On convergence of arbitrary D-solution of steady Navier–Stokes system in $2D$ exterior domains*

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Abstract

We study solutions to stationary Navier–Stokes system in two dimensional exterior domain. We prove that any such solution with finite Dirichlet integral converges to a constant vector at infinity uniformly. No additional condition (on symmetry or smallness, etc.) are assumed. The proofs based on arguments of the classical Amick's article (Acta Math. 1988) and on results of a recent paper by authors (arXiv 1711:02400) where the uniform boundedness of these solutions was established.

1 Introduction

Let Ω be an exterior domain in $\mathbb{R}^2$, in particular,

$$\Omega \supset \mathbb{R}^2 \setminus B, \tag{1.1}$$

where $B = B_{R_0}$ is the disk of radius R_0 centered at the origin with $\partial\Omega \subset B$.

We consider the solutions to the steady Navier–Stokes system

$$\begin{cases} \nu\Delta\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega. \end{cases} \tag{1.2}$$

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Starting from the pioneering papers by J. Leray [9] it is now customary to consider solutions to (1.2) with finite Dirichlet integral

$$\int_{\Omega} |\nabla \mathbf{u}|^2 < +\infty, \tag{1.3}$$

known also as *D–solutions*. As is well known (e.g., [8]), such solutions are real–analytic in Ω. The existence of solutions to (1.2) was also studied in [2], [11], [6], [12].

The problem of the asymptotic behavior at infinity of an arbitrary D–solution $(\mathbf{u}, p)$ to (1.2) was tackled by D. Gilbarg & H. Weinberger [4]–[5] and Ch. Amick [1]. In [5] it is shown that

$$p(z) - p_0 = o(1) \qquad \text{as } r \to \infty, \tag{1.4}$$

i.e., pressure has a limit at infinity (one can choose, say, $p_0 = 0$) and

$$\begin{aligned} \mathbf{u}(z) &= o(\log^{1/2} r), \\ \omega(z) &= o(r^{-3/4} \log^{1/8} r), \\ \nabla \mathbf{u}(z) &= o(r^{-3/4} \log^{9/8} r), \end{aligned} \tag{1.5}$$

where $r = |z|$ and

$$\omega = \partial_2 u_1 - \partial_1 u_2$$

is the vorticity. If, in addition, $\mathbf{u}$ is bounded, then there is a constant vector $\mathbf{u}_\infty$ such that

$$\lim_{r \to +\infty} \int_0^{2\pi} |\mathbf{u}(r, \theta) - \mathbf{u}_\infty|^2 d\theta = 0, \tag{1.6}$$

and

$$\begin{aligned} \omega(z) &= o(r^{-3/4}), \\ \nabla \mathbf{u}(z) &= o(r^{-3/4} \log r). \end{aligned} \tag{1.7}$$

Here if $\mathbf{u}_\infty = \mathbf{0}$, then

$$\mathbf{u}(z) \to 0 \qquad \text{uniformly as } \ |z| \to \infty. \tag{1.8}$$

In the case $\mathbf{u}_\infty \neq 0$ D. Gilbarg & H. Weinberger proved that there exists a sequence of radii $R_n \in (2^n, 2^{n+1})$, $n \geq n_0$, such that

$$\sup_{\theta \in [0, 2\pi]} |\mathbf{u}(R_n, \theta) - \mathbf{u}_\infty| \to 0 \qquad \text{as } \ n \to \infty. \tag{1.9}$$

In the classical and very elegant paper [1] Ch.Amick proved that under zero boundary condition

$$\mathbf{u}|_{\partial\Omega} \equiv 0 \tag{1.10}$$

the solution has the following asymptotic properties:

(i) $\mathbf{u}$ is bounded and, as a consequence, it satisfies (1.6), (1.7);

(ii) the total head pressure $\Phi = p + \frac{1}{2}|\mathbf{u}|^2$ and the absolute value of the velocity $|\mathbf{u}|$ have the uniform limit at infinity, i.e.,

$$|\mathbf{u}(r,\theta)| \to |\mathbf{u}_\infty| \qquad \text{as } \ r \to \infty, \tag{1.11}$$

where $\mathbf{u}_\infty$ is the constant vector from the condition (1.6).

Recently M.Korobkov, K.Pileckas and R.Russo [7] simplified the issue and proved that the first claim (i) holds in the general case of D-solutions without (1.10) assumption:

Theorem 1.1 ([7])**.** *Let $\mathbf{u}$ be a D-solution to the Navier–Stokes system (1.2) in the exterior domain $\Omega \subset \mathbb{R}^2$. Then $\mathbf{u}$ is uniformly bounded in $\Omega_0 = \mathbb{R}^2 \setminus B$, i.e.,*

$$\sup_{z\in\Omega_0} |\mathbf{u}(z)| < \infty, \tag{1.12}$$

where $B = B_{R_0}$ is an open disk with sufficiently large radius: $B \supset \partial\Omega$.

Using the above–mentioned results of D. Gilbarg and H. Weinberger, we obtain immediately

Corollary 1.1. *Let $\mathbf{u}$ be a D-solution to the Navier–Stokes system (1.2) in a neighbourhood of infinity. Then the asymptotic properties (1.4), (1.6)–(1.7) hold.*

The main result of the present paper is as follows.

Theorem 1.2. *Let $\mathbf{u}$ be a D-solution to the Navier–Stokes system (1.2) in the exterior domain $\Omega \subset \mathbb{R}^2$. Then $\mathbf{u}$ converges uniformly at infinity, i.e.,*

$$\mathbf{u}(z) \to \mathbf{u}_\infty \qquad \textit{uniformly as } \ |z| \to \infty, \tag{1.13}$$

where $\mathbf{u}_\infty \in \mathbb{R}^2$ is the constant vector from the equality (1.6).

The proof of Theorem 1.2 is based on a combination of ideas of papers [1], [7] and [5].

If $\mathbf{u}_\infty \neq 0$, then by results of L.I. Sazonov [13], the convergence (1.13) ensures that the solution behaves at infinity as that of the linear Oseen equations (see also [3]).

2 Notations and preliminaries

By *a domain* we mean an open connected set. We use standard notations for Sobolev spaces $W^{k,q}(\Omega)$, where $k \in \mathbb{N}$, $q \in [1,+\infty]$. In our notation we do not distinguish function spaces for scalar and vector valued functions; it is clear from the context whether we use scalar or vector (or tensor) valued function spaces.

For $q \geq 1$ denote by $D^{k,q}(\Omega)$ the set of functions $f \in W^{k,q}_{\mathrm{loc}}(\Omega)$ such that $\|f\|_{D^{k,q}(\Omega)} = \|\nabla^k f\|_{L^q(\Omega)} < \infty$.

We denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure, i.e., $\mathcal{H}^k(F) = \lim\limits_{t\to 0+} \mathcal{H}^k_t(F)$, where

$$\mathcal{H}^1_t(F) = \big(\frac{\alpha_k}{2}\big)^k \inf\{\sum_{i=1}^{\infty}\big(\mathrm{diam} F_i\big)^k : \mathrm{diam} F_i \leq t, F \subset \bigcup_{i=1}^{\infty} F_i\}$$

and α_k is a Lebesgue volume of the unit ball in $\mathbb{R}^k$.

In particular, for a curve S the value $\mathcal{H}^1$ coincides with its length, and for sets $E \subset \mathbb{R}^2$ the $\mathcal{H}^2(E)$ coincides with the usual Lebesgue measure in $\mathbb{R}^2$.

Also, for a curve S by $\int\limits_S f\, ds$ we denote the usual integral with respect to 1-dimensional Hausdorff measure (=length). Further, for a set $E \subset \mathbb{R}^2$ by $\int\limits_E f(x)\, d\mathcal{H}^2$ or simply $\int\limits_E f(x)$ we denote we integral with respect to the two-dimensional Lebesgue measure.

