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# *On the Flux Problem in the Theory of Steady Navier–Stokes Equations with Nonhomogeneous Boundary Conditions*

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**Abstract**

We study the nonhomogeneous boundary value problem for Navier–Stokes equations of steady motion of a viscous incompressible fluid in a two-dimensional, bounded, multiply connected domain $\Omega = \Omega_1 \setminus \overline{\Omega}_2$, $\overline{\Omega}_2 \subset \Omega_1$. We prove that this problem has a solution if the flux $\mathcal{F}$ of the boundary value through $\partial\Omega_2$ is nonnegative (inflow condition). The proof of the main result uses the Bernoulli law for a weak solution to the Euler equations and the one-sided maximum principle for the total head pressure corresponding to this solution.

## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, with multiply connected Lipschitz boundary $\partial\Omega$ consisting of $N$ disjoint components $\Gamma_j$: $\partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_N$ and $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$. Consider in $\Omega$ the stationary Navier–Stokes system with nonhomogeneous boundary conditions

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p = 0 & \text{in } \ \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \ \Omega, \\ \mathbf{u} = \mathbf{a} & \text{on } \ \partial\Omega. \end{cases} \tag{1.1}$$

The continuity equation ($1.1_2$) implies the necessary compatibility condition for the solvability of problem (1.1):

$$\int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S = \sum_{j=1}^{N}\int_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S = \sum_{j=1}^{N}\mathcal{F}_j = 0, \tag{1.2}$$

where $\mathbf{n}$ is a unit vector of the outward (with respect to $\Omega$) normal to $\partial\Omega$. The compatibility condition (1.2) means that the net flux of the fluid over the boundary $\partial\Omega$ is zero.

Starting from the famous paper of LERAY [27] published in 1933, problem (1.1) has been the subject of investigation in many papers. However, in spite of all efforts, the existence of a weak solution $\mathbf{u} \in W^{1,2}(\Omega)$ to problem (1.1) was proved only either under the stronger condition, which requires the fluxes $\mathcal{F}_j$ of the boundary value $\mathbf{a}$ to be zero separately across each component $\Gamma_j$ of the boundary $\partial\Omega$[1],

$$\mathcal{F}_j = \int_{\Gamma_j} \mathbf{a} \cdot \mathbf{n} \, \mathrm{d}S = 0, \qquad j = 1, 2, \ldots, N, \tag{1.3}$$

(for example [19,23,24,27,41]), or for sufficiently small fluxes $\mathcal{F}_j$[2] (for example [3,9,10,13,14,21,34,35], or under certain symmetry conditions on the domain $\Omega$ and the boundary value $\mathbf{a}$ (for example [1,11,31–33,37]).

Another interesting contribution to this problem is due to FUJITA and MORIMOTO [12] (see also [36]). They studied problem (1.1) in a plane domain $\Omega$ with two components of the boundary, $\Gamma_1$ and $\Gamma_2$. Assuming that $\mathbf{a} = \mathcal{F}\nabla u_0 + \boldsymbol{\alpha}$, where $\mathcal{F} \in \mathbb{R}$, $u_0$ is a harmonic function, and $\boldsymbol{\alpha}$ satisfies condition (1.3), they proved that there is a countable subset $\mathscr{N} \subset \mathbb{R}$ such that if $\mathcal{F} \notin \mathscr{N}$ and $\boldsymbol{\alpha}$ is small (in a suitable norm), then problem (1.1) has a weak solution. Moreover, if $\Omega \subset \mathbb{R}^2$ is an annulus and $u_0 = \log|x|$, then $\mathscr{N} = \emptyset$.

Problem (1.1), (1.3) can be reduced to an operator equation in a Hilbert space (with a compact operator) and the existence of a fixed-point to this equation can be proved by using the Leray–Schauder Theorem (for example [14,23,27]). In order to apply the Leray–Schauder Theorem, one needs an a priori estimate of solutions to the operator equation. In [27], LERAY initiated two different approaches of getting this estimate. The first method uses the extension of boundary value $\mathbf{a}$ into $\Omega$ as $\mathbf{A}(\varepsilon, x) = curl\,\big(\zeta(\varepsilon, x)\mathbf{b}(x)\big)$, where $\zeta(\varepsilon, x)$ is Hopf's cut-off function [18]. For this extension the estimate

$$-\int_\Omega \big(\mathbf{v} \cdot \nabla\big)\mathbf{A} \cdot \mathbf{v} \, \mathrm{d}x \leqq \varepsilon c \int_\Omega |\nabla \mathbf{v}|^2 \, \mathrm{d}x \quad \forall \;\; \mathbf{v} \in \mathring{W}^{1,2}(\Omega) \tag{1.4}$$

holds (for example [24]), where $\varepsilon > 0$ can be taken arbitrarily small and the constant $c$ is independent of $\varepsilon$ (in fact, one needs to know only that $\varepsilon c < \nu$). Usually (1.4) is called Leray–Hopf's inequality. It is well known that boundary value can be extended into the domain as a $curl$ only if condition (1.3) is satisfied. The counterexamples in [39] and [17] show that if the net flux across some component of the boundary is nonzero, then it is impossible to extend the boundary value $\mathbf{a}$ in any manner as a solenoidal function $\mathbf{A}$ satisfying Leray–Hopf's inequality (1.4). Thus, this approach may be applied only when condition (1.3) is valid.

The second approach in [27] is to prove an a priori estimate by a contradiction. Such arguments can also be found in the book of LADYZHENSKAYA [24]. In [1] the solvability of (1.1) was proved using this method for arbitrary fluxes $\mathcal{F}_j$, assuming only the necessary condition (1.2). However, the problem has been studied for a special class of plane symmetric domains and symmetric boundary

[1] Condition (1.3) does not allow the presence of sinks and sources.

[2] This condition does not assumes the norm of the boundary value $\mathbf{a}$ to be small.

values. An effective estimate for the solution of the Navier–Stokes problem with the above symmetry conditions was first obtained by Sazonov [37], who constructed a symmetric extension of the boundary data satisfying the Leray–Hopf inequality. Analogous results were independently obtained by Fujita [11] (see also [31]), who called the proposed method the "virtual drains method".

However, the fundamental question whether problem (1.1) is solvable for all values of $\mathcal{F}_j$ (Leray's problem) is still open, despite the efforts of many mathematicians (see the review papers [32,33,42]). In this paper we study problem (1.1) in a plane domain

$$\Omega = \Omega_1 \setminus \overline{\Omega}_2, \quad \overline{\Omega}_2 \subset \Omega_1, \tag{1.5}$$

where $\Omega_1$ and $\Omega_2$ are bounded, simply connected domains of $\mathbb{R}^2$ with Lipschitz boundaries $\partial\Omega_1 = \Gamma_1$, $\partial\Omega_2 = \Gamma_2$. Without loss of generality we may assume that $\Omega_2 \supset \{x \in \mathbb{R}^2 : |x| < 1\}$. Since $\Omega$ has only two components of the boundary, condition (1.2) may be rewritten in the form

$$\mathcal{F} = \int_{\Gamma_2} \mathbf{a} \cdot \mathbf{n} \, \mathrm{d}S = -\int_{\Gamma_1} \mathbf{a} \cdot \mathbf{n} \, \mathrm{d}S, \tag{1.6}$$

where $\mathbf{n}$ is an outward normal with respect to the domain $\Omega$. We prove the solvability of problem (1.1) without any restriction on the value of $|\mathcal{F}|$, provided that $\mathcal{F} > 0$ (inflow condition). Since it is known that problem (1.1) is solvable for sufficiently small $|\mathcal{F}|$ (see [3,9,10,13,21]), we conclude that the solution exists if $\mathcal{F} \in [-\mathcal{F}_0, \infty)$, where $\mathcal{F}_0$ is some positive number. Note that this is the first result on Leray's problem which does not require smallness of the flux or symmetry conditions on the domain and boundary values. The method proposed here works only for $\mathcal{F} > 0$. We have neither physical nor mathematical arguments for the existence or nonexistence of the solution to (1.1) in the case $\mathcal{F} < 0$ with large $|\mathcal{F}|$.

The proof of the existence theorem is based on an a priori estimate which we obtain using the *reductio ad absurdum* argument proposed by Leray [27]. The essentially new part in this argument is the use of the weak one-sided maximum principle for the total head pressure, corresponding to weak solutions of the Euler equations, and a representation of the total head pressure in the divergence form (see formula (4.20)), while the proof of the above maximum principle is based on the Bernoulli Law for a weak solution to the Euler equations. The results concerning the Bernoulli Law and the weak one-sided maximum principle for the total head pressure were obtained in [20]. However, the proofs there were not detailed; some steps were only sketched. Below (Section 3) we publish the first detailed proofs of these results.

The paper is organized as follows. Section 2 contains preliminaries. Basically, this section consists of standard facts[3] and we present them only in order to make the paper self contained. Results of Sections 2.1–2.3 are used in the proof of the Bernoulli Law, etc., results of Section 2.4 are used for reducing of problem (1.1) to

[3] Except for the results of Section 2.2, where we formulate the recent version [4] of the Morse-Sard Theorem for the space $W^{2,1}(\mathbb{R}^2)$, which plays the key role in the Section 3.

an operator equation (Section 4), and results of Section 2.5 are used for the proof of the continuity of the pressure in the Euler equations (Theorem 3.2). Section 3 is devoted to the above-mentioned results for the Euler equation. Finally, Section 4 contains the proof of the existence theorem for problem (1.1).

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## 2. Notation and Auxiliary Results

### *2.1. Function Spaces and Definitions*

By *a domain* we mean an open connected set. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary[4] $\partial\Omega$. We use standard notations for function spaces: $C^k(\overline{\Omega})$, $C^k(\partial\Omega)$, $W^{k,q}(\Omega)$, $\mathring{W}^{k,q}(\Omega)$, $W^{\alpha,q}(\partial\Omega)$, where $\alpha \in (0,1)$, $k \in \mathbb{N}_0$, $q \in [1,+\infty]$. $\mathcal{H}^1(\mathbb{R}^2)$ denotes the Hardy space on $\mathbb{R}^2$. In our notation we do not distinguish function spaces for scalar and vector valued functions; it is clear from the context whether we use scalar or vector (or tensor) valued function spaces. $H(\Omega)$ is subspace of all solenoidal vector fields ($\operatorname{div}\mathbf{u} = 0$) from $\mathring{W}^{1,2}(\Omega)$ with the norm $\|\mathbf{u}\|_{H(\Omega)} = \|\nabla\mathbf{u}\|_{L^2(\Omega)}$. Note that for functions $\mathbf{u} \in H(\Omega)$, the norm $\|\cdot\|_{H(\Omega)}$ is equivalent to $\|\cdot\|_{W^{1,2}(\Omega)}$.

Working with Sobolev functions, we always assume that the "best representatives" are chosen. If $w \in L^1_{\mathrm{loc}}(\Omega)$, then the best representative $w^*$ is defined by

$$w^*(x) = \begin{cases} \lim\limits_{r\to 0} ⨍_{B_r(x)} w(z)\mathrm{d}z, & \text{if the finite limit exists;} \\ 0 & \text{otherwise,} \end{cases}$$

where $⨍_{B_r(x)} w(z)\mathrm{d}z = \frac{1}{\operatorname{meas}(B_r(x))}\int_{B_r(x)} w(z)\mathrm{d}z$, $B_r(x) = \{y : |y-x| < r\}$ is a ball of radius $r$ centered at $x$.

Let us discuss some properties of the best representatives of Sobolev functions.

**Lemma 2.1.** (see, for example, Theorem 1 of Sect. 4.8 and Theorem 2 of Sect. 4.9.2 in [8]) *Let $w \in W^{1,s}(\mathbb{R}^2)$, $s \geqq 1$. Then there exists a set $A_{1,w} \subset \mathbb{R}^2$ such that*

(i) $\mathfrak{H}^1(A_{1,w}) = 0$;
(ii) *for each $x \in \Omega \setminus A_{1,w}$*

$$\lim_{r\to 0} ⨍_{B_r(x)} |w(z) - w(x)|^2\,\mathrm{d}z = 0;$$

(iii) *for all $\varepsilon > 0$ there exists a set $U \subset \mathbb{R}^2$ such that $\mathfrak{H}^1_\infty(U) < \varepsilon$, $A_{1,w} \subset U$ and the function $w$ is continuous in $\overline{\Omega} \setminus U$;*
(iv) *for any unit vector $\mathbf{l} \in \partial B_1(0)$ the restriction $w|_L$ is an absolutely continuous function (of one variable) for almost all straight lines $L$ parallel to the direction* $\mathbf{l}$.

4 $\partial\Omega$ is Lipschitz, if for every $\xi \in \partial\Omega$, there is a neighborhood of $\xi$ in which $\partial\Omega$ is the graph of a Lipschitz continuous function (defined on an open interval).

Here and henceforth we denote by $\mathfrak{H}^1$ the one-dimensional Hausdorff measure, that is, $\mathfrak{H}^1(F) = \lim_{t\to 0+} \mathfrak{H}^1_t(F)$, where $\mathfrak{H}^1_t(F) = \inf\{\sum_{i=1}^{\infty} \operatorname{diam} F_i \,:\, \operatorname{diam} F_i \leqq t,\, F \subset \bigcup_{i=1}^{\infty} F_i\}$.

**Remark 2.1.** Property (iii) above means that $f$ is quasicontinuous with respect to the Hausdorff content $\mathfrak{H}^1_\infty$. Really, Theorem 1(iii) of Sect. 4.8 in [8] asserts that $f \in W^{1,s}(\mathbb{R}^2)$ is quasicontinuous with respect to $s$-capacity. But it is well known that for $s = 1$ the smallness of the 1-capacity of a set $F \subset \mathbb{R}^2$ is equivalent to the smallness of $\mathfrak{H}^1_\infty(F)$ (see, for example, Theorem 3 of Sect. 5.6.3 in [8] and its proof.)

**Lemma 2.2.** *Let $w \in W^{1,s}(\mathbb{R}^2)$, $s \geqq 1$. Take any function $g \in C^1(\mathbb{R}^2)$ and a closed set $F \subset \mathbb{R}^2$ such that $\nabla g \neq 0$ on $F$. Then for almost all $y \in g(F)$ and for all the connected components $K$ of the set $F \cap g^{-1}(y)$, the equality $K \cap A_{1,w} = \emptyset$ holds and the restriction $w|_K$ is an absolutely continuous function.*

Lemma 2.2 follows from Lemma 2.1 (iv) by coordinate transformation (see [29, Sect. 1.1.7]).

