful for our purposes. Consider another field $\widetilde{u}=\omega(p)u$ where ω is a smooth function. Due to the above condition we have

$$\operatorname{div}\widetilde{u}=\omega(p)\operatorname{div}u+\omega'(p)(u\cdot\nabla p)=0$$

and

$$(\widetilde{u}\cdot\nabla)\widetilde{u}=\omega^2(p)(u\cdot\nabla)u+\omega(p)\omega'(p)(u\cdot\nabla p)u=-\omega^2(p)\nabla p.$$

So, regardless of a choice of the function ω, the field $\widetilde{u}$ is also an Euler flow, with the corresponding pressure determined by $d\widetilde{p}=\omega^2(p)\,dp$.

Due to Lemma 4, we may assume that $\omega=\omega(p)$ in a vicinity of the circle $\mathcal{C}$ and $\omega=0$ outside this domain. If $\operatorname{supp}(\omega)\subset[\varepsilon,2\varepsilon]$ (as a function of p) with $\varepsilon>0$ sufficiently small, then we have $\widetilde{u}\in C^\infty(\mathbb{R}^3)$. The new flow is supported in a toroidal domain which can be made arbitrarily close to the circle. This completes the proof of Theorem 1.

Remark 5. It should be noted that the poloidal stream function $\Psi=a$ is a solution[6] of the Grad-Shafranov equation in the following form ($R=1$)

$$(\partial_{\rho\rho}+\partial_{zz}-\frac{1}{\rho}\partial_\rho)\Psi=10\rho^2-\frac{1}{2}H'(\Psi).$$

[6] This is probably what an expert would expect in this situation, but the author does not know an appropriate reference to make it a meaningful discussion.

4 Generalized Beltrami Flows

The condition $u \cdot \nabla p = 0$ mentioned above means that the pressure is constant along a streamline. This is very uncommon, and the only other nontrivial example known to the author is a flow on the 3-sphere constructed in [KKP14]. (The trivial examples are vortices (rotational flows) and their variations.) By the Bernoulli theorem $|u|^2$ must also be a first integral; indeed, from (6), (4) and the definition of G we have

$$|u|^2 = \frac{1}{\rho^2}\left[\left(\frac{\partial p}{\partial z}\right)^2 + \left(\frac{\partial p}{\partial \rho}\right)^2 + b^2\right] = 3p.$$

For the modified flow $\widetilde{u}$ the formula is different but $|\widetilde{u}|^2$ is still a function of the pressure $\widetilde{p}$.

Recall that for a Beltrami flow u the Bernoulli function $B = p + \frac{1}{2}|u|^2$ is constant. The case when the Bernoulli function depends on the pressure may be considered a generalization, and constructed flows belongs to this category ($B = \frac{5}{2}p$ for the original flow). As was pointed out by Arnold [AK99][II.1.B], for a non-constant B both the streamlines and the vortex lines lie on the surfaces $B = \text{const}$; in our situation these are the same as $p = \text{const}$. It makes a difference because in this case the flow sheets become independent in a sense, so the flow may be "modulated" (a trick we used in the previous section).

One of the referees pointed out to the author that a generalized Beltrami flow (with an extremum of pressure at some point) has a peculiar restriction on the behaviour of the pressure. Let (u, p) be such a flow, and assume that $|u|^2 = 3p$ as before (we can do this without loss of generality). By the same recipe as above we may then construct another flow $(\widetilde{u}, \widetilde{p})$. If it has compact support then (e.g. [CC15])

$$\int_{\mathbb{R}^3} (|\widetilde{u}|^2 + 3\widetilde{p})\, dx = 0.$$

To make sense of this it is convenient to introduce a function $V(c) = Vol(\{x \in \mathbb{R}^3 : p(x) \le c\})$. The equality then becomes

$$\int_{\mathbb{R}^3} (p\omega^2(p) + \widetilde{p})\, dx = \int \omega^2(p)(p\, dV(p) - V(p)\, dp) = 0.$$

It must be frue for any function ω which means $V(p) = p \cdot \text{const}$.

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