Below we present some usual results concerning the behaviour of D-functions.

Lemma 2.1. *Let $f \in D^{1,2}(\Omega)$ and assume that*

$$\int\limits_D |\nabla f|^2\, d\mathcal{H}^2 < \varepsilon^2$$

for some $\varepsilon > 0$ and for some ring $D = \{z \in \mathbb{R}^2 : r_1 < |z - z_0| < r_2\} \subset \Omega$. Then the estimate

$$|\bar{f}(r_2) - \bar{f}(r_1)| \leq \varepsilon \sqrt{\ln \frac{r_2}{r_1}} \tag{2.1}$$

holds, where $\bar{f}$ means the mean value of f over the circle $S(z_0, r)$:

$$\bar{f}(r) := \frac{1}{2\pi r} \int\limits_{|z-z_0|=r} f(z)\, ds.$$

Proof. Let (r,θ) be polar coordinates with the center in the point z_0. We have

$$|\bar f(r_2)-\bar f(r_1)| = \Big|\int\limits_{r_1}^{r_2} \bar f'(r)dr\Big| \le \int\limits_{r_1}^{r_2}\int\limits_{0}^{2\pi} \Big|\frac{\partial}{\partial r} f(r,\theta)\Big| d\theta dr \le \int\limits_{r_1}^{r_2}\int\limits_{0}^{2\pi} \big|\nabla f(z)\big| d\theta dr.$$

Estimating the right-hand side by the Cauchy–Schwarz inequality we obtain

$$|\bar f(r_2)-\bar f(r_1)| \le \sqrt{\ln\frac{r_2}{r_1}}\Big(\int\limits_{r_1}^{r_2}\Big(\int\limits_{|z-z_0|=r} \big|\nabla f(z)\big|^2 ds\Big)dr\Big)^{1/2}$$

$$\le \sqrt{\ln\frac{r_2}{r_1}}\Big(\int\limits_{D} |\nabla f|^2\, d\mathcal{H}^2\Big)^{1/2} \le \varepsilon\sqrt{\ln\frac{r_2}{r_1}}.$$

□

Lemma 2.2. *Fix a number* $\beta\in(0,1)$. *Let* $f\in D^{1,2}(\Omega)$ *and assume that*

$$\int\limits_{D} |\nabla f|^2\, d\mathcal{H}^2 < \varepsilon^2$$

for some $\varepsilon>0$ *and for some ring* $D=\{z\in\mathbb{R}^2 : \beta R<|z-z_0|<R\}\subset\Omega$. *Then there exists a number* $r\in[\beta R,R]$ *such that the estimate*

$$\sup_{|z-z_0|=r} |f(z)-\bar f(r)| \le c_\beta\varepsilon \tag{2.2}$$

holds, where the constant c_β *depends on* β *only.*

Proof (see the proof of Lemma 2.2 in [5]). Take the polar coordinate system with the center at the point z_0. Since $\int\limits_{\beta R}^{R}\frac{1}{\rho}\int\limits_{0}^{2\pi}\big|\frac{\partial}{\partial\theta}f(\rho,\theta)\big|^2 d\theta d\rho \le \int\limits_{D}|\nabla f(z)|^2dz$, by the integral mean value theorem, there exists some $r\in[\beta R,R]$ such that

$$\int\limits_{0}^{2\pi}\Big|\frac{\partial}{\partial\theta'}f(r,\theta')\Big|^2 d\theta \le \tilde c_\beta\int\limits_{D}|\nabla f(z)|^2dz.$$

Therefore, by Holder inequality

$$\int\limits_{0}^{2\pi}\Big|\frac{\partial}{\partial\theta}f(r,\theta)\Big|d\theta \le \Big(2\pi\int\limits_{0}^{2\pi}\Big|\frac{\partial}{\partial\theta'}f(r,\theta')\Big|^2 d\theta\Big)^{\frac12} \le c_\beta\varepsilon \tag{2.3}$$

On the other hand,

$$f(r,\theta)-f(r,\varphi)=\int\limits_{\varphi}^{\theta}\frac{\partial}{\partial\theta'}f(r,\theta')d\theta'.$$

Integrating this equality with respect to φ and taking the average, we find

$$|f(r,\theta)-\bar f(r)|\le\int\limits_{0}^{2\pi}\left|\frac{\partial}{\partial\theta'}f(r,\theta')\right|d\theta'\le c_\beta\varepsilon.$$

□

Summarize the results of these lemmas, we receive

Lemma 2.3. *Under conditions of Lemma 2.2, there exists* $r\in[\beta R,R]$ *such that*

$$\sup_{|z-z_0|=r}|f(z)-\bar f(R)|\le\tilde c_\beta\varepsilon. \tag{2.4}$$

3 Proof of the main Theorem 1.2.

Suppose the assumptions of Theorem 1.2 are fulfilled. By classical regularity results for D-solutions to the Navier–Stokes system (e.g., [3]), the functions $\mathbf{u}$ and p are real–analytical on the set $\Omega_0=\mathbb{R}^2\setminus B_{R_0}$. Moreover, it follows from results in [5] and Theorem 1.1, that $\mathbf{u}$ and p are uniformly bounded in Ω_0,

$$\sup_{z\in\Omega_0}\big(|p(z)|+|\mathbf{u}(z)|\big)\le C<+\infty, \tag{3.1}$$

and the pressure p has a limit at infinity; we could assume without loss of generality that

$$p(z)\to 0 \qquad \text{uniformly as } \ |z|\to\infty. \tag{3.2}$$

It is also well known (see [3]) that all derivatives of $\mathbf{u}$ uniformly converge to zero:

$$\forall k=1,2,\dots \qquad \nabla^k\mathbf{u}(z)\to 0 \qquad \text{uniformly as } \ |z|\to\infty. \tag{3.3}$$

Further, it is proved in [5] that there exists a vector $\mathbf{u}_\infty\in\mathbb{R}^2$ such that

$$\lim_{r\to+\infty}\int\limits_{0}^{2\pi}|\mathbf{u}(r,\theta)-\mathbf{u}_\infty|^2d\theta=0, \tag{3.4}$$

moreover, if $\mathbf{u}_\infty = \mathbf{0}$, then

$$\mathbf{u}(z) \to 0 \qquad \text{uniformly as } \; |z| \to \infty. \tag{3.5}$$

Thus if $\mathbf{u}_\infty = 0$, the statement of Theorem 1.2 is known and we need to consider only the case

$$\mathbf{u}_\infty \neq 0. \tag{3.6}$$

Consider the vorticity $\omega = \partial_2 u_1 - \partial_1 u_2$ which will play the key role in our proof. Recall that ω satisfies the elliptic equation

$$\nu \Delta \omega = (\mathbf{u} \cdot \nabla)\omega. \tag{3.7}$$

In particular, ω satisfies two-sided maximum principle in $\mathbb{R}^2$; moreover,

$$\int\limits_{\Omega_0} r|\nabla \omega|^2 < \infty \tag{3.8}$$

(see [5]).

We will need also the following statement.

