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The existence of a solution with finite Dirichlet integral for the steady Navier–Stokes equations in a plane exterior symmetric domain

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Abstract

We study the nonhomogeneous boundary value problem for the Navier–Stokes equations of steady motion of a viscous incompressible fluid in a two-dimensional exterior multiply connected domain $\mathbb{R}^2 \setminus (\bigcup_{j=1}^N \bar{\Omega}_j)$. We prove that this problem has a solution if Ω and the boundary datum are axially symmetric. We have no restriction on fluxes, in particular, they could be arbitrary large.

Résumé

Dans cet article, on étudie le système stationnaire, incompressible de Navier–Stokes dans un domaine extérieur bidimensionnel $\mathbb{R}^2 \setminus (\bigcup_{j=1}^N \bar{\Omega}_j)$, axialement symétrique avec des conditions d'adhérence au bord. On démontre que le problème a une solution dans l'hypothèse unique que les données sont symétriques.

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1. Introduction

Let Ω be the exterior domain of $\mathbb{R}^2$

$$\Omega = \mathbb{R}^2 \setminus \left(\bigcup_{j=1}^N \bar{\Omega}_j \right), \tag{1.1}$$

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where $\Omega_j \subset \mathbb{R}^2$, $j = 1, \dots, N$, are bounded, simply connected domains with Lipschitz boundaries and $\bar{\Omega}_j \cap \bar{\Omega}_i = \emptyset$ for $i \neq j$. The steady-state Navier–Stokes problem in plane exterior domains is to find a solution to the equations

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \quad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{h} \quad & \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

$$\lim_{|x|\to+\infty} \mathbf{u}(x) = \xi\mathbf{e}_1, \tag{1.3}$$

where $\mathbf{u}$, p are the (unknown) velocity and pressure fields respectively, $\nu > 0$ is the coefficient of viscosity, $\xi\mathbf{e}_1$, $\mathbf{h}$, $\mathbf{f}$ are the (assigned) velocity value at infinity, boundary datum, and body force field. We assume for simplicity $\mathbf{f}$ vanishes outside a disk.

In a famous paper published in 1933 J. Leray [19] proved that if the data are sufficiently regular, $\mathbf{f} = 0$ and the fluxes through every $\partial\Omega_i$ vanish

$$F_i = \int_{\partial\Omega_i} \mathbf{h}\cdot\mathbf{n}\, dS = 0, \tag{1.4}$$

where $\mathbf{n}$ is the outward (with respect to Ω) unit normal to $\partial\Omega_i$, then problem (1.2) has a weak solution $(\mathbf{u}, p)$ with finite Dirichlet integral

$$\int_{\Omega} |\nabla\mathbf{u}|^2\, dx < +\infty. \tag{1.5}$$

To show this, Leray introduced an elegant argument, since known as *invading domains method*, which consists in proving first that the Navier–Stokes problem

$$\begin{aligned} -\nu\Delta\mathbf{u}_k + (\mathbf{u}_k\cdot\nabla)\mathbf{u}_k + \nabla p_k = 0 \quad & \text{in } \Omega_k, \\ \operatorname{div}\mathbf{u}_k = 0 \quad & \text{in } \Omega_k, \\ \mathbf{u}_k = \mathbf{h} \quad & \text{on } \partial\Omega, \\ \mathbf{u}_k = \xi\mathbf{e}_1 \quad & \text{on } \partial B_k \end{aligned} \tag{1.6}$$

has a weak solution $\mathbf{u}_k$ for every bounded domain $\Omega_k = \Omega \cap B_k$, $B_k = \{x \in \mathbb{R}^2\colon\ |x| < k\}$, $B_k \Supset \complement\Omega$, then to show that the following estimate holds:

$$\int_{\Omega_k} |\nabla\mathbf{u}_k|^2\, dx \leqslant c, \tag{1.7}$$

for some positive constant c independent of k. While (1.7) is sufficient to assures existence of a subsequence $\mathbf{u}_{k_l}$ which converges weakly to a solution of (1.2) satisfying (1.5), it does not give any information about the behavior at infinity of the velocity $\mathbf{u}$,[1] i.e., we do not know whether $\mathbf{u}$ satisfies the condition at infinity (1.3). In 1961 H. Fujita [7] rediscovered, by means of a different method, Leray's result. Nevertheless, due to the lack of a uniqueness theorem, the solutions constructed by Leray and Fujita are not comparable, even for very small ν. We then call by *Leray's solution* the solution constructed by invading domains method and by *D-solution* any solution to (1.2) which satisfies (1.5). Forty years later after the appearing of Leray's paper, D. Gilbarg and H.F. Weinberger [13] were able to show that the velocity $\mathbf{u}$ in Leray's solution is bounded, p converges uniformly to a constant at infinity and there is a constant vector $\bar{\mathbf{u}}$ such that[2]

$$\lim_{R\to+\infty} \int_0^{2\pi} \left|\mathbf{u}(R,\theta) - \bar{\mathbf{u}}\right|^2 d\theta = 0. \tag{1.8}$$

[1] Indeed, the unbounded function $\log^\alpha |x|$ ($\alpha \in (0, 1/2)$) satisfies (1.6).

[2] (R, θ) denote polar coordinates with center at O.

Moreover, they proved that if $\bar{\mathbf{u}} = 0$, then the convergence is uniform and $\nabla\mathbf{u}$ decays at infinity as $r^{\varepsilon-3/4}$ for every positive ε. In the subsequent paper [14], the same authors proved that a bounded D-solution met the same asymptotic properties as Leray's solution. One of the most difficult and unanswered questions is the relation between $\xi\mathbf{e}_1$ and $\bar{\mathbf{u}}$. To point out the difficulties of the problem, let us recall that even the linearized Navier–Stokes problem

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p = 0 \quad & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \quad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{h} \quad & \text{on } \partial\Omega, \\ \lim_{|x|\to+\infty} \mathbf{u}(x) = \xi\mathbf{e}_1, & \end{aligned} \tag{1.9}$$

does not have, in general, a solution. Indeed, one proves that the solutions of the problem

$$\begin{aligned} -\nu\Delta\mathbf{v} + \nabla Q = 0 \quad & \text{in } \Omega, \\ \operatorname{div}\mathbf{v} = 0 \quad & \text{in } \Omega, \\ \mathbf{v} = 0 \quad & \text{on } \partial\Omega, \\ \lim_{|x|\to+\infty} \frac{\mathbf{v}(x)}{|x|} = 0, & \end{aligned}$$

spans a two-dimensional linear space $\mathfrak{C}$ and that (1.9) is solvable *if and only if* the data satisfy the following compatibility condition (Stokes' paradox)[3]

$$\int_{\partial\Omega} (\mathbf{h} - \xi\mathbf{e}_1) \cdot \left[\nu\left(\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right) \cdot \mathbf{n} - Q\mathbf{n}\right] dS = 0, \quad \forall\mathbf{v} \in \mathfrak{C} \tag{1.10}$$

(see [3,11]).

It sounded then astonishing the discovery of R. Finn and D.R. Smith in 1967 [6] of the existence of a solution to (1.2), (1.3) without any compatibility relation between $\mathbf{h}$ and $\xi \neq 0$, for ν sufficiently large. They also showed that $(\mathbf{u} - \xi\mathbf{e}_1, p)$ behaves at infinity as the fundamental solution of the linear Oseen system (see also [9]). In particular, taking also into account the results in [4,28] one obtains the following behavior[4]

$$\begin{aligned} u_1 - \xi = O\left(r^{-1/2}\right), \qquad & u_2 = O\left(r^{-1}\log r\right), \\ \nabla\mathbf{u} = O\left(r^{-1}\log^2 r\right), \qquad & p = O\left(r^{-1}\log r\right), \end{aligned} \tag{1.11}$$

and outside a parabolic "wake region" around axis $\mathbf{e}_1$ the decay is more rapid, in particular, $\omega = \partial_1 u_2 - \partial_2 u_1$ behaves according to