To receive some specific version of the property (iv) in Lemma 2.1, we need

**Lemma 2.3.** *Let $f \in W^{1,s}(\mathbb{R}^2)$, $s \geqq 1$, and $x_0 \in \mathbb{R}^2 \setminus A_{1,f}$. Suppose*

$$\int_{B_1(x_0)} \frac{|\nabla f(x)|}{|x - x_0|}\, \mathrm{d}x < \infty. \tag{2.1}$$

*Then the restriction $f|_{L_{x_0}}$ is an absolutely continuous function (of one variable) for almost all rays $L_{x_0}$ starting from $x_0$.*

Lemma 2.3 is easily deduced from Lemma 2.1 (iv) using polar coordinates.

**Lemma 2.4.** *Let $f \in W^{1,s}(\mathbb{R}^2)$, $s > 1$. Then there exists a set $A_{2,f} \subset \mathbb{R}^2$ such that $\mathfrak{H}^1(A_{2,f}) = 0$ and for all $x_0 \in \mathbb{R}^2 \setminus A_{2,f}$ the restriction $f|_{L_{x_0}}$ is an absolutely continuous function (of one variable) for almost all rays $L_{x_0}$ starting from $x_0$.*

**Proof.** Put

$$A_{2,f} = A_{1,f} \cup \left\{ y \in \mathbb{R}^2 \,:\, \overline{\lim_{r\to 0}} \frac{\int_{B_r(y)} |\nabla f(x)|^s\, \mathrm{d}x}{r} > 0 \right\}.$$

By [8, Theorem 3, Sect. 2.4.3] the equality $\mathfrak{H}^1(A_{2,f}) = 0$ holds. Now take $x_0 \in \mathbb{R}^2 \setminus A_{2,f}$. Then there exists $C_0 > 0$ such that $\int_{B_r(x_0)} |\nabla f(x)|^s\, \mathrm{d}x \leqq C_0 r$ for all $r \in (0, 1)$. Denote $r_k = \frac{1}{2^k}$ and $B_k = B_{r_k}(x_0)$. By direct calculation,

$$\int_{B_k\setminus B_{k+1}} \frac{|\nabla f(x)|}{|x - x_0|}\, \mathrm{d}x \leqq \frac{2}{r_k} \int_{B_k\setminus B_{k+1}} |\nabla f(x)|\, \mathrm{d}x$$

$$\leqq \frac{2}{r_k} \left( \int_{B_k\setminus B_{k+1}} |\nabla f(x)|^s\, \mathrm{d}x \right)^{\frac{1}{s}} \left( \frac{3\pi}{4} r_k^2 \right)^{\frac{1}{s^*}} \leqq C_1 r_k^{\frac{2}{s^*} - 1 + \frac{1}{s}} = C_1 r_k^{\frac{1}{s^*}} = C_1 \left( 2^{-\frac{1}{s^*}} \right)^k.$$

Hence the convergence (2.1) follows easily from the last estimate. Now the assertion of Lemma 2.4 follows from Lemma 2.3. □

**Remark 2.2.** Because of the Sobolev Extension Theorems, the analogs of Lemmas 2.1–2.4 are true for functions $w \in W^{1,s}(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary. Because of Trace Theorems, each function $w \in W^{1,s}(\Omega)$ is "well-defined" for $\mathfrak{H}^1$-almost all $x \in \partial\Omega$. So henceforth we assume that each function $w \in W^{1,s}(\Omega)$ is defined on $\overline{\Omega}$.

### *2.2. On Morse-Sard and Luzin N-Properties of Sobolev Functions from $W^{2,1}$*

First of all, let us recall some classical differentiability properties of Sobolev functions.

**Lemma 2.5.** (see Proposition 1 in [7]) *Let $\psi \in W^{2,1}(\mathbb{R}^2)$. Then there exists a set $A_\psi \supset A_{1,\nabla\psi}$ such that $\mathfrak{H}^1(A_\psi) = 0$, and for all $x \in \mathbb{R}^2 \setminus A_\psi$, the function $\psi$ is differentiable (in the classical sense) at the point $x$; furthermore, the classical derivative coincides with $\nabla\psi(x)$.*

The theorems below in this subsection were proved by BOURGAIN et al. [4].

**Theorem 2.1.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and $\psi \in W^{2,1}(\Omega)$. Then*

(i) *for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $U \subset \overline{\Omega}$ with $\mathfrak{H}^1_\infty(U) < \delta$, the inequality $\mathfrak{H}^1(\psi(U)) < \varepsilon$ holds;*
(ii) *for every $\varepsilon > 0$ there exists an open set $V \subset \mathbb{R}$ and a function $g \in C^1(\mathbb{R}^2)$ such that $\mathfrak{H}^1(V) < \varepsilon$, and for each $x \in \overline{\Omega}$ if $\psi(x) \notin V$, then $x \notin A_\psi$, the function $\psi$ is differentiable at the point $x$, and $\psi(x) = g(x)$, $\nabla\psi(x) = \nabla g(x) \neq 0$.*

**Theorem 2.2.** *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary and $\psi \in W^{2,1}(\Omega)$. Then for $\mathfrak{H}^1$-almost all $y \in \psi(\overline{\Omega}) \subset \mathbb{R}$ the preimage $\psi^{-1}(y)$ is a finite disjoint family of $C^1$-curves $S_j$, $j = 1, 2, \ldots, N(y)$. Each $S_j$ is either a cycle in $\Omega$ (that is, $S_j \subset \Omega$ is homeomorphic to the unit circle $\mathbb{S}^1$) or it is a simple arc with endpoints on $\partial\Omega$ (in this case $S_j$ is transversal to $\partial\Omega$). Moreover, the tangent vector to each $S_j$ is an absolutely continuous function.*

### *2.3. Some Facts from Topology*

For the further considerations we will also need some topological definitions and results. By *continuum* we mean a compact connected set. We understand connectedness in the sense of general topology. A set is called *an arc* if it is homeomorphic to the unit interval $[0, 1]$. (Sometimes by arc we mean a corresponding parametrization, for example, a continuous injective function $\gamma : [\alpha, \beta] \to \mathbb{R}^2$).

**Lemma 2.6.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and let $K \subset \overline{\Omega}$ be a continuum. Then there exists $\delta > 0$ such that for any continuous injective function $\gamma : I = [0, 1] \to \overline{\Omega}$ with the properties[5] $\gamma(0), \gamma(1) \in K$, and $\gamma((0, 1)) \subset \{x \in \Omega : \mathrm{dist}(x, K) < \delta\}$, the following assertion is valid:*

[5] $\gamma((0, 1))$ can intersect $K$ or even $\gamma([0, 1]) \subset K$.

($\aleph$) *For any interval* $(\alpha, \beta)$ *adjoining the set* $\tilde{I} = \gamma^{-1}(K)$ (*that is,* $\tilde{I}$ *is a compact subset of the interval* $[0, 1]$ *and* $0, 1 \in \tilde{I}$, $(\alpha, \beta)$ *is a connected component of the open set* $(0, 1) \setminus \tilde{I}$) *there exists a continuum* $K_{\alpha\beta} \subset K$ *and a simply connected domain* $\Omega_{\alpha\beta} \subset \Omega$ *such that* $\Omega_{\alpha\beta} \cap K = \emptyset, \gamma(\alpha), \gamma(\beta) \in K_{\alpha\beta}$ *and* $\partial\Omega_{\alpha\beta} = K_{\alpha\beta} \cup \gamma([\alpha, \beta])$.

Lemma 2.6 is proved in the Appendix.

Here we briefly present some results from the classical paper [22] concerning level sets of continuous functions. Let $Q = [0, 1] \times [0, 1]$ be a square in $\mathbb{R}^2$ and let $f$ be a continuous function defined on $Q$. Denote by $E_t$ a level set of the function $f$, that is, $E_t = \{x \in Q : f(x) = t\}$. A component $K$ of the level set $E_t$ containing a point $x_0$ is a maximal connected subset of $E_t$ containing $x_0$. By $T_f$ denote a family of all connected components of level sets of $f$. Kronrod [22] has established that $T_f$ equipped by a natural topology is a tree. More precisely, he proved the following result.

**Lemma 2.7.** *Let* $f \in C(Q)$. *Then for any different* $A, B \in T_f$ *there exists a unique arc* $J = J(A, B) \subset T_f$ *joining* $A$ *to* $B$. *Moreover, for any inner point* $C$ *of this arc the points* $A, B$ *lie in different connected components of the set* $T_f \setminus \{C\}$.

We can reformulate the above Lemma in the following equivalent form.

**Lemma 2.8.** *Let* $f \in C(Q)$. *Then for any different* $A, B \in T_f$ *there exists an injective function* $\varphi : [0, 1] \to T_f$ *such that*

(i) $\varphi(0) = A, \varphi(1) = B$;
(ii) *for any* $t_0 \in [0, 1]$ *the convergence* $\lim_{[0,1]\ni t \to t_0} \sup_{x \in \varphi(t)} \operatorname{dist}(x, \varphi(t_0)) \to 0$ *holds.*
(iii) *for any* $t \in (0, 1)$ *the sets* $A, B$ *lie in the different connected components of the set* $Q \setminus \varphi(t)$.

**Remark 2.3.** Under conditions and notation of Lemma 2.8, define a function $g : [0, 1] \to \mathbb{R}$ by the rule $g(t) = f(x)$, where $x \in \varphi(t)$. Evidently, $g$ will be a continuous function nonconstant on each subinterval. Consequently, if, in addition, $f \in W^{2,1}(Q)$, then by Theorem 2.2 there exists a dense subset $E$ of $(0, 1)$ such that $\varphi(t)$ is a $C^1$-curve for each $t \in E$. Furthermore, $\varphi(t)$ is either a cycle or a simple arc with endpoints on $\partial Q$.

**Remark 2.4.** All results of Lemmas 2.7–2.8 remain valid for level sets of continuous functions defined on some compact set $R \subset \mathbb{R}^2$ whose boundary $\partial R$ is homeomorphic to the unit circle $\mathbb{S}^1$.

### *2.4. Some Facts About Solenoidal Functions*

The next two lemmas concern the existence of solenoidal extensions of boundary values and the integral representation of the bounded linear functionals vanish ing on solenoidal functions.

**Lemma 2.9.** (see [25]) *Let $\Omega$ be a bounded domain with Lipschitz boundary. If $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ and*

$$\int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S = 0,$$

*then there exists a solenoidal extension $\mathbf{A} \in W^{1,2}(\Omega)$ of $\mathbf{a}$ such that*

$$\|\mathbf{A}\|_{W^{1,2}(\Omega)} \leqq c\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}. \tag{2.2}$$

**Lemma 2.10.** (see [38]) *Let $\Omega$ be a bounded domain with Lipschitz boundary and let $R(\boldsymbol{\eta})$ be a continuous linear functional defined on $\mathring{W}^{1,2}(\Omega)$. If*

$$R(\boldsymbol{\eta}) = 0 \qquad \forall\ \boldsymbol{\eta} \in H(\Omega),$$

*then there exists a function $p \in L^2(\Omega)$ with $\int_\Omega p(x)\,\mathrm{d}x = 0$ such that*

$$R(\boldsymbol{\eta}) = \int_\Omega p\,\operatorname{div}\boldsymbol{\eta}\,\mathrm{d}x \qquad \forall\ \boldsymbol{\eta} \in \mathring{W}^{1,2}(\Omega).$$

*Moreover, $\|p\|_{L^2(\Omega)}$ is equivalent to $\|R\|_{(\mathring{W}^{1,2}(\Omega))^*}$.*

## *2.5. Some Facts from Harmonic Analysis*

**Lemma 2.11.** *Let $f \in \mathcal{H}^1(\mathbb{R}^2)$ and*

$$J(x) = \int_{\mathbb{R}^2} \log|x-y|\, f(y)\,\mathrm{d}y. \tag{2.3}$$

*Then*

(i) $J \in C(\mathbb{R}^2)$;
(ii) $\nabla J \in L^2(\mathbb{R}^2)$, $D^\alpha J \in L^1(\mathbb{R}^2)$, $|\alpha| = 2$.

Lemma 2.11 is well known; the proof of property (i) can be found in [40, Theorem 5.12 and Corollary 12.12 at pp. 82–83]; the property (ii) is proved, for example, in [2, Theorem 5.13, p. 208].

**Lemma 2.12.** *Let $\mathbf{w} \in W^{1,2}(\mathbb{R}^2)$ and $\operatorname{div}\mathbf{w} = 0$. Then*

$$\operatorname{div}\big[(\mathbf{w}\cdot\nabla)\mathbf{w}\big] = \sum_{i,j=1}^{2} \frac{\partial w_i}{\partial x_j}\frac{\partial w_j}{\partial x_i} \in \mathcal{H}^1(\mathbb{R}^2).$$

Lemma 2.12 follows from the div-curl lemma with two cancellations (for example, [6, Theorem II.1]).

**Lemma 2.13.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and $h \in C(\partial\Omega)$. If $h$ can be extended into domain $\Omega$ as a function $H \in W^{1,2}(\Omega)$, then there exists a unique weak solution $v \in W^{1,2}(\Omega)$ of the problem*

$$\begin{cases} -\Delta v = 0 & \text{in} \quad \Omega, \\ \quad\ v = h & \text{on} \quad \partial\Omega \end{cases} \tag{2.4}$$

*such that $v \in C(\overline{\Omega})$.*

The proof of Lemma 2.13 can be found in [28] (see also [26, Theorem 4.2]). Note that not every function $h$ continuous on $\partial\Omega$ can be extended into $\Omega$ as a function $H$ from $W^{1,2}(\Omega)$. In this case there exists a weak solution $v$ of (2.4) satisfying only $v \in W^{1,2}_{loc}(\Omega) \cap C(\overline{\Omega})$ (see [26, Chapter II]).