Lemma 3.1. *Let $\mathbf{u}$ be a D-solution to the Navier–Stokes system (1.2) in the exterior domain $\Omega \subset \mathbb{R}^2$. Denoted by $\bar{\mathbf{u}}(z,r)$ the mean value of $\mathbf{u}$ over the circle $S(z,r)$:*

$$\bar{\mathbf{u}}(z,r) = \frac{1}{2\pi r} \int\limits_{|\xi - z| = r} \mathbf{u}(\xi)\, ds \tag{3.9}$$

and let $\varphi(z,r)$ be the argument of the complex number associated to the vector $\bar{\mathbf{u}}(z,r) = (\bar{u}_1(r), \bar{u}_2(r))$, i.e., $\varphi(z,r) = \arg(\bar{u}_1(r) + i\bar{u}_2(r))$. Suppose $|z|$ is large enough so that the disk $D_z = \big\{\xi \in \mathbb{R}^2 : |\xi - z| \le \frac{4}{5}|z|\big\}$ is contained in Ω. Assume also that

$$|\bar{\mathbf{u}}(z,r)| \ge \sigma.$$

for some positive constant $\sigma > 0$ and for all $r \in \big(0, \frac{4}{5}|z|\big]$. Then the estimate

$$\sup_{0<\rho_1\le\rho_2\le\frac{4}{5}|z|} |\varphi(z,\rho_2) - \varphi(z,\rho_1)| \le \frac{1}{4\pi\sigma^2} \int\limits_{D_z} \left(\frac{1}{r}|\nabla\omega| + |\nabla \mathbf{u}|^2\right) d\mathcal{H}^2_\xi \tag{3.10}$$

holds, where $r = |\xi - z|$.

For the proof of the estimate (3.10) see [5, Proof of Theorem 4, page 399].

To apply the last Lemma 3.1, we need also the following simple technical assertion.

Lemma 3.2. *Let* $\mathbf{u}$ *be a D-solution to the Navier–Stokes system (1.2) in the exterior domain* $\Omega \subset \mathbb{R}^2$*. For* $z \in \Omega$ *denote as above*

$$D_z = \big\{\xi \in \mathbb{R}^2 : |\xi - z| \le \frac{4}{5}|z|\big\}.$$

Then the uniform convergence

$$\int\limits_{D_z} \frac{1}{r}|\nabla\omega|\, d\mathcal{H}^2_\xi \to 0 \qquad \text{as } |z| \to \infty \tag{3.11}$$

holds, where again $r = |\xi - z|$.

Proof. Take and fix arbitrary $\varepsilon > 0$. Take also numbers $r_2 > r_1 > 0$ large enough so that

$$2\pi < \varepsilon r_1; \tag{3.12}$$

$$\int\limits_{D_z} r|\nabla\omega|^2\, d\mathcal{H}^2_\xi < \varepsilon \qquad \text{if } |z| > r_2; \tag{3.13}$$

$$2\pi r_1 \max_{|\xi - z| < r_1} |\nabla\omega(\xi)| < \varepsilon \qquad \text{if } |z| > r_2 \tag{3.14}$$

(the existence of such numbers follows from the estimate (3.8) and from the uniform convergence (3.3)).

Now take arbitrary $z \in \mathbb{R}^2$ with $|z| > r_2$. Then the disk D_z is represented as the union of two sets $D_z = D_1 \cup D_2$, where

$$D_1 = \big\{\xi \in \mathbb{R}^2 : |\xi - z| < r_1\big\}, \qquad D_2 = \big\{\xi \in \mathbb{R}^2 : r_1 \le |\xi - z| < \frac{4}{5}|z|\big\}.$$

We have

$$\int\limits_{D_1} \frac{1}{r}|\nabla\omega|\, d\mathcal{H}^2_\xi < \max_{|\xi - z| < r_1} |\nabla\omega(\xi)| \int\limits_{D_1} \frac{1}{r}\, d\mathcal{H}^2_\xi$$

$$= 2\pi r_1 \max_{|\xi - z| < r_1} |\nabla\omega(\xi)| \overset{(3.14)}{<} \varepsilon. \tag{3.15}$$

Further, applying the elementary inequality $\frac{1}{r}|\nabla\omega| < \frac{1}{r^3} + r|\nabla\omega|^2$, for the domain D_2 we have:

$$\int\limits_{D_2} \frac{1}{r}|\nabla\omega|\, d\mathcal{H}^2_\xi < \int\limits_{D_2} \frac{1}{r^3}\, d\mathcal{H}^2_\xi + \int\limits_{D_2} r|\nabla\omega|^2\, d\mathcal{H}^2_\xi$$

$$= 2\pi \int\limits_{r=r_1}^{\frac{4}{5}|z|} \frac{1}{r^2}\, dr + \int\limits_{D_2} r|\nabla\omega|^2 \, d\mathcal{H}^2_\xi \overset{(3.12)\text{-}(3.13)}{<} 2\varepsilon. \tag{3.16}$$

From the inequalities (3.15)–(3.16) it follows that

$$\int\limits_{D_z} \frac{1}{r}|\nabla\omega| \, d\mathcal{H}^2_\xi < 3\varepsilon. \tag{3.17}$$

We proved the last inequality for any $z \in \mathbb{R}^2$ with $|z| > r_2$. Since the number $\varepsilon > 0$ is arbitrary, the required convergence (3.11) is established. □

Further we will use the following two criteria for the uniform convergence of the velocity:

Lemma 3.3. *Let* $\mathbf{u}$ *be a D-solution to the Navier–Stokes system (1.2) in the exterior domain* $\Omega \subset \mathbb{R}^2$*. Suppose that at least one of the following two conditions is fulfilled:*

(i) $\omega(z) = o(|z|^{-1})$ *as* $|z| \to \infty$*;*

(ii) *the absolute value of the velocity has a uniform limit at infinity:*

$$|\mathbf{u}(z)| \to |\mathbf{u}_\infty| \qquad \text{uniformly as } |z| \to \infty, \tag{3.18}$$

where the vector $\mathbf{u}_\infty$ *was specified above.*

Then $\mathbf{u}$ *converges uniformly at infinity as well, i.e., the formula (1.13) holds.*

Proof. Part (i) was established by Amick (see [1], Remark 3(i) on p. 103 and the proof of Theorem 19). Recall, that his argument is based on the classical Cauchy-type representation formula of complex analysis:

$$w(z) = \frac{1}{2\pi i} \oint\limits_{|\xi - z_0| = r} \frac{w(\xi)\, d\xi}{\xi - z} + \frac{1}{2\pi i} \iint\limits_{|\xi - z_0| < r} \frac{\omega(\xi)}{\xi - z_0}\, dx\, dy, \tag{3.19}$$

where $w(\xi) = u_1(\xi) - iu_2(\xi)$ and $\xi = x + iy$.

Let us prove the second part of Lemma 3.3. Suppose that assumption (ii) is fulfilled. If $\mathbf{u}_\infty = 0$, then there is nothing to prove (see the above discussion concerning the results of D. Gilbarg & H. Weinberger [4]–[5]). So we assume without loss of generality that

$$|\mathbf{u}_\infty| > 0. \tag{3.20}$$

From assumption (3.18) and Lemmas 2.1–2.3 it follows that

$$\sup_{0<\rho\leq\frac{4}{5}|z|}\Big||\mathbf{u}_\infty|-|\bar{\mathbf{u}}(z,\rho)|\Big|\to 0 \qquad \text{uniformly as } |z|\to\infty, \tag{3.21}$$

where $\bar{\mathbf{u}}(z,r)$ is the mean value of $\mathbf{u}$ over the circle $S(z,r)$. In particular, because of inequality (3.20), there exist numbers $\sigma>0$ and $R_*>0$ such that

$$|\bar{\mathbf{u}}(z,r)|\geq\sigma \qquad \text{if } |z|\geq R_* \quad \text{and } 0<r\leq\frac{4}{5}|z|. \tag{3.22}$$