$$\omega(x) = O\left(e^{\frac{1}{2}(\xi x_1 - |\xi| r)}\right). \tag{1.12}$$

R. Finn and D.R. Smith called a solution $(\mathbf{u}, p)$ to (1.2), (1.3) *physically reasonable* provided $\mathbf{u} - \xi\mathbf{e}_1 = O(r^{-1/4-\varepsilon})$ for some positive ε. D.R. Smith [28] proved that a physically reasonable solution satisfies (1.11) and D.C. Clark [4] that (1.11) implies (1.12). More recently, V.I. Sazonov [27] showed that a D-solution such that $\mathbf{u} - \xi\mathbf{e}_1 = o(1)$, with $\xi \neq 0$, is physically reasonable (see also [12,10]). Notice that nothing is currently known about the asymptotic behavior, in general, for $\xi = 0$ or for arbitrary ν.

Later, in 1988, problem (1.2), (1.3) was taken up by Ch.J. Amick [2] under the assumption $\mathbf{f} = 0$. He proved that if $\mathbf{h} = 0$, then any D-solution is bounded and converges to $\bar{\mathbf{u}}$ in the sense of (1.8). Moreover, he considered a particular but physically interesting class of solutions $\mathbf{u} = (u_1, u_2)$ such that u_1 is an even function of x_2 and u_2 is an odd function of x_2:

$$u_1(x_1, x_2) = u_1(x_1, -x_2), \qquad u_2(x_1, x_2) = -u_2(x_1, -x_2) \tag{1.13}$$

[3] Let us observe, by the way, that this is not surprising. Indeed, the natural solution to $(1.9)_{1,2,3}$ should behave at infinity as the fundamental solution to (1.9) ($\mathbf{u} = O(\log r)$) and the addition of $(1.9)_4$ makes (1.9) over-determined. Therefore, (1.10) appears to be quite natural.

[4] Here the symbol $f(x) = O(g(r))$ means that there is a positive constant c such that $|f(x)| \leqslant cg(r)$ for large r.

in the symmetric domain

$$(x_1, x_2) \in \Omega \quad \Leftrightarrow \quad (x_1, -x_2) \in \Omega. \tag{1.14}$$

Using Leray's argument Ch. Amick showed that for symmetric solutions the convergence of $\mathbf{u}$ at infinity is uniform, moreover, if $\partial\Omega$ is regular enough and $\mathbf{h} = 0$, then $\mathbf{u}$ is nontrivial.[5] The last results rule out the Stokes paradox for the nonlinear case for symmetric domains and vanishing boundary data. A clear exposition of Amick's results, as well as the results outlined above, can be found in [9]. For an exterior domain condition (1.4) has been replaced in [24] by the weaker assumption that the sum $\sum_i |F_i|$ is sufficiently small. Finally, we mention the recent paper [21] by the authors, where the problem (1.2), (1.3) with $\xi = 0$ was considered in exterior plane domains symmetric with respect to both coordinate axes and a solution was found in the class $\mathfrak{C}_0$ of vector fields $\mathbf{v}$ satisfying the following symmetry conditions

$$\begin{aligned} v_1(x_1, x_2) &= v_1(x_1, -x_2) = -v_1(-x_1, x_2), \\ v_2(x_1, x_2) &= -v_2(x_1, -x_2) = v_2(-x_1, x_2) \end{aligned} \tag{1.15}$$

(the class $\mathfrak{C}_0$ is defined by these conditions). It is proved in [21] that if data $\mathbf{h}, \mathbf{f} \in \mathfrak{C}_0$ satisfy only natural regularity assumptions, then (1.2) has a D-solution in $\mathfrak{C}_0$ which converges uniformly to zero at infinity. The flux of the boundary value $\mathbf{h}$ over $\partial\Omega$ in this case is arbitrary.