## 3. Euler Equation

In this section we prove some properties of a solution to the Euler system

$$\begin{cases} (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla p = 0, \\ \operatorname{div}\mathbf{w} = 0. \end{cases} \tag{3.1}$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. Assume that $\mathbf{w} \in W^{1,2}(\Omega)$ and $p \in W^{1,s}(\Omega)$, $s \in [1,2)$, satisfy the Euler equations (3.1) for almost all $x \in \Omega$, and let $\int_{\Gamma_i} \mathbf{w}\cdot\mathbf{n}\,\mathrm{d}S = 0$, $i = 1, 2, \dots, N$, where $\Gamma_i$ are connected components of the boundary $\partial\Omega$. Then there exists a stream function $\psi \in W^{2,2}(\Omega)$ such that $\nabla\psi = (-w_2, w_1)$ (note that by Sobolev Embedding Theorem $\psi$ is continuous in $\overline{\Omega}$). Denote by $\Phi = p + \dfrac{|\mathbf{w}|^2}{2}$ the total head pressure corresponding to the solution $(\mathbf{w}, p)$. Obviously, $\Phi \in W^{1,s}(\Omega)$ for all $s \in [1,2)$. By direct calculations one easily gets the identity

$$\nabla\Phi \equiv \Big(\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}\Big)\big(w_2, -w_1\big) = (\Delta\psi)\nabla\psi. \tag{3.2}$$

Applying Lemmas 2.1, 2.2, 2.4, 2.5, and Remark 2.2 to the functions $\mathbf{w}$, $\psi$, $\Phi$, we get the following

**Theorem 3.1.** *There exists a set $A_{\mathbf{w}} \subset \overline{\Omega}$ such that:*

(i) $\mathfrak{H}^1(A_{\mathbf{w}}) = 0$;
(ii) *for all $x \in \Omega \setminus A_{\mathbf{w}}$*

$$\lim_{r\to 0} \fint_{B_r(x)} |\mathbf{w}(z) - \mathbf{w}(x)|^2 \mathrm{d}z = \lim_{r\to 0} \fint_{B_r(x)} |\Phi(z) - \Phi(x)|^2 \mathrm{d}z = 0;$$

*moreover, the function $\psi$ is differentiable at $x$ and $\nabla\psi(x) = (-w_2(x), w_1(x))$;*
(iii) *for all $\varepsilon > 0$ there exists a set $U \subset \mathbb{R}^2$ such that $\mathfrak{H}^1_\infty(U) < \varepsilon$, $A_{\mathbf{w}} \subset U$ and the functions $\mathbf{w}$, $\Phi$ are continuous in $\overline{\Omega} \setminus U$;*
(iv) *for any two points $a, b \in \overline{\Omega} \setminus A_{\mathbf{w}}$ there exists a Lipschitz function (an arc) $\gamma : [0,1] \to \overline{\Omega} \setminus A_{\mathbf{w}}$, $\gamma(0) = a$, $\gamma(1) = b$, $\gamma((0,1)) \subset \Omega$ such that $\Phi\circ\gamma$ is an absolutely continuous function and*

$$\big[\Phi(\gamma(t))\big]' \equiv \big[\Delta\psi(\gamma(t))\big]\nabla\psi(\gamma(t))\cdot\gamma'(t) \quad \textit{for almost all} \quad t \in [0,1]. \tag{3.3}$$

(v) *Take any function $g \in C^1(\mathbb{R}^2)$ and a closed set $F \subset \overline{\Omega}$ such that $\nabla g \neq 0$ on $F$. Then for almost all $y \in g(F)$ and for all connected components $K$ of the set $F \cap g^{-1}(y)$ the equality $K \cap A_{\mathbf{w}} = \emptyset$ holds and the restriction $\Phi|_K$ is absolutely continuous. Moreover, for any $C^1$-smooth parametrization $\gamma : [0,1] \to K$, $\gamma' \neq 0$ on $[0,1]$, the identity (3.3) holds.*

If all functions are smooth, then from (3.2) the classical Bernoulli law follows immediately:

*The total head pressure $\Phi(x)$ is constant along any streamline of the flow.*

In the general case the following assertion holds.

**Theorem 3.2.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, multiply connected domain with Lipschitz boundary $\partial\Omega = \cup_{i=1}^N \Gamma_i$. Assume that $\mathbf{w} \in W^{1,2}(\Omega)$ and $p \in W^{1,s}(\Omega)$, $s \in [1,2)$, satisfy Euler equations* (3.1) *for almost all $x \in \Omega$ and $\int_{\Gamma_i} \mathbf{w} \cdot \mathbf{n}\, \mathrm{d}S = 0$, $i = 1, \dots, N$. Then for any connected set $K \subset \overline{\Omega}$ such that*

$$\psi\big|_K = \text{const}, \tag{3.4}$$

*the assertion*

$$\exists C = C(K) \quad \Phi(x) = C \quad \textit{for} \quad \mathfrak{H}^1\textit{-almost all } x \in K \tag{3.5}$$

*holds.*

Theorem 3.2 was obtained in [20, Theorem 1]. But the proofs in [20] were not detailed, some steps were only sketched. Here we publish the first detailed proof of this result.

**Proof.** (i) Fix any $\varepsilon > 0$ and consider a function $g \in C^1(\mathbb{R}^2)$ and an open set $V$ with $\mathfrak{H}^1(V) < \varepsilon$ from Theorem 2.1 (ii) applied to the function $\psi$. Put $F = \overline{\Omega} \setminus \psi^{-1}(V)$. Then $\psi(x) = g(x)$ and $\nabla\psi(x) = \nabla g(x) \neq 0$ for any $x \in F$. Thus, by Theorem 3.1 (v) for almost all $y \in \psi(\overline{\Omega}) \setminus V = g(F)$, for any connected component $K$ of the set $\psi^{-1}(y)$ (that is, for any streamline) and for any $C^1$-smooth parametrization $\gamma : [0,1] \to K$ the restriction $\Phi|_K$ is absolutely continuous, and identity (3.3) gives

$$[\Phi(\gamma(t))]' = [\Delta\psi(\gamma(t))]\nabla\psi(\gamma(t)) \cdot \gamma'(t) = [\Delta\psi(\gamma(t))]\nabla g(\gamma(t)) \cdot \gamma'(t) = 0.$$

The last equality is valid because $g(x) = \text{const}$ on $K$ and, hence, $\nabla g(\gamma(t)) \cdot \gamma'(t) = 0$, so we have $\Phi(x) = \text{const}$ on $K$. In view of arbitrariness of $\varepsilon > 0$, we have proved that for almost all $y \in \psi(\overline{\Omega})$ and for all connected components $K$ of the set $\psi^{-1}(y)$, the equality $K \cap A_{\mathbf{w}} = \emptyset$ holds[6] and $\Phi(x) = \text{const}$ on $K$. Notice that the last identity is valid everywhere on $K$, instead of almost everywhere, but for almost all $y \in \psi(\overline{\Omega})$ only. Here (only during this proof!) such components $K$ will be called *regular components*.

(ii) Now take an arbitrary value $y \in \psi(\overline{\Omega})$ and a connected component $K$ of the level set $\psi^{-1}(y)$ and fix them. Take also any pair of points $a, b \in K \setminus A_{\mathbf{w}}$. We shall prove that

$$\Phi(a) = \Phi(b). \tag{3.6}$$

Consider a Lipschitz function $\gamma : [0,1] \to \overline{\Omega} \setminus A_{\mathbf{w}}$ from the assertion (iv) of Theorem 3.1. There is considerable arbitrariness in the choice of this $\gamma$ (recall, that $\Phi$ is absolutely continuous along almost all straight segments) and we can choose $\gamma$ to satisfy the condition ($\aleph$) of Lemma 2.6. Now take any interval $(\alpha, \beta)$ adjoining the set $\tilde{I} = \gamma^{-1}(K)$, and consider the corresponding subdomain $\Omega_{\alpha\beta}$. Denote by $T$ the family of all connected components of level sets of the function

[6] For this equality see Theorems 3.1 (i) and 2.1 (i).

$\psi_{\alpha\beta} = \psi|_{\overline{\Omega_{\alpha\beta}}}$. According to Lemma 2.7 the topological space $T$ is a tree. Let $t_0 \in (\alpha, \beta)$, $K_0 \ni \gamma(t_0)$ be a connected component of the level set of $\psi_{\alpha\beta}$, and let $K_{\alpha\beta} \subset K$ be a continuum from the property ($\aleph$). Denote by $J = J(K_{\alpha\beta}, K_0)$ the arc (of the tree $T$) joining the points $K_{\alpha\beta}$ and $K_0$ (see Lemmas 2.7–2.8). Take a sequence of regular components $C_i \in J \setminus \{K_{\alpha\beta}, K_0\}$, $C_i \to K_{\alpha\beta}$ (this is possible because of Remark 2.3). Then the sets $K_0 \ni \gamma(t_0)$, $K_{\alpha\beta} \supset \{\gamma(\alpha), \gamma(\beta)\}$ lie in different connected components of the set $\overline{\Omega_{\alpha\beta}} \setminus C_i$ for all $i$. Therefore, there exist $t_i \in (\alpha, t_0)$ and $s_i \in (t_0, \beta)$ such that $\gamma(t_i), \gamma(s_i) \in C_i$. Since $C_i \to K_{\alpha\beta}$, we obtain $t_i \to \alpha, s_i \to \beta$. By paragraph (i) $\Phi(x) \equiv$ const on $C_i$. In particular, $G(t_i) = G(s_i)$, where by $G$ we denote the absolutely continuous function $G(t) = \Phi(\gamma(t))$. Since $G$ is continuous, it follows that $G(\alpha) = G(\beta)$ for any interval $(\alpha, \beta)$ adjoining the set $\tilde{I}$ and, since the absolutely continuous function $G(t)$ is differentiable almost everywhere, we obtain

$$\int_\alpha^\beta G'(t)\mathrm{d}t = 0.$$

Hence,

$$\int_\mu^\nu G'(t)\mathrm{d}t = 0 \tag{3.7}$$

if $\mu, \nu \in \tilde{I}$ and the interval $(\mu, \nu)$ contains only a finite number of points from $\tilde{I}$.

Now consider the closed set $I_\infty = \{t \in [0, 1]$: *in any neighborhood of the point $t$ there exist infinitely many points from $\tilde{I}$*$\}$. It follows from (3.7) that

$$\int_{[0,1]\setminus I_\infty} G'(t)\mathrm{d}t = 0. \tag{3.8}$$

According to properties (ii) and (iv) in Theorem 3.1, the function $\psi$ is differentiable at any point $\gamma(t)$, $t \in (0, 1)$. Furthermore, the properties of Lipschitz functions imply that the function $\gamma(t)$ is differentiable for almost all $t \in [0, 1]$. Clearly, if for $t \in I_\infty$ there exists $\gamma'(t)$, then $\gamma'(t) \cdot \nabla\psi(\gamma(t)) = 0$ (since $\psi$ is equal to a constant at points $\gamma(I_\infty) \subset \gamma(\tilde{I}) \subset K$). In view of formula (3.3), we then immediately derive

$$\int_{I_\infty} G'(t)\mathrm{d}t = 0. \tag{3.9}$$

Summing formulas (3.8) and (3.9) we get

$$G(1) - G(0) = \int_0^1 G'(t)\mathrm{d}t = 0.$$

The last relation is equivalent to equality (3.6). The theorem is proved. □

**Remark 3.1.** Really, we have proved in Theorem 3.2 that the identity

$$\Phi(a) = \Phi(b) \quad \forall a, b \in K \setminus A_{\mathbf{w}}$$

holds for any connected set $K \subset \overline{\Omega}$ with $\psi|_K =$ const.

**Remark 3.2.** In particular, if $\mathbf{w} = 0$ on $\partial\Omega$ (in the sense of trace), then the pressure $p(x)$ is constant on $\partial\Omega$. Note that $p(x)$ could take different constant values $p_j = p(x)\big|_{\Gamma_j}$ on different connected components $\Gamma_j$ of the boundary $\partial\Omega$. This statement was also proved in [19, Lemma 4] and in [1, Theorem 2.2].

**Theorem 3.3.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, multiply connected domain with Lipschitz boundary $\partial\Omega = \bigcup_{i=1}^N \Gamma_i$. Assume that $(\mathbf{w}, p)$ satisfy the Euler equations (3.1) for almost all $x \in \Omega$, $\mathbf{w} \in W^{1,2}(\Omega)$ and $\mathbf{w}(x)\big|_{\partial\Omega} = 0$. Then*

$$p \in C(\overline{\Omega}) \cap W^{1,2}(\Omega). \tag{3.10}$$

**Proof.** From the Euler equations (3.1) it follows that $p \in W^{1,s}(\Omega)$ for any $s \in [1, 2)$ and

$$\|p\|_{W^{1,s}(\Omega)} \leqq c\|\mathbf{w}\|^2_{H(\Omega)}.$$

Multiply (3.1) by $\boldsymbol{\varphi} = \nabla\xi$, where $\xi \in C_0^\infty(\Omega)$:

$$\int\limits_\Omega \nabla p \cdot \nabla\xi \, \mathrm{d}x = -\int_\Omega \big(\mathbf{w} \cdot \nabla\big)\mathbf{w} \cdot \nabla\xi \, \mathrm{d}x \quad \forall\xi \in C_0^\infty(\Omega).$$

Thus, $p \in W^{1,s}(\Omega)$ can be interpreted as the unique weak solution of the Dirichlet boundary value problem for the Poisson equations

$$\begin{cases} -\Delta p = \operatorname{div}\big[\big(\mathbf{w} \cdot \nabla\big)\mathbf{w}\big] & \text{in} \quad \Omega, \\ p(x) = p_i & \text{on} \quad \Gamma_i, \ \ i = 1, \dots, N, \end{cases} \tag{3.11}$$

where $p_i$ are constants. According to Lemma 2.12, $\operatorname{div}\big[\big(\mathbf{w} \cdot \nabla\big)\mathbf{w}\big] \in \mathcal{H}^1(\mathbb{R}^2)$ (here we assume that $\mathbf{w} \in H(\Omega)$ is extended by zero to $\mathbb{R}^2$). Define the function $J_1(x)$ by the formula

$$J_1(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x - y| \operatorname{div}_y \big[\big(\mathbf{w}(y) \cdot \nabla_y\big)\mathbf{w}(y)\big] dy.$$

In virtue of Lemma 2.11, $J_1 \in C(\mathbb{R}^2)$, $\nabla J_1 \in L^2(\mathbb{R}^2)$, $D^\alpha J_1 \in L^1(\mathbb{R}^2)$, $|\alpha| = 2$. Since $-\Delta J_1(x) = \operatorname{div}\big[\big(\mathbf{w} \cdot \nabla\big)\mathbf{w}\big]$ in $\mathbb{R}^2$, we get for $J_2(x) = p(x) - J_1(x)$ the following problem

$$\begin{cases} -\Delta J_2 = 0 & \text{in} \quad \Omega, \\ J_2\big|_{\partial\Omega} = j_2 - j_1 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.12}$$

where $j_1(x) = J_1(x)\big|_{\partial\Omega}$, $j_2(x) = p_i$ on $\Gamma_i$, $i = 1, \dots, N$. The function $j_1$ is a trace on $\partial\Omega$ of $J_1 \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$, while $j_2 \in C(\partial\Omega)$ and $j_2$ obviously could be extended to $\Omega$ as a function from $W^{1,2}(\Omega)$. Thus, by Lemma 2.13, problem (3.12) has a unique weak solution $J_2 \in W^{1,2}(\Omega)$ such that $J_2 \in C(\overline{\Omega})$. By uniqueness $p(x) = J_1(x) + J_2(x)$. Hence, $p \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$. The lemma is proved. □

Let $\Omega$ be a bounded domain with Lipschitz boundary. We say that the function $f \in W^{1,s}(\Omega)$ satisfies a *one-sided maximum principle locally* in $\Omega$, if

$$\operatorname*{ess\,sup}_{x\in\Omega'} f(x) \leqq \operatorname*{ess\,sup}_{x\in\partial\Omega'} f(x) \tag{3.13}$$

holds for any strictly interior subdomain $\Omega'$ ($\overline{\Omega}' \subset \Omega$) with the boundary $\partial\Omega'$ not containing singleton connected components. (In (3.13) negligible sets are the sets of two-dimensional Lebesgue measure zero in the left ess sup, and the sets of one-dimensional Hausdorff measure zero in the right ess sup.)

