Liouville theorems:

Problem 1. Assume that for $-\infty < t < 0, \quad \mathbf{x} \in \mathbb{R}^3$,

$$\begin{cases} \mathbf{v}_t - \Delta \mathbf{v} = -\mathbf{v} \cdot \nabla \mathbf{v} - \nabla p, \\ \nabla \cdot \mathbf{v} = 0, \\ |\mathbf{v}| \le 1. \end{cases}$$

Prove that $\mathbf{v} \equiv const.$

Problem 2. Add the decay condition $|\mathbf{v}| \le \frac{C}{|\mathbf{x}| + \sqrt{|t|}}$ to Problem 1. Prove that $\mathbf{v} \equiv const.$

Problem 3. Assume that $\mathbf{v}(\mathbf{x})$ satisfies

$$\begin{cases} \Delta \mathbf{v} = \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p, \\ \nabla \cdot \mathbf{v} = 0, \\ \int |\nabla \mathbf{v}|^2 dx < \infty, \\ \mathbf{v} \to 0, \quad \mathbf{x} \to \infty. \end{cases}$$

Prove that $\mathbf{v} \equiv 0$.

Liouville theorems for axi-symmetric case:

Set $r = \sqrt{x^2 + y^2}$. $\mathbf{e}_r = \begin{pmatrix} \frac{x}{r} \\ \frac{y}{r} \\ 0 \end{pmatrix}$, $\mathbf{e}_\theta = \begin{pmatrix} -\frac{y}{r} \\ \frac{x}{r} \\ 0 \end{pmatrix}$, $\mathbf{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Let $\mathbf{v} = v^r \mathbf{e}_r + v^\theta \mathbf{e}_\theta + v^z \mathbf{e}_z$. $\mathbf{b} = v^r \mathbf{e}_r + v^z \mathbf{e}_z$. $\Gamma = r v^\theta$. Then from Navier-Stokes one gets

$$\partial_t \Gamma + \mathbf{b} \cdot \nabla \Gamma + \frac{2}{r} \partial_r \Gamma = \Delta \Gamma \tag{0.1}$$

$$\partial_t \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{b} + \nabla p = \Delta \mathbf{b} + \frac{(u^\theta)^2}{r} \mathbf{e}_r \tag{0.2}$$

Problem 4. For (0.1), assume that $|\mathbf{v}| \lesssim 1$ and $|\Gamma| \lesssim 1$. Prove that Γ is constant($= 0$).

Problem 5. Add the condition $|\mathbf{v}| \le \frac{C}{|x| + \sqrt{|t|}}$ to Problem 4. Prove that Γ is constant. This is already solved.

Problem 6. Consider stationary $\mathbf{v}(x)$ satisfying Problem 4. Prove that Γ is constant. (In a recent preprint of Lei, the case when $\mathbf{v}$ is periodic in z is solved.)

Other problems:

Problem 7. Show energy equality for 3D non-stationary Navier-Stokes. For now only local energy inequality is valid.

Problem 8. Improve the Caffarelli-Kohn-Nirenberg theorem $\mathcal{P}^1(\mathcal{S}) = 0$ in the axially symmetric situation.