All mentioned above results (except [21]) were proved either under the condition that all fluxes F_i are equal to zero (see (1.4)) or assuming that fluxes F_i are "small". Besides the relation between $\xi\mathbf{e}_1$ and $\bar{\mathbf{u}}$, another relevant problem in the theory of the stationary Navier–Stokes equations is to ascertain whether a solution to problem (1.2) exists without any restriction on the fluxes F_i. Even in the case of bounded domains (see, for example, [10,22,23]) this problem, in general, is unsolved until now. The first result in this direction for bounded symmetric domains $\Omega_0 \setminus \bigcup_{i=1}^N \bar{\Omega}_i$ such that every Ω_i is intersecting the x_1-axis is due to C.J. Amick [1] (see also [8,20,5,26]). In the recent paper [15] we have proved that in a bounded two-dimensional domain a weak solution of (1.2) exists for every data, provided $N = 1$ and $F_1 > 0$ (see also [16,17] where the axially symmetric three-dimensional case is studied).

The goal of this paper is to prove for arbitrary fluxes F_i the existence of a solution to problem (1.2) for exterior plane domains in the case when only Amick's symmetry conditions (1.13), (1.14) are satisfied and every Ω_i intersects the x_1-axis, i.e.,

$$\Omega_i \cap O_{x_1} \neq \emptyset \quad \text{for all } i = 1, \dots, N.$$

We also do not require the total flux

$$F = \int_{\partial\Omega} \mathbf{h}(x) \cdot \mathbf{n}(x)\, dS = \sum_{i=1}^N F_i \tag{1.16}$$

to be zero or small. By what was said before, if $\mathbf{f}$ has a compact support, then the solution converges uniformly at infinity to a constant vector $\alpha\mathbf{e}_1$, moreover, for $\alpha \neq 0$, it behaves at large distance according to (1.11), (1.12). The proof of this result is based on the Leray–Hopf inequality which is obtained by applying a new inequality of Poincaré type (see Lemma 2.3) that could be useful also in other contexts. The existence theorem is proved in Section 4. In Section 2 we collect the main results that we need to prove in Section 3 the Leray–Hopf inequality (see Lemma 3.4).

2. Main notations and auxiliary results

We use standard notations for function spaces: $C^k(\bar{\Omega})$, $L^q(\Omega)$, $W^{k,q}(\Omega)$, $\mathring{W}^{k,q}(\Omega)$, $W^{\alpha,q}(\partial\Omega)$, where $\alpha \in (0,1)$, $k \in \mathbb{N}_0$, $q \in [1, +\infty]$. In our notation we do not distinguish function spaces for scalar and vector valued functions; it is clear from the context whether we use scalar or vector (or tensor) valued function spaces.

For any set of functions $V(\Omega)$ defined in the symmetric domain Ω satisfying (1.14), we denote by $V_S(\Omega)$ the subspace of symmetric functions (i.e., satisfying (1.13)) from $V(\Omega)$. If the continuous function $\mathbf{u}(x)$ is symmetric, then

[5] Amick assumes Ω to be of class C^3. Recently, this result has been extended to Lipschitz domains [25].

$$u_2(x_1,0)=0. \tag{2.1}$$

For Sobolev functions $\mathbf{v}\in W^{1,2}_S(\Omega)$ the equality (2.1) is valid in the sense of traces.

Let $D(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in the Dirichlet norm

$$\|v\|_{D(\Omega)}=\|\nabla v\|_{L^2(\Omega)}.$$

Lemma 2.1. *Let Ω be the exterior domain (1.1), $v\in D(\Omega)$. Then the following inequality*

$$\int\limits_\Omega \frac{|v(x)|^2}{|x|^2\log^2|x|}\,dx\leqslant c\int\limits_\Omega \bigl|\nabla v(x)\bigr|^2\,dx \tag{2.2}$$

holds.

Inequality (2.2) is well known (e.g., [18]).

As follows from (2.2), functions $v\in D(\Omega)$ do not have to vanish at infinity. The next assertion gives some information about the behavior of a function of $D(\Omega)$ as $|x|\to\infty$.

Lemma 2.2. *Let Ω be the exterior domain (1.1), $v\in D(\Omega)$. Then*

$$\limsup_{r\to\infty}\frac{1}{\log r}\int\limits_0^{2\pi}\bigl|v(r,\theta)\bigr|^2\,d\theta\leqslant 2\int\limits_\Omega\bigl|\nabla v(x)\bigr|^2\,dx. \tag{2.3}$$

Here (r,θ) are polar coordinates in $\mathbb{R}^2$.