If (3.13) holds for any $\Omega' \subset \Omega$ (not necessary strictly interior) with the boundary $\partial\Omega'$ not containing singleton connected components, then we say that $f \in W^{1,s}(\Omega)$ satisfies a *one-sided maximum principle* in $\Omega$ (in particular, we can take $\Omega' = \Omega$ in (3.13)).

**Theorem 3.4.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, multiply connected domain with Lipschitz boundary $\partial\Omega = \cup_{i=1}^N \Gamma_i$. Let $\mathbf{w} \in W^{1,2}(\Omega)$ and $p \in W^{1,s}(\Omega)$, $s \in [1,2)$, satisfy Euler equations (3.1) for almost all $x \in \Omega$ and $\int_{\Gamma_i} \mathbf{w}\cdot\mathbf{n}\,dS = 0$, $i = 1,\dots,N$. Assume that there exists a sequence of functions $\{\Phi_\mu\}$ such that $\Phi_\mu \in W^{1,s}_{\rm loc}(\Omega)$ and $\Phi_\mu \rightharpoonup \Phi$ in the space $W^{1,s}_{\rm loc}(\Omega)$ for some $s \in [4/3, 2)$. If all $\Phi_\mu$ satisfy the one-sided maximum principle locally in $\Omega$, then $\Phi$ satisfies the one-sided maximum principle in $\Omega$.*

Theorem 3.4 was obtained in [20, Theorem 2]. But the proofs in [20] were not detailed, some steps were only sketched. Below we publish the first detailed proof of this result, but first we need some auxiliary results.

**Lemma 3.1.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and $\psi \in W^{2,1}(\Omega)$. Suppose that $\Omega' \subset \Omega$ is a domain, $\Gamma \subset \partial\Omega'$ is a compact connected set, $\psi|_\Gamma \neq \text{const}$, and $K_0 \subset \overline{\Omega}'$ is a connected component of some level set of $\psi|_{\bar{\Omega}'}$ such that*

$$K_0 \cap \Omega' \neq \emptyset, \quad K_0 \cap \Gamma \neq \emptyset. \tag{3.14}$$

*Then for any set $E \subset \overline{\Omega}$ with $\mathfrak{H}^1(E) = 0$ there exists a sequence of connected compact sets $K_i \subset \overline{\Omega}'\setminus E$ and points $x_i \in K_i$ such that $\psi|_{K_i} \equiv c_i = \text{const}$, $K_i \cap \partial\Omega' \neq \emptyset$, $\operatorname{diam} K_i \geqq \delta$ for some $\delta > 0$, and $x_i \to x_0 \in K_0$.*

**Proof.** Fix a set $E \subset \overline{\Omega}$ with $\mathfrak{H}^1(E) = 0$. Here (during this proof) a value $y \in \psi(\overline{\Omega})$ is called *admissible* if $\psi^{-1}(y) \cap E = \emptyset$ and the assertions of Theorem 2.2 are fulfilled. From Theorems 2.1 (i), 2.2 it follows that $\mathfrak{H}^1$-almost all $y \in \psi(\overline{\Omega})$ are admissible.

Let $\psi|_{K_0} \equiv y_0$. Take a sequence of admissible values $y_i^+ \to y_0 + 0$, $y_i^- \to y_0 - 0$. Denote by $C_i$ the connected component of the compact set $\{x \in \overline{\Omega}' : \psi(x) \in [y_i^-, y_i^+]\}$ such that $K_0 \subset C_i$. Then, obviously, $C_i \to K_0$ with respect to the Hausdorff distance.[7]

[7] Recall that the Hausdorff distance $d_H$ between two compact sets $A, B \subset \mathbb{R}^n$ is defined as follows: $d_H(A,B) = \max\bigl(\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\bigr)$ (see, for example, Sect. 7.3.1

Denote $T_i = \Omega' \cap \partial C_i$. Clearly,

$$\psi(x) = y_i^+ \quad \textit{or} \quad \psi(x) = y_i^- \quad \text{for any} \quad x \in T_i.$$

We shall say that a connected component $S$ of the set $T_i$ is *interesting* if $\bar{S} \cap \partial\Omega' \neq \emptyset$; otherwise $S$ is called *uninteresting*. Since the values $y_i^+$, $y_i^-$ are admissible (see the beginning of the proof), we see that

(**) Each uninteresting component $S$ of the set $T_i$ is a $C^1$-smooth curve homeomorphic to the unit circle. Moreover the set $\mathbb{R}^2 \setminus S$ has two connected components $U, V$ such that $U \cap C_i = \emptyset$ and $V \cap \{x \in \mathbb{R}^2 : \operatorname{dist}(x, S) < \delta_S\} \subset C_i$, where $\delta_S > 0$.

Denote by $\{T_j^i\}$ the family of all interesting connected components of $T_i$. To finish the proof of the Lemma, it is sufficient to check the convergence

$$\varlimsup_{i\to\infty} \sup_j \operatorname{diam} T_j^i > 0. \tag{3.15}$$

Suppose the convergence (3.15) does not hold, that is,

$$\lim_{i\to\infty} \sup_j \operatorname{diam} T_j^i = 0. \tag{3.16}$$

Take any point $z \in \Gamma$ such that $\psi(z) \neq y_0$. Then there exists $\delta > 0$ such that

$$B_\delta(z) \cap C_i = \emptyset \quad \text{for sufficiently large } i, \tag{3.17}$$

where $B_\delta(z)$ is a ball with center $z$ and radius $\delta$. Now fix points $z_0 \in K_0 \cap \Omega'$, $z_1 \in B_\delta(z) \cap \Omega'$. Take an arc $\gamma$ joining $z_0$ to $z_1$ in $\Omega'$. More precisely, $\gamma : [0, 1] \to \Omega'$ is a continuous injective function such that $\gamma(0) = z_0, \gamma(1) = z_1$. Denote $t_i = \sup\{t \in [0, 1] : \gamma(t) \in C_i\}$. Evidently, $\gamma(t_i) \in T_i$ for sufficiently large $i$. From (3.16) and the inequality $\inf_{t\in[0,1]} \operatorname{dist}(\gamma(t), \partial\Omega') > 0$ it follows that $\gamma(t_i)$ belongs to the uninteresting component $S_i$ of the set $T_i$ for $i \geqq i_0$. Denote by $U_i, V_i$ the corresponding components of $\mathbb{R}^2 \setminus S_i$ (see (**)). Then by construction we see that $\gamma(t) \in U_i$ for all $i \geqq i_0$ and $t \in (t_i, 1]$; in particular, $z_1 \in U_i$. From (3.17) it follows that $B_\delta(z) \cap S_i = \emptyset$. Hence $B_\delta(z) \subset U_i$. From the definition of uninteresting components it follows that $S_i \cap \Gamma = \emptyset$, consequently, $\Gamma \subset U_i \subset \mathbb{R}^2 \setminus C_i$. The last inclusions contradict assumption (3.14) and, hence, inequality (3.15) is valid. The lemma is proved. □

Footnote 7 continued
in [5]). By the Blaschke Selection Theorem, if $X \subset \mathbb{R}^n$ is a compact set, then the space of all compact subsets of $X$ equipped with the Hausdorff distance is a compact set, as well [ibid]. In other words, for any uniformly bounded sequence of compact sets $A_i \subset \mathbb{R}^n$, there exists a subsequence $A_{i_j}$ which converges to some compact set $A_0$ with respect to the Hausdorff distance. Of course, if all $A_i$ are compact connected sets and $\operatorname{diam} A_i \geqq \delta$ for some $\delta > 0$, then the limit set $A_0$ is also connected and $\operatorname{diam} A_0 \geqq \delta$ (we will use these elementary properties below).

**Lemma 3.2.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and let* $\mathbf{w}$, *$p$ satisfy the conditions of Theorem 3.2. Assume that $K_i \subset \overline{\Omega}$ is a sequence of connected compact sets such that* $\operatorname{diam} K_i \geqq \delta > 0$ *and* $\psi|_{K_i} \equiv c_i = \text{const}$. *Take any converging sequence of points $K_i \ni x_i \to x_0$, and denote by $K_{x_0}$ the connected component of the level set $\{x \in \overline{\Omega} : \psi(x) = \psi(x_0)\}$ containing $x_0$. Then for any $y_i \in K_i \setminus A_{\mathbf{w}}$ and for any $y_0 \in K_{x_0} \setminus A_{\mathbf{w}}$ the equality*

$$\lim_{i\to\infty} \Phi(y_i) = \Phi(y_0) \tag{3.18}$$

*holds.*

**Proof.** We may assume without loss of generality that

$$K_i \to K_0 \quad \text{with respect to the Hausdorff distance} \tag{3.19}$$

(see the footnote in the previous proof of Lemma 3.1). Then, by our assumptions, $K_0$ is a compact connected set, $K_0 \subset K_{x_0}$, and $\operatorname{diam} K_0 \geqq \delta > 0$. Take a straight line $L$ such that the projection of $K_0$ on $L$ is not a singleton. Since $K_0$ is a connected set, we see that this projection is a segment. By $I_0$ denote the interior of this segment. For $z \in I_0$, by $L_z$ denote the straight line such that $z \in L_z$ and $L_z \perp L$. From Lemma 2.1 (iv) it follows that for $\mathfrak{H}^1$-almost all $z \in I_0$, we have $L_z \cap \overline{\Omega} \subset \overline{\Omega} \setminus A_{\mathbf{w}}$ and the restriction $\Phi|_{\overline{\Omega}\cap L_z}$ is continuous. Fix a point $z \in I_0$ with the above properties. Then by construction $\emptyset \neq K_i \cap L_z \subset \overline{\Omega} \setminus A_{\mathbf{w}}$ for sufficiently large $i$. Now take a sequence $y_i \in K_i \setminus A_{\mathbf{w}}$ and a point $y_0 \in K_{x_0} \setminus A_{\mathbf{w}}$. From Remark 3.1 it follows that $\Phi(y_i) = \Phi(x)$ for any $x \in K_i \setminus A_{\mathbf{w}}$. Hence, we may assume without loss of generality that $y_i \in L_z$ and $\lim_{i\to\infty} y_i = y_* \in L_z \cap K_0 \subset \overline{\Omega} \setminus A_{\mathbf{w}}$. By continuity of $\Phi|_{\overline{\Omega}\cap L_z}$,

$$\lim_{i\to\infty} \Phi(y_i) = \Phi(y_*).$$

By Remark 3.1, $\Phi(y_0) = \Phi(y_*)$ and we get the required equality (3.18). □

Under conditions of Theorem 3.2, let $\Omega'$ be an arbitrary subdomain of $\Omega$ and let $K_x$ be a connected component of the level set $\{z \in \overline{\Omega} : \psi(z) = \psi(x)\}$ containing the point $x$. Denote $X = X_{\Omega'} = \{x \in \Omega' : K_x \cap \partial\Omega' = \emptyset\}$. By Theorems 2.1 (i), 2.2, for almost all $y \in \psi(X)$ and for any $x \in X \cap \psi^{-1}(y)$, the component $K_x \subset \Omega' \setminus A_{\mathbf{w}}$ is a $C^1$-smooth curve homeomorphic to the circle and $\nabla\psi \neq 0$ on $K_x$. Below, we call such $K_x$ an *admissible cycle* (note that if $X \neq \emptyset$ then the set $\psi(X)$ contains an interval; hence, it has positive measure and the family of admissible cycles is nonempty).

**Lemma 3.3.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, multiply connected domain with Lipschitz boundary. Let $\mathbf{w} \in W^{1,2}(\Omega)$ and $p \in W^{1,s}(\Omega)$ satisfy Euler equations (3.1) for almost all $x \in \Omega$ and $\int_{\Gamma_i} \mathbf{w}\cdot\mathbf{n}\,dS = 0,\ i = 1,\dots,N$. Assume that there exists a sequence of functions $\{\Phi_\mu\}$ such that $\Phi_\mu \in W^{1,s}_{\mathrm{loc}}(\Omega)$ and $\Phi_\mu \rightharpoonup \Phi$ in $W^{1,s}_{\mathrm{loc}}(\Omega)$ for some $s \in [4/3, 2)$. Then for any subdomain $\Omega' \subset \Omega$ with $X = X_{\Omega'} \neq \emptyset$ the functions $\Phi_\mu|_K$ are continuous on almost all*[8] *admissible cycles $K$ and the sequence $\{\Phi_\mu|_K\}$ converges to $\Phi|_K$ uniformly: $\Phi_\mu|_K \rightrightarrows \Phi|_K$.*

[8] "Almost all cycles" means cycles in preimages $\psi^{-1}(y)$ for almost all values $y \in \psi(X)$.