Then, by Lemma 3.1, the argument $\varphi(z,r)$ of the complex number associated to $\bar{\mathbf{u}}(z,r)$ satisfies the estimate (3.10). From (3.10)–(3.11) it follows immediately that

$$\sup_{0<\rho_1\leq\rho_2\leq\frac{4}{5}|z|}|\varphi(z,\rho_2)-\varphi(z,\rho_1)|\to 0 \tag{3.23}$$

uniformly as $|z|\to\infty$. In particular,

$$\sup_{0<\rho\leq\frac{4}{5}|z|}|\arg\mathbf{u}(z)-\arg\bar{\mathbf{u}}(z,\rho)|\to 0 \tag{3.24}$$

uniformly as $|z|\to\infty$. From the assumption (3.18) and (3.21) we have

$$\sup_{0<\rho\leq\frac{4}{5}|z|}\Big||\mathbf{u}(z)|-|\bar{\mathbf{u}}(z,\rho)|\Big|\to 0 \qquad \text{uniformly as } |z|\to\infty. \tag{3.25}$$

Summarizing the information from formulas (3.24)–(3.25), we obtain

$$\sup_{0<\rho\leq\frac{4}{5}|z|}|\mathbf{u}(z)-\bar{\mathbf{u}}(z,\rho)|\to 0 \qquad \text{uniformly as } |z|\to\infty. \tag{3.26}$$

Consider the sequence of circles $S_{R_n}=\{\xi\in\mathbb{R}^2:|\xi|=R_n\}$ such that $2^n<R_n<2^{n+1}$ and

$$\sup_{|\xi|=R_n}|\mathbf{u}(\xi)-\mathbf{u}_\infty|=\varepsilon_n\to 0 \qquad \text{as } n\to\infty \tag{3.27}$$

(the existence of such sequence is guaranteed by above mentioned results of D. Gilbarg and H. Weinberger, see (1.9)).

Now take a point $z\in\mathbb{R}^2$ with sufficiently large $|z|$ and take also the natural number $n=n_z$ such that

$$2^{n+1}\leq|z|<2^{n+2}.$$

Then by construction and by the triangle inequality we have

$$S_{R_n} \cap S_{z,\rho} \neq \emptyset \qquad \text{if} \quad \frac{3}{4}|z| < \rho < \frac{4}{5}|z|, \tag{3.28}$$

where $S_{z,\rho} = \{\xi \in \mathbb{R}^2 : |\xi - z| = \rho\}$. From Lemma 2.2 it follows that there exists $\rho_* \in \left(\frac{3}{4}|z|, \frac{4}{5}|z|\right)$ such that

$$\sup_{|\xi - z| = \rho_*} |\mathbf{u}(\xi) - \bar{\mathbf{u}}(z, \rho_*)| = \varepsilon_z, \tag{3.29}$$

where $\varepsilon_z \to 0$ uniformly as $|z| \to \infty$. Summarizing the information from formulas (3.27)–(3.29), we obtain that

$$|\mathbf{u}_\infty - \bar{\mathbf{u}}(z, \rho_*)| = \varepsilon'_z \to 0 \qquad \text{uniformly as} \quad |z| \to \infty. \tag{3.30}$$

Finally, from the last formula and from (3.26) we conclude that

$$|\mathbf{u}_\infty - \mathbf{u}(z)| \to 0 \qquad \text{uniformly as} \quad |z| \to \infty, \tag{3.31}$$

as required. The Lemma 3.3 is proved completely. □

Proof of Theorem 1.2. For a point $z \in \Omega_0$ denote by $K(z)$ the connected component of the level set of the vorticity ω containing z, i.e., $K(z) \subset \{x \in \Omega_0 : \omega(x) = \omega(z)\}$. Here we understand the notion of connectedness in the sense of general topology.

We consider two possible cases:

Case I. Level sets of ω separate infinity from the origin:

$$\exists z_* \in \Omega_0 : \quad \omega(z_*) \neq 0 \quad \text{and} \quad K(z_*) \cap \partial\Omega_0 = \emptyset. \tag{3.32}$$

Case II. Level sets of ω do not separate infinity from the origin:

$$K(z) \cap \partial\Omega_0 \neq \emptyset \qquad \forall z \in \Omega_0, \tag{3.33}$$

In Case I, we shall show that

$$|z|\omega(z) \to 0 \qquad \text{uniformly as} \quad |z| \to \infty \tag{3.34}$$

and we obtain the statement of Theorem applying Lemma 3.3(i).

In Case II, we prove that

$$|\mathbf{u}(z)| \to |\mathbf{u}_\infty| \qquad \text{uniformly as} \quad |z| \to \infty, \tag{3.35}$$

where $\mathbf{u}_\infty$ is the vector defined in (3.4). In this case the statement of Theorem will follow from Lemma 3.3(ii).

Consider the case (3.32). Note that then the set $K(z_*)$ is compact. Indeed, the set $K(z_*)$ is connected and if it is not compact, it should "reach" infinity. Since the vorticity tends to zero at infinity, $\omega(z)$ has to be zero on $K(z_*)$, but this contradicts the assumption (3.32).

Next, by elementary compactness and continuity arguments we have that there exists $\delta_0 > 0$ such that

$$K(z) \text{ is a compact set satisfying } \ K(z) \cap \partial\Omega_0 = \emptyset \ \text{ whenever } \ |z - z_*| < \delta_0. \tag{3.36}$$

Note, that since ω is an analytical nonconstant function, we have that $\omega(z) \neq$ const in any open neighborhood of z_*.

Recall, that a real number t is called *a regular value of* ω, if the set $\{z \in \Omega_0 : \omega(z) = t\}$ is nonempty and $\nabla\omega(z) \neq 0$ whenever $\omega(z) = t$. By the classical Morse–Sard theorem, almost all values of ω are regular. Now take a point z_1 satisfying $|z_1 - z_*| < \delta_0$ with regular value $t_1 = \omega(z_1)$. Then by definition and regularity assumptions the set $K(z_1)$ is a smooth compact curve (="*compact one dimensional manifold without boundary*"). By obvious topological reasons, $K(z_1)$ is a smooth curve homeomorphic to the circle. Since ω satisfies maximum principle, this circle surrounds the origin. Therefore, the curve $K(z_1)$ separates the boundary $\partial\Omega_0$ from infinity[1].

Denote $R_* = \max\{|z| : z \in K(z_1)\}$ and $\Omega_* = \{z \in \mathbb{R}^2 : |z| > R_*\}$. Then by construction we have

$$K(z) \cap \partial\Omega_0 = \emptyset \qquad \forall z \in \Omega_*. \tag{3.37}$$

Applying again the same Morse–Sard theorem, we obtain that for almost all $t \in \mathbb{R} \setminus \{0\}$ if $z \in \Omega_*$ and $\omega(z) = t$, then $K(z)$ is a smooth curve homeomorphic to the circle. Since ω satisfies maximum principle, we conclude that this circle surrounds the origin, moreover,

$$K(z_1) = K(z_2) \qquad \text{if } z_1, z_2 \in \Omega_* \quad \text{and } \omega(z_1) = \omega(z_2) \neq 0. \tag{3.38}$$

This implies that

$$\omega(z) \ \text{ does not change sign in } \Omega_*. \tag{3.39}$$

Indeed, let there are points $z_1, z_2 \in \Omega_*$ with regular values $\omega(z_1) < 0$ and $\omega(z_2) > 0$. Taking into account that $\omega(z)$ is vanishing at the infinity, by

[1]It means that infinity and the set $\partial\Omega_0$ lie in the different connected components of the set $\mathbb{R}^2 \setminus K(z_1)$.

maximum principle, $\omega(z)$ is negative in the exterior of $K(z_1)$ and $\omega(z)$ is positive in the exterior of $K(z_2)$. Since this is impossible, $\omega(z)$ cannot change the sign.