Inequality (2.3) is proved in [14] (see Lemma 2.1).

If Ω is the exterior domain (1.1), then by construction there exists a positive number R_0 such that $B_{R_0}\Supset \complement\Omega$. Without loss of generality, we may assume that $R_0>1$.

Lemma 2.3. *Let Ω be the exterior domain (1.1), $v\in D(\Omega)$, $\kappa>0$, $\alpha\in(0,1)$, $R_*\geqslant R_0>1$. Then the following inequality*

$$\int\limits_{\mathbb{R}\setminus(-R_*,R_*)}\int\limits_0^{\kappa|x_1|^\alpha}\frac{|v(x_1,x_2)|^2}{|x|^2}\,dx_1\,dx_2\leqslant c\int\limits_\Omega\bigl|\nabla v(x)\bigr|^2\,dx \tag{2.4}$$

holds. The constant c in (2.4) depends only on R_0,κ and α.

Proof. Consider first the domain $\omega_+=\{x:\ x_1>R_*,\ 0<x_2<\kappa x_1^\alpha\}$. Obviously,

$$\begin{aligned}\omega_+\subset\Xi&=\bigl\{(r,\theta):\ r>R_*,\ 0\leqslant\theta\leqslant\vartheta(r)=\arctan\bigl(c_*\kappa r^{\alpha-1}\bigr)\bigr\}\\ &\subset\Xi_*=\bigl\{(r,\theta):\ r>R_*,\ 0\leqslant\theta\leqslant\vartheta(R_*)\bigr\}\end{aligned}$$

with some constant $c_*>0$ depending only on R_0 and α. Let

$$\bar v(r)=\frac{1}{\vartheta(R_*)}\int\limits_0^{\vartheta(R_*)}v(r,\theta)\,d\theta,\qquad \hat v(r,\theta)=v(r,\theta)-\bar v(r).$$

Then, by Poincaré's inequality

$$\begin{aligned}\int\limits_{\omega_+}\frac{|\hat v(r,\theta)|^2}{r^2}\,dx &\leqslant \int\limits_{\Xi_*}\frac{|\hat v(r,\theta)|^2}{r^2}\,dx = \int\limits_{R_*}^{\infty}\frac{1}{r}\int\limits_0^{\vartheta(R_*)}\left|\hat v(r,\theta)\right|^2 d\theta\,dr\\ &\leqslant c\vartheta(R_*)^2\int\limits_{r_0}^{\infty}\frac{1}{r}\int\limits_0^{\vartheta(R_*)}\left|\frac{\partial\hat v(r,\theta)}{\partial\theta}\right|^2 d\theta\,dr = c\vartheta(R_*)^2\int\limits_{r_0}^{\infty}\frac{1}{r}\int\limits_0^{\vartheta(R_*)}\left|\frac{\partial v(r,\theta)}{\partial\theta}\right|^2 d\theta\,dr\\ &\leqslant c\vartheta(R_*)^2\int\limits_{\Omega}|\nabla v|^2\,dx. \end{aligned}\tag{2.5}$$

Consider the integral

$$\begin{aligned}\int\limits_{\omega_+}\frac{|\bar v(r)|^2}{r^{2-\gamma}}\,dx &\leqslant \int\limits_{\Xi}\frac{|\bar v(r)|^2}{r^{2-\gamma}}\,dx = \int\limits_{R_*}^{\infty}\frac{|\bar v(r)|^2}{r^{1-\gamma}}dr\int\limits_0^{\vartheta(r)}d\theta\\ &= \int\limits_{R_*}^{\infty}\frac{\vartheta(r)|\bar v(r)|^2}{r^{1-\gamma}}\,dr \leqslant \frac{1}{\vartheta(R_*)}\int\limits_{R_*}^{\infty}\frac{\vartheta(r)}{r^{1-\gamma}}\int\limits_0^{\vartheta(R_*)}\left|v(r,\theta)\right|^2 d\theta\,dr\\ &\leqslant \frac{c}{\vartheta(R_*)}\int\limits_{R_*}^{\infty}\frac{1}{r^{2-\gamma-\alpha}}\int\limits_0^{2\pi}\left|v(r,\theta)\right|^2 d\theta\,dr. \end{aligned}\tag{2.6}$$