**Proof.** Fix an arbitrary $\varepsilon > 0$ and take a set $V \subset \mathbb{R}$ and a function $g \in C^1(\mathbb{R}^2)$ from Theorem 2.1 (ii). Put $X_g = X_{\Omega'} \setminus \psi^{-1}(V)$ and take arbitrary point $x_0 \in X_g$, $\psi(x_0) = y_0$. Then by construction we have $g(x_0) = y_0$, $\nabla\psi(x) = \nabla g(x) \neq 0$ for $x \in K_{x_0}$, and the $C^1$-smooth cycle $K_{x_0}$ coincides with the connected component of the level set $\{z \in \mathbb{R}^2 : g(z) = y_0\}$ containing the point $x_0$. Take small $\varepsilon_0 > 0$ and denote by $X_0$ the connected component of the preimage $\{z \in \mathbb{R}^2 : g(z) \in [y_0 - \varepsilon_0, y_0 + \varepsilon_0]\}$ containing $x_0$. By construction, $\forall y \in [y_0 - \varepsilon_0, y_0 + \varepsilon_0]$, the preimage $g^{-1}(y) \cap X_0$ is a $C^1$-smooth cycle where $\nabla g \neq 0$. Now it is easy to construct a $C^1$-smooth diffeomorphism $F = (f_1, f_2) : X_0 \to [0, 1] \times \mathbb{S}^1$ such that $f_1 = l \circ g$, where $l : \mathbb{R} \to \mathbb{R}$ is a linear function which maps the segment $[y_0 - \varepsilon_0, y_0 + \varepsilon_0]$ onto $[0, 1]$ (in particular, the level sets of $f_1$ coincides with the level sets of $g$).[9]

Let $G = G(t, \theta) = F^{-1}$, that is, $G : [0, 1] \times \mathbb{S}^1 \to X_0$ is a $C^1$-diffeomorphism such that for any $t \in [0, 1]$ the image $\{G(t, \theta) : \theta \in [0, 2\pi)\}$ coincides with the connected component of the level set $\{z \in \mathbb{R}^2 : g(z) = g(G(t, 0))\}$ containing $G(t, 0)$. Put $\widetilde{\Phi}(t, \theta) = \Phi(G(t, \theta))$, etc.

The rest of the proof repeats the argument of the proof of Theorem 3.2 in [1]. For the reader's convenience we repeat these arguments. Denote

$$z_\mu(t) = \int_0^{2\pi} |\widetilde{\Phi}_\mu(t,\theta) - \widetilde{\Phi}(t,\theta)| \left| \frac{\partial}{\partial\theta}\widetilde{\Phi}_\mu(t,\theta) - \frac{\partial}{\partial\theta}\widetilde{\Phi}(t,\theta)\right| \mathrm{d}\theta.$$

Then

$$\begin{aligned}\int_0^1 z_\mu(t)\mathrm{d}t &\leqq \Big(\int_0^1\int_0^{2\pi} |\widetilde{\Phi}_\mu(t,\theta) - \widetilde{\Phi}(t,\theta)|^q \mathrm{d}\theta dt\Big)^{\frac{1}{q}} \\ &\quad \times \Big(\int_0^1\int_0^{2\pi} \left|\frac{\partial}{\partial\theta}\widetilde{\Phi}_\mu(t,\theta) - \frac{\partial}{\partial\theta}\widetilde{\Phi}(t,\theta)\right|^s \mathrm{d}\theta dt\Big)^{\frac{1}{s}} \\ &\leqq c\|\Phi_\mu - \Phi\|_{L^q(X_0)} \|\nabla(\Phi_\mu - \Phi)\|_{L^s(X_0)},\end{aligned} \tag{3.20}$$

where $\dfrac{1}{q} + \dfrac{1}{s} = 1$, $X_0 = \overline{X}_0 \subset \Omega$. Since $\Phi_\mu \rightharpoonup \Phi$ in $W^{1,s}_{\mathrm{loc}}(\Omega)$, by the Embedding Theorem $\Phi_\mu \to \Phi$ in $L^{q_*}(X_0)$ for $q_* = \frac{2s}{2-s} \geqq \frac{s}{s-1} = q$ (the last inequality follows from the assumption $s \geqq 4/3$), and it follows from (3.20) that $z_\mu \to 0$ in $L^1([0, 1])$. Thus, there exists a subsequence (we denote it again by $\{z_\mu\}$) converging to zero almost everywhere on $[0, 1]$.

Define

$$H_\mu(t) = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{\Phi}_\mu(t,\theta)\mathrm{d}\theta, \quad H(t) = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{\Phi}(t,\theta)\mathrm{d}\theta.$$

[9] The second function $f_2 : X_0 \to \mathbb{S}^1$ can be constructed as follows. First of all, let $h : K_{x_0} \to \mathbb{S}^1$ be $C^1$-diffeomorphism. Then take a $C^\infty$-smooth vector field $\xi(x)$, $x \in X_0$, which is close to $\nabla g(x)$ in $C$-norm. Consider integral lines of this vector field. Then each integral line intersects the cycle $K_{x_0}$ at one point. So we have a $C^1$-smooth function $\tilde{f} : X_0 \to K_{x_0}$. Finally take $f_2 = h \circ \tilde{f}$. Because level sets of $f_2$ are "almost orthogonal" to the level sets of $g$, the mapping $F = (f_1, f_2)$ is a diffeomorphism.

Since $\varPhi_\mu \rightharpoonup \varPhi$ in $W^{1,s}(X_0)$, by the Embedding Theorem we conclude that $H_\mu \to H$ in $C([0,1])$ as $\mu \to \infty$. Moreover, $\widetilde{\varPhi}_\mu, \widetilde{\varPhi} \in W^{1,s}([0,1]\times\mathbb{S}^1)$ and, hence, $\widetilde{\varPhi}_\mu(t,\cdot), \widetilde{\varPhi}(t,\cdot)$ are absolutely continuous functions with respect to $\theta$ for almost all $t \in [0,1]$.

Let us fix arbitrary $t_* \in [0,1]$ such that $z_\mu(t_*) \to 0$ and that the functions $\widetilde{\varPhi}_\mu(t_*,\cdot), \widetilde{\varPhi}(t_*,\cdot)$ are continuous. Let $\theta_\mu \in [0,2\pi]$ be such that

$$\widetilde{\varPhi}_\mu(t_*,\theta_\mu) - \widetilde{\varPhi}(t_*,\theta_\mu) = H_\mu(t_*) - H(t_*).$$

Then

$$\max_{\theta\in[0,2\pi]} |\widetilde{\varPhi}_\mu(t_*,\theta) - \widetilde{\varPhi}(t_*,\theta)|^2 \leqq |\widetilde{\varPhi}_\mu(t_*,\theta_\mu) - \widetilde{\varPhi}(t_*,\theta_\mu)|^2$$
$$+ \int_0^{2\pi} \left|\frac{\partial}{\partial\theta}\big(\widetilde{\varPhi}_\mu(t_*,\theta) - \widetilde{\varPhi}(t_*,\theta)\big)^2\right| d\theta = |H_\mu(t_*) - H(t_*)|^2 + 2z_\mu(t_*) \to 0$$

as $\mu \to \infty$. Thus, the continuity of $\widetilde{\varPhi}_\mu(t,\cdot)$ and the uniform convergence $\widetilde{\varPhi}_\mu(t,\cdot) \rightrightarrows \widetilde{\varPhi}(t,\cdot)$ are proved for almost all $t \in [0,1]$, that is, for almost all level sets of function $g|_{X_0}$. Since $X_0$ was a neighborhood of an arbitrary admissible cycle $K_{x_0} \subset X_g$, the claim of the lemma is proved for almost all admissible cycles $K \subset X_g = X_{\varOmega'} \setminus \psi^{-1}(V)$. Because $\mathfrak{H}^1(V) < \varepsilon$ and $\varepsilon > 0$ is arbitrary, the lemma is proved completely. □

**Proof of Theorem 3.4.** Let $\varOmega' \subset \varOmega$ be an arbitrary domain with the boundary $\partial\varOmega'$ not containing a singleton connected component (in particular, we can take $\varOmega' = \varOmega$). Denote

$$\sigma_1 = \operatorname*{ess\,sup}_{x\in\partial\varOmega'} \varPhi(x), \quad \sigma_2 = \operatorname*{ess\,sup}_{x\in\varOmega'} \varPhi(x). \tag{3.21}$$

Assume that the claim of Theorem 3.4 is false, that is,

$$\sigma_1 < \sigma_2. \tag{3.22}$$

Then for all $\sigma \in (\sigma_1,\sigma_2)$ the set $\{x \in \varOmega'\setminus A_{\mathbf{w}} : \varPhi(x) > \sigma\}$ is nonempty. Denote by $K_x$ the connected component of the level set $\{y \in \overline{\varOmega} : \psi(y) = \psi(x)\}$ containing the point $x$. There are two possibilities:

(i) there exists a point $y_0 \in \varOmega'\setminus A_{\mathbf{w}}$ such that $\varPhi(y_0) > \sigma_1$ and $K_{y_0} \cap \partial\varOmega' \neq \emptyset$;
(ii) for any $x \in \varOmega' \setminus A_{\mathbf{w}}$ such that $\varPhi(x) > \sigma_1$, the equality $K_x \cap \partial\varOmega' = \emptyset$ holds.

We shall prove that in both cases (i) and (ii) the assumption (3.22) leads to a contradiction.

(i) Let $\Gamma'$ be a connected component of $\partial\varOmega'$ such that $K_{y_0} \cap \Gamma' \neq \emptyset$. If $\psi(x) = \text{const}$ on $\Gamma'$, then by Remark 3.1 $\varPhi(x) = \varPhi(y_0) > \sigma_1$ for any $x \in \Gamma'\setminus A_{\mathbf{w}} \subset \partial\varOmega'$. However, the last equality contradicts the definition (3.21) of $\sigma_1$.

Now let $\psi(x) \neq \text{const}$ on $\Gamma'$. Take any set $E \subset \partial\Omega'$ with $\mathfrak{H}^1(E) = 0$. In virtue of Lemma 3.1, there exists a sequence of connected compact sets $K_i \subset \overline{\Omega}' \setminus (A_{\mathbf{w}} \cup E)$ and points $x_i \in K_i$ such that $\psi|_{K_i} \equiv c_i = \text{const}$, $K_i \cap \partial\Omega' \neq \emptyset$, $\operatorname{diam} K_i \geqq \delta > 0$ and $x_i \to x_0 \in K_{y_0}$. Let $y_i \in K_i \cap \partial\Omega'$. By Lemma 3.2,

$$\lim_{i\to\infty} \Phi(y_i) = \Phi(y_0) > \sigma_1.$$

Thus $\sup_{x\in\partial\Omega'\setminus(A_{\mathbf{w}}\cup E)} \Phi(x) \geqq \Phi(y_0) > \sigma_1$. Because of arbitrariness of $E \subset \partial\Omega'$ with $\mathfrak{H}^1(E) = 0$ we have $\operatorname{ess\,sup}_{x\in\partial\Omega'} \Phi(x) > \sigma_1$, and we obtain the contradiction.[10]

(ii) These assumptions imply that the family of all admissible cycles is non-empty (the definition of an admissible cycle is given in the commentary above Lemma 3.3). By Lemma 3.3, on almost all admissible cycles $K$, the functions $\Phi_\mu|_K$ are continuous and the sequence $\{\Phi_\mu|_K\}$ converges to $\Phi|_K$ uniformly. Here (during this proof) admissible cycles having this property are called *uniformly regular*.

Let us fix $\sigma \in (\sigma_1, \sigma_2)$ such that there exists an admissible cycle $K^\sigma$ with the property $\Phi(x) \equiv \sigma$ for all $x \in K^\sigma$.[11] A uniformly regular cycle $K$ is called *red*, if $\Phi(x) < \sigma$ for $x \in K$. We need the following claim.

(***) For any $z \in \partial\Omega'$ there exists a red cycle $K$ and an open neighborhood $U(z)$ such that $U(z)$ and $K^\sigma$ lie in different connected components of the set $\mathbb{R}^2 \setminus K$.

To prove (***), take any $z \in \partial\Omega'$. Because of the Sobolev Extension Theorem we can assume that $\psi \in W^{2,1}(\mathbb{R}^2)$. Fix a square $Q \supset \overline{\Omega}'$ and apply Lemmas 2.7–2.8 to the function $f = \psi$. Consider the corresponding arc $J = J(K^\sigma, K_z)$ from these Lemmas, parameterized by the injective function $\varphi : [0,1] \to J$ such that $\varphi(0) = K^\sigma$ and $\varphi(1) = K_z$ (below during this proof by $K_x$ we denote the connected component of the level set $\{y \in Q : \psi(y) = \psi(x)\}$ containing $x$). From the definition of admissible cycles it follows that $K^\sigma \cap \partial\Omega' = \emptyset$. By continuity of $\varphi$ we have $\varphi(t) \cap \partial\Omega' = \emptyset$ for $t \in [0,\delta]$, $\delta$ is sufficiently small. Put $t_* = \sup\{t \in [0,1] :\ \varphi(\tau) \cap \partial\Omega' = \emptyset\ \forall \tau \in [0,t)\}$. Clearly,

$$\varphi(t_*) \cap \partial\Omega' \neq \emptyset. \tag{3.23}$$

Take a sequence $t_i \to t_* - 0$ such that $K_i = \varphi(t_i)$ are uniformly regular cycles. If there exists $i$ such that the cycle $K_i$ is red, then the assertion of (***) is true and there is nothing to prove. Assume now that $K_i$ are not red for all $i$, that is,

$$\Phi \geqq \sigma \quad \text{on all } K_i. \tag{3.24}$$

To finish the proof of (***), we need to obtain a contradiction. By $\Gamma'$ denote the connected component of $\partial\Omega'$ containing $z$. Since $K_i \cap \partial\Omega' = \emptyset$, we see that the

[10] Note that in arguments of this paragraph we do not use the fact that the functions $\Phi_\mu$ converge to $\Phi$.

[11] It follows from Remark 3.1 that for any admissible cycle $K_x$ the identity $\Phi(z) = \Phi(x)$ holds $\forall z \in K_x$.

sets $\Gamma'$, $K^\sigma$ lie in the different connected components of the set $\mathbb{R}^2 \setminus K_i$ for all $i$. Hence $\operatorname{diam} K_i \geqq \min(\operatorname{diam} K^\sigma, \operatorname{diam} \Gamma') > 0$. We may assume without loss of generality that $K_i$ converges to some set $K_0 \subset \overline{\Omega}'$ with respect to the Hausdorff distance (see the footnote in the proof of Lemma 3.1), and $x_i \to x_0$, where $x_i \in K_i$, $x_0 \in K_0$. From the previous properties we have $K_0 \subset \varphi(t_*)$, $\operatorname{diam} K_0 \geqq \min(\operatorname{diam} K^\sigma, \operatorname{diam} \Gamma') > 0$, $\psi \equiv \text{const}$ on $K_0$. Now from Lemma 3.2 and (3.24) we conclude that

$$\Phi(x) \geqq \sigma \quad \text{for all} \quad x \in K_0 \setminus A_{\mathbf{w}}.$$

From the last inequality and from the assumption $\operatorname{ess\,sup}_{x\in\partial\Omega'} \Phi(x) = \sigma_1 < \sigma$ it follows that there exists a point $x_* \in K_0 \cap \Omega' \setminus A_{\mathbf{w}}$; moreover, $\Phi(x_*) > \sigma_1$. Then by assumption (ii) we obtain the equality $K_{x_*} \cap \partial\Omega' = \emptyset$.[12] Thus $K_{x_*} = \varphi(t_*)$, but the last equality contradicts (3.23). This contradiction finishes the proof of (***).