Thus we may suppose without loss of generality that

$$\omega(z) \geq 0 \qquad \text{in } \Omega_*. \tag{3.40}$$

Then by the maximum principle we have the strict inequality

$$\omega(z) > 0 \qquad \text{in } \Omega_*. \tag{3.41}$$

Moreover, from (3.38) and from the uniform convergence (see (3.3))

$$\omega(z) \to 0 \qquad \text{as } |z| \to \infty \tag{3.42}$$

and from Morse–Sard theorem we conclude that there exists a number $\delta > 0$ such that

$$\begin{aligned} &\text{for almost all } t \in (0,\delta) \text{ the set } K_t := \{z \in \Omega_* : \omega(z) = t\} \\ &\text{coincides with the smooth curve homeomorphic to the circle} \\ &\text{such that } K_t \cap \partial\Omega_* = \emptyset \text{ and } \nabla\omega \neq 0 \text{ on } K_t. \end{aligned} \tag{3.43}$$

Denote by $\mathscr{T}$ the set of full measure in the interval $(0,\delta)$ consisting of values t satisfying (3.43). Denote also by Ω_t the unbounded connected component of the set $\mathbb{R}^2 \setminus K_t$. Since ω satisfies the maximum principle, the sets K_t have the following monotonicity property:

$$\Omega_{t_1} \subset \Omega_{t_2} \qquad \text{if} \quad 0 < t_1 < t_2. \tag{3.44}$$

Moreover, from the uniform convergence (3.42), it follows that

$$\inf\{|z| : z \in \Omega_t\} \to \infty \qquad \text{as} \quad t \to 0+. \tag{3.45}$$

Our task is to show the property (i) of Lemma 3.3, i.e., to show that

$$|z|\omega(z) \to 0 \qquad \text{uniformly as} \quad |z| \to \infty. \tag{3.46}$$

The last condition is equivalent to

$$tg(t) \to 0 \qquad \text{as} \quad t \to 0+, \tag{3.47}$$

where the function $g(t)$ is defined by

$$g(t) := \sup\{|z| : z \in K_t\}. \tag{3.48}$$

Obviously, $g(t) \le \mathcal{H}^1(K_t)$, where, recall, $\mathcal{H}^1$ is the one-dimensional Hausdorff measure (=length).

For $t \in \mathscr{T}$ and $R > R_*$ denote $\Omega_{t,R} = \Omega_t \cap B_R = \{z \in \Omega_t : |z| < R\}$. Then for sufficiently large R

$$\partial\Omega_{t,R} = K_t \cup S_R,$$

where $S_R = \{z \in \mathbb{R}^2 : |z| = R\}$ is the corresponding circle. Integrating the equation (3.7) over the domain $\Omega_{t,R}$ and taking into account that $(\mathbf{u}\cdot\nabla)\omega = \operatorname{div}(\mathbf{u}\omega)$, we obtain

$$\int\limits_{K_t} |\nabla\omega|\,ds + \int\limits_{S_R} \nabla\omega\cdot\mathbf{n}\,ds = t\int\limits_{K_t} \mathbf{u}\cdot\mathbf{n}\,ds + \int\limits_{S_R} \omega\,\mathbf{u}\cdot\mathbf{n}\,ds. \tag{3.49}$$

Here $\mathbf{n}$ is a unit vector of the outward with respect to $\Omega_{t,R}$ normal to $\partial\Omega_{t,R}$. Note also that the unit normal to the level set $K_t = \{z \in \Omega_* : \omega(z) = t\}$ is given by the formula $\mathbf{n} = \dfrac{\nabla\omega}{|\nabla\omega|}$.

Since $\operatorname{div}\mathbf{u} = 0$, we have $\int\limits_{K_t} \mathbf{u}\cdot\mathbf{n}\,ds = \int\limits_{\partial\Omega_*} \mathbf{u}\cdot\mathbf{n}\,ds = C_*$, i.e., this value does not depend on t. On the other hand, the estimate $\int\limits_{\Omega_0} \big(|\omega|^2 + |\nabla\omega|^2\big)\,d\mathcal{H}^2 < \infty$ implies that there is a sequence $R_k \to +\infty$ such that

$$\int\limits_{S_{R_k}} \big(|\omega| + |\nabla\omega|\big)\,ds \to 0.$$

Taking $R = R_k$ in the equality (3.49) and having in mind the uniform boundedness of the velocity (see (1.12)), we deduce, passing $R_k \to +\infty$, that

$$\int\limits_{K_t} |\nabla\omega|\,ds = C_* t. \tag{3.50}$$

Further, for $t \in (0, \frac{1}{2}\delta)$ denote $E_t = \{z \in \Omega_* : \omega(z) \in (t, 2t)\}$. By construction,

$$\partial E_t = K_t \cup K_{2t}.$$

Applying the classical Coarea formula (see, e.g., [10])

$$\int\limits_{E_t} f\,|\nabla\omega|\,d\mathcal{H}^2 = \int\limits_t^{2t} \Big(\int\limits_{K_\tau} f\,ds\Big)\,d\tau$$

for $f = |\nabla\omega|$ we obtain

$$\int\limits_{E_t} |\nabla\omega|^2\, d\mathcal{H}^2 = \int\limits_t^{2t} \Big(\int\limits_{K_\tau} |\nabla\omega|\, ds \Big)\, d\tau \overset{(3.50)}{=} \int\limits_t^{2t} C_*\tau\, d\tau = 3C_* t^2. \tag{3.51}$$

Applying now the same Coarea formula for $f = 1$ and using the Cauchy–Schwarz inequality, we get

$$\int\limits_t^{2t} \mathcal{H}^1(K_\tau)\, d\tau = \int\limits_{E_t} |\nabla\omega|\, d\mathcal{H}^2 \le \Big(\int\limits_{E_t} |\nabla\omega|^2\, d\mathcal{H}^2 \Big)^{\frac12} \Big(\operatorname{meas} E_t \Big)^{\frac12}$$

$$\overset{(3.51)}{=} \sqrt{3C_*} \Big(t^2 \operatorname{meas}(E_t) \Big)^{\frac12} \le \sqrt{\tfrac34 C_*} \Big(\int\limits_{E_t} \omega^2\, d\mathcal{H}^2 \Big)^{\frac12} \le \varepsilon_t \to 0 \quad \text{as } t \to 0. \tag{3.52}$$

Here we have used also the fact that $t \le |\omega(z)| \le 2t$ in E_t. By virtue of the mean-value theorem, this implies that for any sufficiently small $t \in \mathscr{T}$ there exists a number $\tau \in [t, 2t]$ such that

$$t\mathcal{H}^1(K_\tau) \le \varepsilon_t.$$

By construction, the closed curve K_τ surrounds K_{2t}. Therefore,

$$\sup\{|z| : z \in K_{2t}\} \le \mathcal{H}^1(K_\tau) \le \frac{\varepsilon_t}{t}$$

with $\varepsilon_t \to 0$ as $t \to 0$. From the last inequality we receive the relation (3.47) which is equivalent to (3.46). According to Lemma 3.3(i), this finishes the proof of Theorem 1.2 in the considered Case I.

Consider Case II, i.e., the when

$$K(z) \cap \partial\Omega_0 \ne \emptyset \qquad \forall z \in \Omega_0. \tag{3.53}$$

Now we shall prove that the assertion (3.35) is valid.

Let us recall that Ch. Amick [1] has proved the convergence (3.35) under the assumption that

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{3.54}$$

The condition (3.35) was used in [1] in order to define the stream function ψ in the neighborhood of infinity:

$$\nabla\psi = \mathbf{u}^\perp = (-v, u), \tag{3.55}$$

where $\mathbf{u} = (u, v)$. Using the stream function ψ, Amick introduced an auxiliary function $\gamma = \Phi - \omega\psi$, where $\Phi := p + \frac{1}{2}|\mathbf{u}|^2$ is the Bernoulli pressure. The gradient of this auxiliary function γ satisfies the identity

$$\nabla\gamma = -\nu\nabla^{\perp}\omega - \psi\nabla\omega.$$

Then $\nabla\gamma \cdot \nabla^{\perp}\omega = -\nu|\nabla^{\perp}\omega|^2$, and therefore, γ has the following monotonicity properties:

$$\begin{aligned}&\gamma \text{ is monotone along level sets of the vorticity } \omega = c \text{ and}\\ &\text{vice versa — the vorticity } \omega \text{ is monotone along level sets of } \gamma = c,\end{aligned} \tag{3.56}$$

see [1].