Combining (***) with the compactness of $\partial\Omega'$ (that is, for any open covering of the compact set $\partial\Omega'$ it is possible to extract a finite subcovering), we get that there exist finitely many red cycles separating $K^\sigma$ from all the points of $\partial\Omega'$. In other words, there is a strictly interior subdomain $\Omega^* \subset \Omega'$ ($\overline{\Omega}^* \subset \Omega'$) such that $K^\sigma \subset \Omega^*$ and the boundary $\partial\Omega^*$ is the union of a finite number of uniformly regular cycles $S^{(1)}, \dots, S^{(M)}$ such that

$$\Phi|_{S^{(j)}} < \sigma, \quad j = 1, \dots, M.$$

Therefore,

$$\begin{aligned}\sigma > \sup_{x\in\partial\Omega^*} \Phi(x) &= \max_{j=1,\dots,M} \sup_{x\in S^{(j)}} \Phi(x) = \lim_{\mu\to\infty} \left( \max_{j=1,\dots,M} \sup_{x\in S^{(j)}} \Phi_\mu(x) \right) \\ &= \lim_{\mu\to\infty} \sup_{x\in\partial\Omega^*} \Phi_\mu(x).\end{aligned}$$

Since by the assumptions the functions $\Phi_\mu$ satisfy the one-sided maximum principle locally in $\Omega$, we have the estimate

$$\lim_{\mu\to\infty} \operatorname*{ess\,sup}_{x\in\Omega^*} \Phi_\mu(x) \leqq \lim_{\mu\to\infty} \operatorname*{ess\,sup}_{x\in\partial\Omega^*} \Phi_\mu(x) < \sigma,$$

and because of the weak convergence $\Phi_\mu \rightharpoonup \Phi$ in $W^{1,s}(\Omega^*)$, the inequality

$$\operatorname*{ess\,sup}_{x\in\Omega^*} \Phi(x) < \sigma \tag{3.25}$$

is valid. However, by the construction $K^{(\sigma)} \subset \Omega^* \setminus A_{\mathbf{w}}$, thus the inequality (3.25) contradicts the identity $\Phi|_{K^{(\sigma)}} = \sigma$. The contradiction obtained completes the proof. □

[12] Remark, that in this case the connected component of the level set $\{y \in Q : \psi(y) = \psi(x_*)\}$ containing $x_*$ coincides with the connected component of the level set $\{y \in \overline{\Omega}' : \psi(y) = \psi(x_*)\}$ containing $x_*$.

**Remark 3.3.** In particular, it follows from Theorem 3.4 and Remark 3.2, that if $\mathbf{w}\big|_{\partial\Omega} = 0$ (in the sense of traces), then

$$\operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) \leqq \operatorname*{ess\,sup}_{x\in\partial\Omega} \Phi(x) = \max\{p_1, p_2, \ldots, p_N\}, \tag{3.26}$$

where $p(x)|_{\Gamma_i} = p_i = \text{const}$.

**Remark 3.4.** Note that some version of a local weak one-sided maximum principle was proved by Amick [1] (see Theorem 3.2 and Remark thereafter).

## 4. Existence Theorem

Now consider Navier–Stokes problem (1.1) in the domain $\Omega \subset \mathbb{R}^2$ defined by (1.5) and assume that $\partial\Omega$ is Lipschitz. If the boundary datum $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ satisfies condition (1.2), that is,

$$\int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\Gamma_1} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S + \int_{\Gamma_2} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S = 0,$$

then by Lemma 2.9 there exists a solenoidal extension $\mathbf{A} \in W^{1,2}(\Omega)$ of $\mathbf{a}$ and estimate (2.2) holds. Using this fact and standard results (for example [24]) we can find a weak solution $\mathbf{U} \in W^{1,2}(\Omega)$ of the Stokes problem such that $\mathbf{U}-\mathbf{A} \in H(\Omega)$ and

$$\nu \int_{\Omega} \nabla\mathbf{U}\cdot\nabla\boldsymbol{\eta}\,\mathrm{d}x = 0 \quad \forall\, \boldsymbol{\eta} \in H(\Omega). \tag{4.1}$$

Moreover,

$$\|\mathbf{U}\|_{W^{1,2}(\Omega)} \leqq c\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}. \tag{4.2}$$

By a *weak solution* of problem (1.1) we understand a function $\mathbf{u}$ such that $\mathbf{w} = \mathbf{u} - \mathbf{A} \in H(\Omega)$ and

$$\nu \int_{\Omega} \nabla\mathbf{w}\cdot\nabla\boldsymbol{\eta}\,\mathrm{d}x - \int_{\Omega} \big((\mathbf{w}+\mathbf{U})\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{w}\,\mathrm{d}x - \int_{\Omega} \big(\mathbf{w}\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{U}\,\mathrm{d}x$$

$$= \int_{\Omega} \big(\mathbf{U}\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{U}\,\mathrm{d}x \qquad \forall\boldsymbol{\eta} \in H(\Omega). \tag{4.3}$$

We shall prove the following

**Theorem 4.1.** *Assume that* $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ *and let condition* (1.6) *be fulfilled. If* $\mathcal{F} = \int_{\Gamma_2} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S \geqq 0$, *then problem* (1.1) *admits at least one weak solution.*

**Proof.**

1. We follow a contradiction argument of Leray [27]. Although this argument has also been used in many other papers (for example [1,19,23,24]), we reproduce, for the reader's convenience, some of its details. It is well known (for example [24]) that integral identity (4.3) is equivalent to an operator equation in the space $H(\Omega)$ with a compact operator. Therefore, by virtue of the Leray–Schauder Theorem, to prove the existence of a weak solution to Navier–Stokes problem (1.1) it is sufficient to show that all possible solutions of the integral identity

$$\nu \int_\Omega \nabla \mathbf{w} \cdot \nabla \boldsymbol{\eta} \, \mathrm{d}x - \lambda \int_\Omega \big((\mathbf{w}+\mathbf{U}) \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{w} \, \mathrm{d}x - \lambda \int_\Omega \big(\mathbf{w} \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{U} \, \mathrm{d}x$$
$$= \lambda \int_\Omega \big(\mathbf{U} \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{U} \, \mathrm{d}x \qquad \forall\, \boldsymbol{\eta} \in H(\Omega) \tag{4.4}$$

are uniformly bounded (with respect to $\lambda \in [0, 1]$) in $H(\Omega)$. Assume this is false. Then there exist sequences $\{\lambda_k\}_{k\in\mathbb{N}} \subset [0, 1]$ and $\{\mathbf{w}_k\}_{k\in\mathbb{N}} \in H(\Omega)$ such that

$$\nu \int_\Omega \nabla \mathbf{w}_k \cdot \nabla \boldsymbol{\eta} \, \mathrm{d}x - \lambda_k \int_\Omega \big((\mathbf{w}_k+\mathbf{U}) \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{w}_k \, \mathrm{d}x - \lambda_k \int_\Omega \big(\mathbf{w}_k \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{U} \, \mathrm{d}x$$
$$= \lambda_k \int_\Omega \big(\mathbf{U} \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{U} \, \mathrm{d}x \qquad \forall\, \boldsymbol{\eta} \in H(\Omega), \tag{4.5}$$

and

$$\lim_{k\to\infty} \lambda_k = \lambda_0 \in [0, 1], \quad \lim_{k\to\infty} J_k = \lim_{k\to\infty} \|\mathbf{w}_k\|_{H(\Omega)} = \infty. \tag{4.6}$$

Let us take in (4.5) $\boldsymbol{\eta} = J_k^{-2}\mathbf{w}_k$ and denote $\widehat{\mathbf{w}}_k = J_k^{-1}\mathbf{w}_k$. Since

$$\int_\Omega \big((\mathbf{w}_k+\mathbf{U}) \cdot \nabla\big)\mathbf{w}_k \cdot \mathbf{w}_k \, \mathrm{d}x = 0,$$

we get

$$\nu \int_\Omega |\nabla \widehat{\mathbf{w}}_k|^2 \, \mathrm{d}x = \lambda_k \int_\Omega \big(\widehat{\mathbf{w}}_k \cdot \nabla\big)\widehat{\mathbf{w}}_k \cdot \mathbf{U} \, \mathrm{d}x + J_k^{-1}\lambda_k \int_\Omega \big(\mathbf{U} \cdot \nabla\big)\widehat{\mathbf{w}}_k \cdot \mathbf{U} \, \mathrm{d}x. \tag{4.7}$$

Since $\|\widehat{\mathbf{w}}_k\|_{H(\Omega)} = 1$, there exists a subsequence $\{\widehat{\mathbf{w}}_{k_l}\}$ converging weakly in $H(\Omega)$ to a vector field $\widehat{\mathbf{w}} \in H(\Omega)$. Because of the compact embedding

$$H(\Omega) \hookrightarrow L^r(\Omega) \quad \forall\, r \in (1, \infty),$$

the subsequence $\{\widehat{\mathbf{w}}_{k_l}\}$ converges strongly in $L^r(\Omega)$. Therefore, passing to a limit as $k_l \to \infty$ in equality (4.7) we obtain

$$\nu = \lambda_0 \int_\Omega \big(\widehat{\mathbf{w}} \cdot \nabla\big)\widehat{\mathbf{w}} \cdot \mathbf{U} \, \mathrm{d}x. \tag{4.8}$$

2. Let us return to integral identity (4.5). Consider the functional

$$R_k(\boldsymbol{\eta}) = \int_\Omega \Big(\nu \nabla \mathbf{w}_k \cdot \nabla \boldsymbol{\eta} - \lambda_k\big((\mathbf{w}_k+\mathbf{U}) \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{w}_k - \lambda_k\big(\mathbf{w}_k \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{U}\Big) \, \mathrm{d}x$$
$$-\lambda_k \int_\Omega \big(\mathbf{U} \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{U} \, \mathrm{d}x \qquad \forall\, \boldsymbol{\eta} \in \mathring{W}^{1,2}(\Omega).$$

Obviously, $R_k(\boldsymbol{\eta})$ is a linear functional, and using (4.2) and Sobolev Embedding Theorem, we get the estimate

$$\big|R_k(\boldsymbol{\eta})\big| \leqq c\Big(\|\mathbf{w}_k\|_{H(\Omega)} + \|\mathbf{w}_k\|^2_{H(\Omega)} + \|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}\Big)\|\boldsymbol{\eta}\|_{H(\Omega)},$$

with constant $c$ independent of $k$. It follows from (4.5) that

$$R_k(\boldsymbol{\eta}) = 0 \qquad \forall\, \boldsymbol{\eta} \in H(\Omega).$$

Therefore, by Lemma 2.10, there exist functions $p_k \in \widehat{L}^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q(x)\,\mathrm{d}x = 0\}$ such that

$$R_k(\boldsymbol{\eta}) = \int_\Omega p_k \operatorname{div}\boldsymbol{\eta}\,\mathrm{d}x \qquad \forall\, \boldsymbol{\eta} \in \mathring{W}^{1,2}(\Omega)$$

and

$$\|p_k\|_{L^2(\Omega)} \leqq c\Big(\|\mathbf{w}_k\|_{H(\Omega)} + \|\mathbf{w}_k\|^2_{H(\Omega)} + \|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}\Big). \tag{4.9}$$

The pair $(\mathbf{w}_k, p_k)$ satisfies the integral identity

$$\nu\int_\Omega \nabla\mathbf{w}_k\cdot\nabla\boldsymbol{\eta}\,\mathrm{d}x - \lambda_k\int_\Omega \big((\mathbf{w}_k+\mathbf{U})\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{w}_k\,\mathrm{d}x - \lambda_k\int_\Omega \big(\mathbf{w}_k\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{U}\,\mathrm{d}x$$
$$-\lambda_k\int_\Omega \big(\mathbf{U}\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{U}\,\mathrm{d}x = \int_\Omega p_k\operatorname{div}\boldsymbol{\eta}\,\mathrm{d}x \qquad \forall\,\boldsymbol{\eta}\in\mathring{W}^{1,2}(\Omega). \tag{4.10}$$

Let $\mathbf{u}_k = \mathbf{w}_k + \mathbf{U}$. Then identity (4.10) takes the form (see (4.1))

$$\nu\int_\Omega \nabla\mathbf{u}_k\cdot\nabla\boldsymbol{\eta}\,\mathrm{d}x - \int_\Omega p_k\operatorname{div}\boldsymbol{\eta}\,\mathrm{d}x = -\lambda_k\int_\Omega (\mathbf{u}_k\cdot\nabla\big)\mathbf{u}_k\cdot\boldsymbol{\eta}\,\mathrm{d}x \quad \forall\,\boldsymbol{\eta}\in\mathring{W}^{1,2}(\Omega).$$

Thus, $(\mathbf{u}_k, p_k)$ might be considered as a weak solution to the Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{u}_k + \nabla p_k = \mathbf{f}_k & \text{in } \ \Omega, \\ \operatorname{div}\mathbf{u}_k = 0 & \text{in } \ \Omega, \\ \mathbf{u}_k = \mathbf{a} & \text{on } \ \partial\Omega, \end{cases}$$

with the right-hand side $\mathbf{f}_k = -\lambda_k\big(\mathbf{u}_k\cdot\nabla\big)\mathbf{u}_k$. Obviously, $\mathbf{f}_k \in L^s(\Omega)$ for $s \in (1,2)$ and

$$\|\mathbf{f}_k\|_{L^s(\Omega)} \leqq c\|\big(\mathbf{u}_k\cdot\nabla\big)\mathbf{u}_k\|_{L^s(\Omega)} \leqq c\|\mathbf{u}_k\|_{L^{2s/(2-s)}(\Omega)}\|\nabla\mathbf{u}_k\|_{L^2(\Omega)}$$
$$\leqq c\Big(\big(\|\mathbf{w}_k\|_{H(\Omega)} + \|\mathbf{U}\|_{W^{1,2}(\Omega)}\big)^2\Big) \leqq c\Big(\|\mathbf{w}_k\|^2_{H(\Omega)} + \|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}\Big),$$

where $c$ is independent of $k$. By well known local regularity results for the Stokes system (see [14,24]) we have $\mathbf{w}_k \in W^{2,s}_{loc}(\Omega)$, $p_k \in W^{1,s}_{loc}(\Omega)$, and the estimate

$$\|\mathbf{w}_k\|_{W^{2,s}(\Omega')} + \|p_k\|_{W^{1,s}(\Omega')} \leqq c\Big(\|\mathbf{f}_k\|_{L^s(\Omega)} + \|\mathbf{u}_k\|_{W^{1,2}(\Omega)} + \|p_k\|_{L^2(\Omega)}\Big)$$
$$\leqq c\Big(\|\mathbf{w}_k\|^2_{H(\Omega)} + \|\mathbf{w}_k\|_{H(\Omega)} + \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}\Big) \tag{4.11}$$

holds, where $\Omega'$ is an arbitrary domain with $\overline{\Omega}' \subset \Omega$ and the constant $c$ depends on dist $(\Omega', \partial\Omega)$ but not on $k$.

Denote $\widehat{p}_k = J_k^{-2} p_k$. It follows from (4.9) and (4.11) that

$$\|\widehat{p}_k\|_{L^2(\Omega)} \leqq const, \quad \|\widehat{p}_k\|_{W^{1,s}(\Omega')} \leqq const$$

for any $\overline{\Omega}' \subset \Omega$ and $s \in (1, 2)$. Hence, from the sequence $\{\widehat{p}_{k_l}\}$ we can extract a subsequence, still denoted by $\{\widehat{p}_{k_l}\}$, which converges weakly in $\widehat{L}^2(\Omega)$ and $W^{1,s}_{loc}(\Omega)$ to some function $\widehat{p} \in W^{1,s}_{loc}(\Omega) \cap \widehat{L}^2(\Omega)$. Let $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$. Taking in (4.10) $\boldsymbol{\eta} = J_{k_l}^{-2}\boldsymbol{\varphi}$ and letting $k_l \to \infty$ yields

$$-\lambda_0 \int_\Omega \big(\widehat{\mathbf{w}} \cdot \nabla\big)\boldsymbol{\varphi} \cdot \widehat{\mathbf{w}}\, \mathrm{d}x = \int_\Omega \widehat{p}\, \mathrm{div}\varphi\, \mathrm{d}x \quad \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega).$$

Integrating by parts in the last equality, we derive

$$\lambda_0 \int_\Omega \big(\widehat{\mathbf{w}} \cdot \nabla\big)\widehat{\mathbf{w}} \cdot \boldsymbol{\varphi}\, \mathrm{d}x = -\int_\Omega \nabla\widehat{p} \cdot \varphi\, \mathrm{d}x \quad \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega). \tag{4.12}$$

Hence, the pair $\big(\widehat{\mathbf{w}}, \widehat{p}\big)$ satisfies, for almost all $x \in \Omega$, the Euler equations

$$\begin{cases} \lambda_0\big(\widehat{\mathbf{w}} \cdot \nabla\big)\widehat{\mathbf{w}} + \nabla\widehat{p} = 0, \\ \qquad\qquad \mathrm{div}\, \widehat{\mathbf{w}} = 0, \end{cases} \tag{4.13}$$

and $\widehat{\mathbf{w}}\big|_{\partial\Omega} = 0$. By Theorem 3.3, $\widehat{p} \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ and the pressure $\widehat{p}(x)$ is constant on $\Gamma_1$ and $\Gamma_2$ (see Remark 3.2). Denote by $\widehat{p}_1$ and $\widehat{p}_2$ values of $\widehat{p}(x)$ on $\Gamma_1$ and $\Gamma_2$, respectively.

Multiplying equations (4.13) by $\mathbf{U}$ and integrating by parts we derive

$$\begin{aligned} \lambda_0 \int_\Omega \big(\widehat{\mathbf{w}} \cdot \nabla\big)\widehat{\mathbf{w}} \cdot \mathbf{U}\, \mathrm{d}x &= -\int_\Omega \nabla\widehat{p} \cdot \mathbf{U}\, \mathrm{d}x = -\int_{\partial\Omega} \widehat{p}\,\mathbf{a} \cdot \mathbf{n}\, \mathrm{d}S \\ &= -\widehat{p}_1 \int_{\Gamma_1} \mathbf{a} \cdot \mathbf{n}\, \mathrm{d}S - \widehat{p}_2 \int_{\Gamma_2} \mathbf{a} \cdot \mathbf{n}\, \mathrm{d}S = \mathcal{F}(\widehat{p}_1 - \widehat{p}_2) \end{aligned} \tag{4.14}$$

(see formula (1.6)). If either $\mathcal{F} = 0$ or $\widehat{p}_1 = \widehat{p}_2$, then from (4.14) it follows that

$$\lambda_0 \int_\Omega \big(\widehat{\mathbf{w}} \cdot \nabla\big)\widehat{\mathbf{w}} \cdot \mathbf{U}\, \mathrm{d}x = 0. \tag{4.15}$$

This last relation contradicts equality (4.8). Therefore, the norms $\|\mathbf{w}\|_{H(\Omega)}$ of all possible solutions to identity (4.4) are uniformly bounded with respect to $\lambda \in [0, 1]$ and by the Leray–Schauder Theorem problem (1.1) admits at least one weak solution $\mathbf{u} \in W^{1,2}(\Omega)$.

3. Up to this point our arguments have been standard and have followed those of Leray [27] (see also [19] and [1]). However, by our assumptions $\mathcal{F} > 0$ and, in general, $\widehat{p}_2 \neq \widehat{p}_1$ (see a counterexample in [1]). Thus, (4.15) may be false. In order to prove that $\widehat{p}_1$ and $\widehat{p}_2$ do coincide in the case $\mathcal{F} > 0$, we use the fact that $\big(\widehat{\mathbf{w}}, \widehat{p}\big)$ was obtained as a limit (in some sense) of solutions to the Navier–Stokes

equations. Note that the possibility of using this fact was already pointed up by AMICK [1].

Let $\Phi_{k_l} = p_{k_l} + \frac{\lambda_{k_l}}{2}|\mathbf{u}_{k_l}|^2$, where $\mathbf{u}_{k_l} = \mathbf{w}_{k_l} + \mathbf{U}$, be total head pressures corresponding to the solutions $(\mathbf{w}_{k_l}, p_{k_l})$ of integral identities (4.10). Then $\Phi_{k_l} \in W^{2,s}_{loc}(\Omega),\ s \in (1,2)$, satisfy almost everywhere in $\Omega$ the equations

$$\nu\Delta\Phi_{k_l} - \lambda_{k_l}\big(\mathbf{u}_{k_l}\cdot\nabla\big)\Phi_{k_l} = \nu\Big(\frac{\partial u_{1k_l}}{\partial x_2} - \frac{\partial u_{2k_l}}{\partial x_1}\Big)^2.$$

It is well known [15,16] (see also [30]) that $\Phi_{k_l}$ satisfy the one-sided maximum principle locally in $\Omega$ (the boundary $\partial\Omega$ is only Lipschitz and functions $\Phi_{k_l}$ do not have second derivatives up to the boundary). Set $\widehat{\Phi}_{k_l} = J_{k_l}^{-2}\Phi_{k_l}$. From (4.9), (4.11) it follows that the sequence $\{\widehat{\Phi}_{k_l}\}$ weakly converges to $\widehat{\Phi} = \widehat{p} + \frac{\lambda_0}{2}|\widehat{\mathbf{w}}|^2$ in the space $L^2(\Omega)\cap W^{1,s}_{loc}(\Omega),\ s\in(1,2)$. Therefore, by Theorem 3.4, $\widehat{\Phi}$ satisfies the weak one-sided maximum principle in $\Omega$:

$$\operatorname*{ess\,sup}_{x\in\Omega}\widehat{\Phi}(x) \leqq \operatorname*{ess\,sup}_{x\in\partial\Omega}\widehat{\Phi}(x) = \max\{\widehat{p}_1,\widehat{p}_2\}. \tag{4.16}$$

From equalities (4.8) and (4.14) we conclude that

$$(\widehat{p}_1 - \widehat{p}_2)\mathcal{F} = \nu > 0. \tag{4.17}$$

So, if $\mathcal{F} > 0$, then

$$\widehat{p}_2 < \widehat{p}_1. \tag{4.18}$$

Now, from (4.16), (4.18) it follows that

$$\int_\Omega \widehat{\Phi}(x)\,\mathrm{d}x \leqq \operatorname*{ess\,sup}_{x\in\Omega}\widehat{\Phi}(x)|\Omega| \leqq \widehat{p}_1|\Omega|, \tag{4.19}$$

where $|\Omega| = \mathrm{meas}(\Omega)$.

On the other hand, from equation (4.13$_1$) we obtain the identity

$$\begin{aligned}0 &= x\cdot\nabla\widehat{p}(x) + \lambda_0 x\cdot\big(\widehat{\mathbf{w}}(x)\cdot\nabla\big)\widehat{\mathbf{w}}(x) = \operatorname{div}\big[x\,\widehat{p}(x) + \lambda_0\big(\widehat{\mathbf{w}}(x)\cdot x\big)\widehat{\mathbf{w}}(x)\big]\\ &\quad -\widehat{p}(x)\,\operatorname{div}x - \lambda_0|\widehat{\mathbf{w}}(x)|^2 = \operatorname{div}\big[x\,\widehat{p}(x) + \lambda_0\big(\widehat{\mathbf{w}}(x)\cdot x\big)\widehat{\mathbf{w}}(x)\big] - 2\widehat{\Phi}(x).\end{aligned} \tag{4.20}$$

Integrating this identity over $\Omega$ we derive

$$\begin{aligned}2\int_\Omega\widehat{\Phi}(x)\,\mathrm{d}x &= \int_{\partial\Omega}\widehat{p}(x)\big(x\cdot\mathbf{n}\big)\,\mathrm{d}S = \widehat{p}_1\int_{\Gamma_1}\big(x\cdot\mathbf{n}\big)\,\mathrm{d}S + \widehat{p}_2\int_{\Gamma_2}\big(x\cdot\mathbf{n}\big)\,\mathrm{d}S\\ &= \widehat{p}_1\int_{\Omega_1}\operatorname{div}x\,\mathrm{d}x - \widehat{p}_2\int_{\Omega_2}\operatorname{div}x\,\mathrm{d}x = 2\big(\widehat{p}_1|\Omega_1| - \widehat{p}_2|\Omega_2|\big).\end{aligned}$$

Hence,

$$\int_\Omega\widehat{\Phi}(x)\,\mathrm{d}x = \widehat{p}_1|\Omega_1| - \widehat{p}_2|\Omega_2| = \widehat{p}_1|\Omega| + \big(\widehat{p}_1 - \widehat{p}_2\big)|\Omega_2|. \tag{4.21}$$

Inequalities (4.19) and (4.21) yield

$$\widehat{p}_1 \leqq \widehat{p}_2.$$

This contradicts inequality (4.18). Thus, all solutions of integral identity (4.4) are uniformly bounded in $H(\Omega)$ and by the Leray–Schauder Theorem there exists at least one weak solution of problem (1.1). $\square$

**Remark 4.1.** The arguments of the first two steps in the proof of Theorem 4.1 also remain valid (with obvious changes) in the three-dimension case.

**Remark 4.2.** Let $\Omega = \{x : 1 < |x| < 2\}$ be the annulus and $(r, \theta)$ be the polar coordinates in $\mathbb{R}^2$. If $f \in C_0^\infty(1, 2)$, then the pair $\widehat{\mathbf{w}} = \big(\widehat{w}_r, \widehat{w}_\theta\big)$ and $\widehat{p}$ with

$$\widehat{w}_r(r,\theta) = 0, \quad \widehat{w}_\theta(r,\theta) = f(r), \quad \widehat{p}(r,\theta) = \lambda_0 \int\limits_1^r \frac{f^2(t)}{t}\,\mathrm{d}t \tag{4.22}$$

satisfy both equations (4.13) and the boundary condition $\widehat{\mathbf{w}}\big|_{\partial\Omega} = 0$ ($\widehat{w}_r$ and $\widehat{w}_\theta$ are components of the velocity field in polar coordinate system). However,

$$0 = \widehat{p}(x)\big|_{r=1} \neq \widehat{p}(x)\big|_{r=2} = \lambda_0 \int_1^2 \frac{f^2(t)}{t}\,\mathrm{d}t > 0.$$

This simple example, due to Amick [1] (see also [13], v. II, p. 59), shows that, in general, the pressure $\widehat{p}$ corresponding to the solution of Euler equations (4.13) could have different constant values on different components of the boundary.

It is interesting to observe that for a solution like (4.22) the inequality $\widehat{p}_1 = \widehat{p}(x)\big|_{r=2} > \widehat{p}_2 = \widehat{p}(x)\big|_{r=1}$ necessarily holds. Thus the solution (4.22) cannot be a limit of solutions to the Navier–Stokes problem (in the sense described in the proof of Theorem 4.1). If it were, then we would conclude from (4.8), (4.14) that $\mathcal{F} > 0$. But this, as proved in Theorem 4.1, leads to a contradiction.

We emphasize that in the case when $\mathcal{F} < 0$, problem (1.1) remains unsolved. However, in this case we do not know a counterexample showing that for the solution to Euler equations (4.13) the inequality $\widehat{p}_2 > \widehat{p}_1$ holds.

It is well known (see [3, 13]) that independently of the sign of the flux $\mathcal{F}$, problem (1.1) has a solution, if $|\mathcal{F}|$ is sufficiently small. Using this result Theorem 4.1 can be strengthened as follows

**Theorem 4.2.** *Assume that* $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ *and let condition* (1.6) *be fulfilled. Then there exists* $\mathcal{F}_0 > 0$ *such that for any* $\mathcal{F} \in [-\mathcal{F}_0, +\infty)$, *problem* (1.1) *admits at least one weak solution.*

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## Appendix (Proof of Lemma 2.6)

In order to prove Lemma 2.6 we need some simple additional statements.