Obviously, the stream function ψ (and, consequently, the corresponding auxiliary function γ) is well defined in the neighborhood of infinity under the more general condition

$$\int\limits_{\partial\Omega_0} \mathbf{u} \cdot \mathbf{n}\, ds = 0 \tag{3.57}$$

instead of (3.54). However, in the general case the flow-rate of the velocity field is not zero,

$$\int\limits_{\partial\Omega_0} \mathbf{u} \cdot \mathbf{n}\, ds \neq 0, \tag{3.58}$$

and, therefore, the stream function ψ can not be defined in the neighborhood of infinity.

We will overcome this difficulty using the assumption (3.53). Take and fix a radius $R_* > R_0$ (R_* could be chosen arbitrary large) and consider the domain $\Omega_* = \{z \in \mathbb{R}^2 : |z| > R_*\}$. Denote by U_i the connected components of the open set $\{z \in \Omega_* : \omega(z) \neq 0\}$. Then there holds the following

Lemma 3.4. *Under assumption (3.53) the following assertions are fulfilled:*

(i) *There are only finitely many components U_i, $i = 1, \ldots, N$;*

(ii) *Every U_i is a simply connected open set;*

(iii) *The vorticity $\omega(z)$ change sign in every neighborhood of infinity, i.e., there exist two sequences of points z_n^+ and z_n^- such that $\omega(z_n^+) > 0$, $\omega(z_n^-) < 0$ and $\lim\limits_{n\to\infty} |z_n^+| = \lim\limits_{n\to\infty} |z_n^-| = \infty$.*

We shall prove Lemma 3.4 below. Let us finish the proof of the theorem using this lemma. The components U_i play also an important role in the arguments of Amick. In particular, he proves in [1] the same properties (i)–(iii) using the boundary condition $\mathbf{u}|_{\partial\Omega} = 0$. Here, in Lemma 3.4, we get the properties (i)–(iii) because of the assumption (3.53). Since U_i are simply connected, this allows us to define the stream function ψ in every component U_i. Moreover, since $\omega = 0$ on $\Omega_* \cap \partial U_i$, the auxiliary function $\gamma = \Phi - \omega\psi$ is well defined and continuous on the whole domain Ω_*. After the functions ψ and γ are defined, we can repeat the arguments of the paper [1] and to prove the convergence (3.35) of absolute value of the velocity at infinity. By Lemma 3.3(ii) this implies the statement of Theorem 1.2. For the reader convenience we recall the corresponding arguments of Amick [1] in Appendix (we also simplify some of his proofs). $\square$

PROOF OF LEMMA 3.4. Let us prove (iii) first. Suppose this is not true, i.e., there exists $R_1 > 0$ such that $\omega(z)$ does not change sign in $\Omega_1 = \{z : |z| > R_1\}$. Without loss of generality assume that $\omega(z) \geq 0$ in Ω_1. Then, by maximum principle,

$$\omega(z) > 0 \qquad \text{in } \Omega_1. \tag{3.59}$$

Take arbitrary $R_2 > R_1$ and denote

$$\delta := \inf_{z \in S_{R_2}} \omega(z), \tag{3.60}$$

where, recall, $S_{R_2} = \{z \in \mathbb{R}^2 : |z| = R_2\}$. By (3.59), $\delta > 0$. Now take any z_2 such that $|z_2| > R_2$ and $\omega(z_2) < \delta$. Then by construction $K(z_2) \cap S_{R_2} = \emptyset$. Therefore, $K(z_2) \cap S_{R_0} = K(z_2) \cap \partial\Omega_0 = \emptyset$, a contradiction with (3.53).

(ii). Fix a component U_i and take an arbitrary curve $S \subset U_i$ homeomorphic to the unit circle. By construction, there exists $\delta > 0$ such that

$$\omega(z) > \delta \qquad \forall z \in S.$$

The curve S split the plane $\mathbb{R}^2$ into the two components: $\mathbb{R}^2 \setminus S = \Omega_S \cup \Omega_\infty$, where $\partial\Omega_S = \partial\Omega_\infty = S$, Ω_S is a bounded domain homeomorphic to the disk, and Ω_∞ is a neighborhood of infinity. Now we have to consider two cases:

(α) the curve S surrounds the origin. Then $\Omega_\infty \subset \Omega_*$, and, by maximum principle, $\omega \geq 0$ in Ω_∞. Thus, we received the contradiction with property (iii) proved just above.

($\alpha\alpha$) the curve S does not surround the origin. Then $\Omega_S \subset \Omega_*$, and, by maximum principle, $\omega > 0$ in Ω_S. Therefore, $\Omega_S \subset U_i$. Since S was arbitrary, it means that U_i is a simply connected set.

Let us prove (i). Since ω is a nonzero analytical function, the set $Z_* = \{z \in S_{R_*} : \omega(z) = 0\}$ is finite (recall, that S_{R_*} is a circle of radius R_*). Let S_j, $j = 1, \dots, M$, be the connected components of the set $S_{R_*} \setminus Z_*$.

Fix arbitrary component U_i. By maximum principle, $\omega(z)$ is not identically zero on ∂U_i, i.e., there exists a point z_0 such that

$$z_0 \in \partial U_i \qquad \text{and} \qquad \omega(z_0) \neq 0.$$

On the other hand, by definition U_i is a connected component of the open set

$$\{z \in \Omega_* : \omega(z) \neq 0\},$$

in particular, we have the identity $\omega(z) \equiv 0$ on the set $\Omega_* \cap \partial U_i$. Therefore,

$$z_0 \in \partial\Omega_* = S_{R_*}.$$

It means, using the above notation, that there exists a number $j(i) \in \{1, \dots, M\}$ such that

$$z_0 \in S_{j(i)}.$$

Then by elementary properties of connected sets and by definitions of S_j and U_i, we have

$$S_{j(i)} \subset \partial U_i,$$

and

$$\Big[j(i_1) = j(i_2) \Big] \Rightarrow U_{i_1} = U_{i_2},$$

i.e., the function $i \mapsto j(i)$ is injective. Finally, since the family of components S_j is finite, we conclude that the family U_i is finite as well. This finishes the proof of Lemma 3.4. □

4 Appendix

For reader's convenience we recall here some steps of the corresponding arguments of Amick [1] for the proof of the convergence (3.35).

Our Lemma 3.4 implies, in particular, that there exists at least one unbounded component U_{k_1} where ω is strictly positive and at least one unbounded component U_{k_2} where ω is strictly negative (cf. with [1, Theorem 8, page 84]).

First of all we mention, that by [1, Theorem 15, page 95], if we take the number R_* large enough, then there holds the following statement

$$\nabla\omega(z) \neq 0 \quad \text{if } \omega(z) = 0 \text{ and } |z| \geq R_*. \tag{4.1}$$

This gives the possibility to clarify the geometrical and topological structure of the components U_i. Namely, $\Omega_* \cap \partial U_i$ consists of finitely many smooth (even analytical) curves.

Let U_i, $i = 1, \dots, M$ be a family of *unbounded* components U_i. Then Amick proved the following geometrical and analytical characterization for them:

Theorem 4.1 (see Theorem 11, page 89 in [1])**.** *For every U_i, $i = 1, \dots, M$,*

- *(α) The set $\Omega_* \cap \partial U_i$ has precisely two unbounded components which may be parametrised as $\{(x_j(s), y_j(s)) : s \in (0, \infty)\}$, $j = 1, 2$. In addition, $(x_j(0), y_j(0)) \in \{|z| = R_*\}$, s denotes the arc-length measure from these points, and the functions $x_j(\cdot)$ and $y_j(\cdot)$ are real-analytical (if we choose R_* large enough to have (4.1)). The function ω vanishes on these arcs and $|(x_j(s), y_j(s))| \to \infty$ as $s \to \infty$.*
- *($\alpha\alpha$) The maps $s \mapsto \Phi(x_j(s), y_j(s))$ are monotone decreasing and increasing on $(0, \infty)$, respectively, for $j = 1$ and $j = 2$.*

Since the Bernoulli pressure Φ is uniformly bounded, by Weierstrass Monotone convergence theorem we have that the functions $s \mapsto \Phi(x_j(s), y_j(s))$ have some limits as $s \to \infty$ for $j = 1, 2$. After the usual agreement that

$$p(z) \to 0 \qquad \text{as } |z| \to \infty, \tag{4.2}$$

and taking into account the convergence on the family of circles (1.9) we obtain

Corollary 4.1. *Functions from item ($\alpha\alpha$) of Theorem 4.1 have the same limit*

$$\Phi(x_j(s), y_j(s)) \to \frac{1}{2}|\mathbf{u}_\infty|^2 \qquad \text{as } s \to \infty. \tag{4.3}$$

The next step concerns the auxiliary function γ. One of the most important tool in [1] is the following assertion.

Theorem 4.2 (see Theorem 14, page 92 in [1])**.** *For every U_i, $i = 1, \dots, M$, the convergence*

$$\gamma(z) \to \frac{1}{2}|\mathbf{u}_\infty|^2 \qquad \text{uniformly as } |z| \to \infty,\ z \in U_i \tag{4.4}$$

holds.

Proof. We reproduce here a simplified version of the proof of Theorem 14 in [1, pages 92–94].

Take and fix an unbounded component U_i. We assume without loss of generality that $|\mathbf{u}_\infty| = 1$ and $\omega(z) > 0$ in U_i. By construction, we have

$$\omega \equiv 0 \text{ and } \gamma \equiv \Phi \text{ on } \Omega_* \cap \partial U_i. \tag{4.5}$$

Therefore, the convergence (4.4) for $z \in \partial U_i$ follows immediately from (4.3). Take arbitrary $\varepsilon > 0$ and consider the sufficiently large radius $R_\varepsilon > R_*$ such that

$$\big|\gamma(z) - \frac{1}{2}\big| < \varepsilon/2 \qquad \text{if } \ z \in \partial U_i \text{ and } \ |z| \geq R_\varepsilon. \tag{4.6}$$

Since $\omega(z) > 0$ in U_i and $\omega(z) = 0$ on $S_{R_\varepsilon} \cap \partial U_i$, we deduce from (4.6), by continuity of γ and by compactness arguments, that there exists $\delta = \delta_\varepsilon > 0$ satisfying the condition

$$|\gamma(z) - \frac{1}{2}| < \varepsilon/2 \qquad \text{if } \ z \in U_i, \ \ |z| = R_\varepsilon\,, \ \text{ and } \ \omega(z) < \delta. \tag{4.7}$$

Now take $R_2 > R_\varepsilon$ such that

$$\omega(z) < \delta \qquad \text{if } z \in U_i \text{ and } |z| > R_2. \tag{4.8}$$

Consider an arbitrary point $z_0 \in U_i$ with $|z_0| > R_2$. Since ω is an analytical nonconstant function, by the classical Morse–Sard theorem on critical values and by continuity of γ, there exists $z_1 \in U_i$ such that

$$|z_1| > R_2, \qquad |\gamma(z_1) - \gamma(z_0)| < \frac{\varepsilon}{2} \tag{4.9}$$

and

$$\nabla\omega(z) \neq 0 \qquad \text{if } \ \omega(z) = \omega(z_1) \ \text{ and } z \in U_i. \tag{4.10}$$

Denote $t_1 = \omega(z_1)$, then the connected component L of the level set $\{z \in U_i : \omega(z) = t_1\}$ containing the point z_1, is a smooth curve homeomorphic to the open interval $(-1, 1)$ (indeed, this curve could not be closed because of maximum principle for the vorticity ω). Evidently, the intersection of the curve L with the circle $S_{R_\varepsilon} = \{z : |z| = R_\varepsilon\}$ contains at least two points A and B such that z_1 lies between A and B with respect to L.[2] By construction, $\omega|_L \equiv t_1 < \delta$,

[2]Indeed, take an arbitrary diffeomorphic parametrization $f : (-1, 1) \to L$. Then $f(s_1) = z_1$ for some $s_1 \in (-1, 1)$, further, by construction we have

$$\omega(f(s)) \equiv t_1 > 0. \tag{4.11}$$

Then the closure of L is a compact set and, of course,

$$\text{dist}\,(f(s), \partial U_i) \to 0 \text{ as } |s| \to 1. \tag{4.12}$$

The property (4.11) guaranties that L is separated from the closed set $\{z \in \partial U_i : |z| \geq R_\varepsilon\}$. Therefore, by (4.12) we have $|f(s)| < R_\varepsilon$ when $|s|$ is sufficiently close to 1, and this (together with the assumption $|f(s_1)| = |z_1| > R_\varepsilon$) implies the existence of $s', s'' \in (-1, 1)$ such that $s' < s_1 < s''$ and $|f(s')| = |f(s'')| = R_\varepsilon$. Now we can take $A = f(s')$ and $B = f(s'')$.

thus by (4.7) we have

$$|\gamma(A) - \frac{1}{2}| < \varepsilon/2, \qquad |\gamma(B) - \frac{1}{2}| < \varepsilon/2. \tag{4.13}$$

This implies, by virtue of the monotonicity of γ along the curve L (see (3.56)), that $|\gamma(z_1) - \frac{1}{2}| < \varepsilon/2$. Taking into account the second inequality in (4.9), we obtain

$$\big|\gamma(z_0) - \frac{1}{2}\big| < \varepsilon. \tag{4.14}$$

In other words, for every point $z_0 \in U_i$ with $|z_0| > R_2$ we proved the estimate (4.14). Since $\varepsilon > 0$ was arbitrary, the required convergence (4.4) is established. □

Since there exist only finitely many components U_i, from Theorem 4.2 we obtain immediately

Corollary 4.2. *The convergence*

$$\gamma(z) \to \frac{1}{2}|\mathbf{u}_\infty|^2 \qquad \textit{uniformly as } \ |z| \to \infty \tag{4.15}$$

holds.

The function $\gamma = \Phi - \omega\psi$ is closely related to Φ; in particular, $\gamma = \Phi$ if $\omega = 0$ or $\psi = 0$. Having this in mind, it is possible to prove the same convergence as (4.15) for Φ instead of γ.

We assume without loss of generality that

$$\mathbf{u}_\infty = (1, 0). \tag{4.16}$$

Recall that by D. Gilbarg & H. Weinberger results [5] the convergence

$$\lim_{r\to+\infty} \int\limits_0^{2\pi} |\mathbf{u}(r,\theta) - \mathbf{u}_\infty|^2 d\theta = 0 \tag{4.17}$$

holds. In other words, since $\nabla\psi = \mathbf{u}^\perp = (-v, u)$, we have

$$\lim_{r\to+\infty} \frac{1}{r} \int\limits_{|z|=r} |\nabla\psi(z) - (0,1)|^2 \, ds = 0. \tag{4.18}$$

Form this fact and from the finiteness of the Dirichlet integral $\int\limits_{\Omega} |\nabla \mathbf{u}|^2 < \infty$ we obtain (see [1, pages 99–100] for details) the following asymptotic behaviour of the stream function ψ:[3]

$$\lim_{r\to+\infty} \frac{1}{r}|\psi(x,y) - y| = 0, \tag{4.19}$$

where $r = \sqrt{x^2+y^2}$. For any $\alpha > 0$ denote by Sect_α the sector

$$\mathrm{Sect}_\alpha = \{z = (x,y) \in \Omega_* : \frac{|y|}{|x|} \geq \alpha\}.$$

Since $r \leq c_\alpha |y|$ for $z \in \mathrm{Sect}_\alpha$, from (4.19) it follows that

$$\lim_{(x,y)\in S_\alpha,\ \sqrt{x^2+y^2}\to\infty} \left| \frac{\psi(x,y)}{y} - 1 \right| = 0. \tag{4.20}$$

Let us prove the convergence of Φ in any sector Sect_α.