**Lemma A.1.** *Let $U \subset \mathbb{R}^2$ be a bounded, simply connected domain and let $\gamma : [\alpha, \beta] \to \overline{U}$ be a continuous injective function (an arc) such that $\gamma(\alpha), \gamma(\beta) \in \partial U$ and $\gamma((\alpha, \beta)) \subset U$. Then $\gamma$ divides $U$ into two simply connected domains. More precisely, $U \setminus \gamma((\alpha, \beta)) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, each $V_j$ is a simply connected domain, $\partial V_j = \gamma((\alpha, \beta)) \cup K_j$, where $K_j \subset \partial U$ are continua.*

**Lemma A.2.** *Let $\Omega_* \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega_*$ homeomorphic to the unit circle, let $K \subset \overline{\Omega}_*$ be a continuum, and let an arc $\gamma : [\alpha, \beta] \to \overline{\Omega}_*$ have the properties $\gamma(\alpha), \gamma(\beta) \in K$ and $\gamma((\alpha, \beta)) \subset \Omega_* \setminus K$. Then there exists a connected component $\Omega_{\gamma,K}$ of the open set $\mathbb{R}^2 \setminus \big(K \cup \gamma([\alpha, \beta])\big)$ such that $\Omega_{\gamma,K}$ is a bounded, simply connected domain, $\Omega_{\gamma,K} \subset \Omega_*$ and $\partial\Omega_{\gamma,K} = \gamma([\alpha, \beta]) \cup K_\gamma$, where $K_\gamma \subset K$ is a continuum.*

**Lemma A.3.** *Let $\Omega_* \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega_*$ homeomorphic to the unit circle, let $K \subset \overline{\Omega}_*$ be a continuum, and let $\Omega_1 \subset \Omega_* \setminus K$ be a subdomain. Suppose $\Omega_1$ is contained in the unbounded connected component of the open set $\mathbb{R}^2 \setminus K$. Then there exists $\delta > 0$ such that for any arc $\gamma : [\alpha, \beta] \to \overline{\Omega}_*$ with the properties $\gamma(\alpha), \gamma(\beta) \in K$ and $\gamma((\alpha, \beta)) \subset \{x \in \Omega_* : \operatorname{dist}(x, K) < \delta\} \setminus (K \cup \Omega_1)$, the equality $\Omega_{\gamma,K} \cap \Omega_1 = \emptyset$ holds, where $\Omega_{\gamma,K}$ is an arbitrary simply connected domain from Lemma A.2.*

**Lemma A.4.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary and let $K \subset \overline{\Omega}$ be a continuum. Then for any $\delta > 0$ and for any pair $a, b \in K$ there exists an arc $\gamma : [0, 1] \to \overline{\Omega}$ with the properties $\gamma(0) = a$, $\gamma(1) = b$, and $\gamma((0, 1)) \subset \{x \in \Omega : \operatorname{dist}(x, K) < \delta\}$.*

Lemmas A.1–A.4 are easy consequences of the classical well-known facts of general topology, so we omit their proofs.

**Proof of Lemma 2.6.** Because of the definition of a domain with a Lipschitz boundary, the following representation $\Omega = \Omega_* \setminus \left(\bigcup_{i=1}^k \overline{\Omega}_i\right)$ holds, where $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ for $i \neq j$, $\overline{\Omega}_i \subset \Omega_*$, and each set $\Omega_*$, $\Omega_i$ is a bounded domain whose boundary is homeomorphic to the unit circle. Denote by $U_j$ the connected components of the open set $\Omega_* \setminus K$. Then for any $\Omega_i$ there exists an index $j(i)$ such that $\Omega_i \subset U_{j(i)}$. We may assume without loss of generality that

($\circledast$) *For each $U_j$ there exists at most one $\Omega_i \subset U_j$.*

(Really, if ($\circledast$) is not true, for example if there exist two domains $\Omega_1 \cup \Omega_2 \subset U_j$, then we can take a simply connected Lipschitz domain $\widetilde{\Omega}_1$ such that $\Omega_1 \cup \Omega_2 \subset \widetilde{\Omega}_1 \subset U_j$, $\widetilde{\Omega}_1 \Subset \Omega_*$, and consider (instead of $\Omega$ ) the Lipschitz domain $\widetilde{\Omega} = \Omega_* \setminus \operatorname{Cl}\big(\widetilde{\Omega}_1 \cup \Omega_3 \cup \dots \cup \Omega_k\big)$, etc.)

Take $\delta_1 > 0$ and a continuous injective function $\gamma : [0, 1] \to \overline{\Omega}$ with the properties $\gamma(0), \gamma(1) \in K$, and $\gamma((0, 1)) \subset \{x \in \Omega : \operatorname{dist}(x, K) < \delta_1\}$. Let $(\alpha, \beta)$ be an interval adjoining the set $\tilde{K} = \gamma^{-1}(K)$. Then there exists $U_j$ such that $\gamma((\alpha, \beta)) \subset U_j$. Now we have the following possibilities.

(I) $U_j \cap \Omega_i = \emptyset$ for all $i = 1, \dots, k$. Then we put $\Omega_{\alpha\beta} = \Omega_{\gamma_{\alpha\beta},K}$, $K_{\alpha\beta} = K_{\gamma_{\alpha\beta}}$, where $\gamma_{\alpha\beta}$ is the restriction $\gamma|_{[\alpha,\beta]}$ and $\Omega_{\gamma_{\alpha\beta},K}$, $K_{\gamma_{\alpha\beta}}$ are objects from Lemma A.2.

(II) $U_j \supset \Omega_i$ for some $i = 1, \dots, k$. This possibility splits into two cases.

(II a) $\Omega_i$ is contained in the unbounded connected component of the set $\mathbb{R}^2 \setminus K$. Then $\Omega_{\alpha,\beta}$, $K_{\alpha\beta}$ are the same as in case (I), above. This definition satisfies all the requirements in ($\aleph$) for sufficiently small $\delta_1 > 0$ because of Lemma A.3.

(II b) $\Omega_i$ is contained in the bounded connected component of the set $\mathbb{R}^2 \setminus K$. Of course, in our case this component coincides with $U_j$. Then, obviously, $U_j$ is a simply connected domain, $\partial U_j \subset K$. Take a decomposition $U_j \setminus \gamma((\alpha, \beta)) = V_1 \cup V_2$ from Lemma A.1. Since $\gamma((\alpha, \beta)) \cap \Omega_i = \emptyset$, we have either $\Omega_i \subset V_1$ or $\Omega_i \subset V_2$. Suppose, for definiteness, that the first equality is valid. Then take $\Omega_{\alpha\beta} = V_2$, $K_{\alpha\beta} = K_2$ (see Lemma A.1).

We have consider all possible cases, so Lemma 2.6 is proved. □

## References


1. Amick, Ch.J.: Existence of solutions to the nonhomogeneous steady Navier–Stokes equations. *Indiana Univ. Math. J.* **33**, 817–830 (1984)
2. Bensoussan, A., Frehse, J.: *Regularity Results for Nonlinear Elliptic Systems and Applications*. Springer, Berlin, 2002
3. Borchers, W., Pileckas, K.: Note on the flux problem for stationary Navier–Stokes equations in domains with multiply connected boundary. *Acta Appl. Math.* **37**, 21–30 (1994)
4. Bourgain, J. Korobkov, M.V., Kristensen, J.: On the Morse–Sard property and level sets of Sobolev and BV functions. *Revista Mat. Iberoamericana* (to appear). arXiv:1007.4408v1, [math.AP], 26 July 2010
5. Burago, D., Burago, Yu., Ivanov, S.: A course in metric geometry. *Graduate Studies in Mathematics*, vol. 33. AMS, 2001
6. Coifman, R.R., Lions, J.L., Meier, Y., Semmes, S.: Compensated compactness and Hardy spaces. *J. Math. Pures App. IX Sér.* **72**, 247–286 (1993)
7. Dorronsoro, J.R.: Differentiability properties of functions with bounded variation. *Indiana U. Math. J.* **38**(4), 1027–1045 (1989)
8. Evans, L.C., Gariepy, R.F.: Measure theory and fine properties of functions. *Studies in Advanced Mathematics*. CRC Press, Boca Raton, 1992
9. Finn, R.: On the steady-state solutions of the Navier–Stokes equations. III. *Acta Math.* **105**, 197–244 (1961)
10. Fujita, H.: On the existence and regularity of the steady-state solutions of the Navier–Stokes theorem. *J. Fac. Sci. Univ. Tokyo Sect. I* **9**, 59–102 (1961)
11. Fujita, H.: On stationary solutions to Navier–Stokes equation in symmetric plane domain under general outflow condition. *Pitman Research Notes in Mathematics. Proceedings of International Conference on Navier–Stokes Equations. Theory and Numerical Methods*, vol. 388, pp. 16–30. Varenna, Italy, June 1997
12. Fujita, H., Morimoto, H.: A remark on the existence of the Navier–Stokes flow with non-vanishing outflow condition. *GAKUTO Int. Ser. Math. Sci. Appl.* **10**, 53–61 (1997)
13. Galdi, G.P.: On the existence of steady motions of a viscous flow with non-homogeneous conditions. *Le Matematiche* **66**, 503–524 (1991)
14. Galdi, G.P.: *An Introduction to the Mathematical Theory of the Navier–Stokes Equations,* vol. I, II revised edition (Ed. Truesdell, C.). Springer Tracts in Natural Philosophy, vols. 38, 39. Springer, Berlin, 1998
15. Gilbarg, D., Weinberger, H.F.: Asymptotic properties of Leray's solution of the stationary two-dimensional Navier–Stokes equations. *Russian Math. Surveys* **29**, 109–123 (1974)

16. Gilbarg, D., Weinberger, H.F.: Asymptotic properties of steady plane solutions of the Navier–Stokes equations with bounded Dirichlet integral. *Ann. Scuola Norm. Pisa (4)* **5**, 381–404 (1978)
17. Heywood, J.G.: On the impossibility, in some cases, of the Leray-Hopf condition for energy estimates. *J. Math. Fluid Mech.* doi:10.1007/s00021-010-0028-8
18. Hopf, E.: Ein allgemeiner Endlichkeitssats der Hydrodynamik. *Math. Ann.* **117**, 764–775 (1941)
19. Kapitanskii, L.V., Pileckas, K.: On spaces of solenoidal vector fields and boundary value problems for the Navier–Stokes equations in domains with noncompact boundaries. *Trudy Mat. Inst. Steklov* **159**, 5–36 (1983). English Transl.: *Proc. Math. Inst. Steklov* **159**, 3–34 (1984)
20. Korobkov, M.V.: Bernoulli law under minimal smoothness assumptions. *Dokl. Math.* **83**, 107–110 (2011)
21. Kozono, H., Janagisawa, T.: Leray's problem on the Navier–Stokes equations with nonhomogeneous boundary data. *Math. Z.* **262**, 27–39 (2009)
22. Kronrod, A.S.: On functions of two variables. *Uspechi Matem. Nauk (N.S.)* **5**, 24–134 (1950, in Russian)
23. Ladyzhenskaya, O.A.: Investigation of the Navier–Stokes equations in the case of stationary motion of an incompressible fluid. *Uspech Mat. Nauk.* **3**, 75–97 (1959, in Russian)
24. Ladyzhenskaya, O.A.: *The Mathematical Theory of Viscous Incompressible Fluid*. Gordon and Breach, 1969
25. Ladyzhenskaya, O.A., Solonnikov, V.A.: On some problems of vector analysis and generalized formulations of boundary value problems for the Navier–Stokes equations. *Zapiski Nauchn. Sem. LOMI* **59**, 81–116 (1976, in Russian)
26. Landis, E.M.: *Second Order Equations of Elliptic and Parabolic Type*. Nauka, 1971 (in Russian)
27. Leray, J.: Étude de diverses équations intégrales non linéaire et de quelques problèmes que pose l'hydrodynamique. *J. Math. Pures Appl.* **12**, 1–82 (1933)
28. Littman, W., Stampacchia, G., Weinberger, H.F.: Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Pisa (3)* **17**, 43–77 (1963)
29. Maz'ya, V.G.: *Sobolev Spaces*. Springer, Berlin, 1985
30. Miranda, C.: *Partial Differential Equations of Elliptic Type*. Springer, Berlin, 1970
31. Morimoto, H.: A remark on the existence of 2-D steady Navier–Stokes flow in bounded symmetric domain under general outflow condition. *J. Math. Fluid Mech.* **9**(3), 411–418 (2007)
32. Pukhnachev, V.V.: Viscous flows in domains with a multiply connected boundary. *New Directions in Mathematical Fluid Mechanics. The Alexander V. Kazhikhov Memorial Volume* (Eds. Fursikov, A.V., Galdi, G.P., Pukhnachev, V.V.). Birkhäuser, Basel, pp. 333–348, 2009
33. Pukhnachev, V.V.: The Leray problem and the Yudovich hypothesis. *Izv. vuzov. Sev.-Kavk. region. Natural sciences. The special issue "Actual Problems of Mathematical Hydrodynamics"*, pp. 185–194, 2009 (in Russian)
34. Russo, R.: On the existence of solutions to the stationary Navier–Stokes equations. *Ricerche Mat.* **52**, 285–348 (2003)
35. Russo, A.: A note on the two-dimensional steady-state Navier–Stokes problem. *J. Math. Fluid Mech.* **52**, 407–414 (2009)
36. Russo, A., Starita, G.: On the existence of steady-state solutions to the Navier–Stokes system for large fluxes. *Ann. Scuola Norm. Sup. Pisa* **7**, 171–180 (2008)
37. Sazonov, L.I.: On the existence of a stationary symmetric solution of the two-dimensional fluid flow problem. *Mat. Zametki* **54**(6), 138–141 (1993, in Russian). English Transl.: *Math. Notes* **54**(6), 1280–1283 (1993)
38. Solonnikov, V.A., Scadilov, V.E.: On a boundary value problem for a stationary system of Navier–Stokes equations. *Proc. Steklov Inst. Math.* **125**, 186–199 (1973, in Russian)

39. Takashita, A.: A remark on Leray's inequality. *Pac. J. Math.* **157**, 151–158 (1993)
40. Taylor, M.E.: *Partial Differential Equations III. Nonlinear Equations*. Springer, Berlin, 1996
41. Vorovich, I.I., Yudovich, V.I.: Stationary flows of a viscous incompres-sible fluid. *Mat. Sbornik* **53**, 393–428 (1961, in Russian)
42. Yudovich, V.I.: Eleven great problems of mathematical hydrodynamics. *Mosc. Math. J.* **3**(2), 711–737 (2003)

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