Lemma 4.1 (see Theorem 17 and Corollary 18 on page 101 in [1])**.** *For any $\alpha > 0$ the uniform convergences*

$$|z|\omega(z) \to 0 \qquad \text{as } |z| \to \infty,\ z \in \mathrm{Sect}_\alpha, \tag{4.21}$$

$$\Phi(z) \to \frac{1}{2}|\mathbf{u}_\infty|^2 \qquad \text{as } |z| \to \infty,\ z \in \mathrm{Sect}_\alpha. \tag{4.22}$$

hold.

Proof. Fix $\alpha > 0$. Then

$$\forall z = (x,y) \in \mathrm{Sect}_{\frac{\alpha}{3}} : |z| \leq \tilde{c}_\alpha |y|. \tag{4.23}$$

Take $z_0 = (x_0, y_0) \in \mathrm{Sect}_\alpha$. Without loss of generality assume that $y_0 > 0$. Since

$$\int\limits_{\Omega_*} |\nabla\Phi|^2 < \infty,$$

[3] Stream function ψ is well defined by identity $\nabla\psi = \mathbf{u}^\perp$ in every simply–connected subdomain of Ω_*; in particular, ψ is well-defined in intersection of Ω_* with every of the four half spaces $\{(x,y) \in \mathbb{R}^2 : x > 0\}$, $\{(x,y) \in \mathbb{R}^2 : x \leq 0\}$, $\{(x,y) \in \mathbb{R}^2 : y > 0\}$, $\{(x,y) \in \mathbb{R}^2 : y \leq 0\}$. Since these definitions of ψ differ only by some additive constants, they have no influence on the asymptotic properties discussed here.

from Lemma 2.2, from the uniform convergence of the pressure to zero (see (4.2)) and from average convergence of the velocity to $\mathbf{u}_\infty = (1,0)$ (see (4.17)), we have that

$$\exists r \in [\frac{1}{4}y_0, \frac{1}{2}y_0] : \qquad \sup_{|z-z_0|=r} \left|\Phi(z) - \frac{1}{2}\right| \le \varepsilon_1(r_0), \tag{4.24}$$

where $r_0 = |z_0|$ and $\varepsilon_1(r_0) \to 0$ uniformly as $r_0 \to \infty$ (of course, this function $\varepsilon_1(r_0)$ depends also on the parameter α fixed above).

From (4.24) and from Corollary 4.2 we have

$$\sup_{|z-z_0|=r} |\omega(z)\psi(z)| \le \varepsilon_2(r_0), \tag{4.25}$$

where again $\varepsilon_2(r_0) \to 0$ uniformly as $r_0 \to \infty$. Denote by B_0 the disk $\{z \in \mathbb{R}^2 : |z - z_0| \le r\}$. By construction,

$$B_0 \subset \mathrm{Sect}_{\frac{\alpha}{3}}.$$

Then by (4.20),

$$\sup_{(x,y)\in B_0} \left|\frac{\psi(x,y)}{y} - 1\right| \to 0 \text{ as } r_0 \to \infty. \tag{4.26}$$

In particular,

$$\psi(y) \ge d_\alpha r_0 \tag{4.27}$$

if r_0 is sufficiently large, here the constant d_α depends on α only. From (4.27) and (4.25) we obtain immediately that

$$\sup_{|z-z_0|=r} |\omega(z)| \le \frac{1}{r_0}\varepsilon_3(r_0), \tag{4.28}$$

where again $\varepsilon_3(r_0) \to 0$ uniformly as $r_0 \to \infty$. By maximum principle,

$$|\omega(z_0)| \le \frac{1}{r_0}\varepsilon_3(r_0). \tag{4.29}$$

Thus, we have proved the asymptotic estimate (4.21). Then the convergence (4.22) follows immediately from (4.21) and (4.15). □

The convergence of Φ *outside* of the sectors Sect_α is more delicate and subtle question. Ch. Amick solved this problem [1] using level sets of the stream function ψ.

Define the stream function in the half-domain $\Omega_+ = \{(x,y) : x \ge 0,\ x^2 + y^2 \ge R_*^2\}$ and consider the set $C_+ = \{z \in \Omega_+ : \psi(z) = 0\}$ [4]. Then $\gamma =$

[4] The asymptotic behavior of $\psi(x,y)$ is similar to that of the linear function $g(x,y) = y$. Since the level set $\{(x,y) \in \Omega_+ : g(x,y) = 0\}$ is a ray $\{(x,y) \in \Omega_+ : y = 0\}$, the set C_+ goes to infinity as well, see also Lemma 4.2 for the precise formulation.

Φ on C_+ and from the convergence of γ (4.15) we obtain immediately that $\frac{1}{2}|\nabla\psi(z)|^2 = \frac{1}{2}|\mathbf{u}(z)|^2 \to \frac{1}{2}$ when $|z| \to \infty$, $z \in C_+$. In particular, $\nabla\psi \neq 0$ on C_+ if we choose the parameter R_* sufficiently large. Using similar arguments, Amick proved that the set C_+ has very simple geometrical structure.

Lemma 4.2 (see Lemma 20 on page 104 in [1]). *If the number R_* is chosen large enough, then the set C_+ is a smooth curve*

$$C_+ = \big\{(p_+(s), q_+(s)) : s \in [0, +\infty)\big\},$$

here p_+ and q_+ are real-analytic functions on $[0, \infty)$, $p_+(s) \to \infty$ and $\frac{q_+(s)}{p_+(s)} \to 0$ as $s \to \infty$. In addition,

$$|\mathbf{u}(p_+(s), q_+(s))| \to |\mathbf{u}_\infty| \quad \text{as } s \to \infty. \tag{4.30}$$

Of course, the similar assertion holds for another half-domain $\Omega_- = \{(x, y) : x \leq 0,\ x^2 + y^2 \geq R_*^2\}$. Using this Lemma and some classical estimates for the Laplace operator (recall, that $\omega = \Delta\psi$), Amick proved the required assertion:

Theorem 4.3 (see Theorem 21 (a) on page 1045 in [1]). *The convergence*

$$|\mathbf{u}(z)| \to |\mathbf{u}_\infty| \qquad \textit{uniformly as } \ |z| \to \infty. \tag{4.31}$$

holds.

Remark 4.1. The proof of Theorem 4.3 could be essentially simplified in comparison with the original version of [1]. Indeed, from the convergence (4.30) on the curve C_+, using the Lemmas 2.1–2.2 it is very easy to derive that there exists $\sigma > 0$ such that for any $z \in C_+$ with sufficiently large value $|z|$ we have

$$\left| \frac{1}{r} \int\limits_{|\xi - z| = r} \mathbf{u}(\xi)\, ds \right| > \sigma$$

for all $r \in (0, \frac{4}{5}|z|]$. Then the arguments of the proof of Lemma 3.3 (ii) of the present paper give us that

$$\mathbf{u}(p_+(s), q_+(s)) \to \mathbf{u}_\infty \quad \text{as } s \to \infty \tag{4.32}$$

instead of (4.30). This more strong convergence allows to simplify some technical moments in the proof of [1, Theorem 21 (a)], see also [1, Theorem 21 (c)].

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