Here we have used the obvious inequality $|\vartheta(r)| \leqslant cr^{\alpha-1}$ for $r \geqslant R_*$. From (2.3) we have

$$\frac{1}{\log r}\int\limits_0^{2\pi}\left|v(r,\theta)\right|^2 d\theta \leqslant c\int\limits_{\Omega}\left|\nabla v(x)\right|^2 dx \quad \text{for } r > R_*,$$

and, therefore, the right-hand side of (2.6) can be estimated as follows

$$\begin{aligned}\int\limits_{R_*}^{\infty}\frac{1}{r^{2-\gamma-\alpha}}\int\limits_0^{2\pi}\left|v(r,\theta)\right|^2 d\theta\,dr &\leqslant c\int\limits_{\Omega}|\nabla v|^2\,dx\left(\int\limits_{R_*}^{\infty}\frac{\log r}{r^{2-\gamma-\alpha}}\,dr\right)\\ &\leqslant c\int\limits_{\Omega}|\nabla v|^2\,dx \quad \text{if } \gamma+\alpha < 1. \end{aligned}\tag{2.7}$$

From (2.6) and (2.7) we obtain the inequality

$$\int\limits_{\omega_+}\frac{|\bar v(r)|^2}{r^{2-\gamma}}\,dx \leqslant \frac{c}{\vartheta(R_*)}\int\limits_{\Omega}\left|\nabla v(x)\right|^2 dx \quad \forall\gamma\in[0,1-\alpha).\tag{2.8}$$

In virtue of (2.5) and (2.8) we have

$$\int\limits_{\omega_+}\frac{|v(x)|^2}{r^2}\,dx \leqslant 2\int\limits_{\omega_+}\frac{|\hat v(x)|^2}{r^2}\,dx + 2\int\limits_{\omega_+}\frac{|\bar v(x)|^2}{r^2}\,dx \leqslant c\int\limits_{\Omega}\left|\nabla v(x)\right|^2 dx.\tag{2.9}$$

Analogously it can be proved that

$$\int\limits_{\omega_-}\frac{|v(x)|^2}{r^2}\,dx \leqslant c\int\limits_{\Omega}\left|\nabla v(x)\right|^2 dx,\tag{2.10}$$

where $\omega_- = \{x\colon\ x_1 < -R_*,\ 0 < x_2 < \kappa(-x_1)^\alpha\}$. Inequality (2.4) follows from (2.9), (2.10). □

Remark 2.1. In fact we have proved a stronger result than that stated in Lemma 2.3. Namely, any function $v \in D(\Omega)$ can be represented as a sum

$$v(r,\theta) = \hat{v}(r,\theta) + \bar{v}(r),$$

where for $\hat{v}$ and $\bar{v}$ the estimates

$$\int\limits_{\omega} \frac{|\hat{v}(r,\theta)|^2}{r^2}\,dx \leqslant c\vartheta(R_*)^2 \int\limits_{\Omega} |\nabla v|^2\,dx, \tag{2.11}$$

$$\int\limits_{\omega_+} \frac{|\bar{v}(r)|^2}{r^{2-\gamma}}\,dx \leqslant \frac{c}{\vartheta(R_*)} \int\limits_{\Omega} \left|\nabla v(x)\right|^2 dx \quad \forall \gamma \in [0, 1-\alpha) \tag{2.12}$$

hold, with $\omega = \omega_+ \cup \omega_-$. In this paper only inequality (2.4) is used and we do not need more precise estimates (2.11), (2.12). However, they may be interesting by themselves since the constant $\vartheta(R_*)$ in (2.11) becomes arbitrary small as $R_* \to \infty$, while in (2.12) we have additional decay exponent r^γ, $\gamma \in [0, 1-\alpha)$.

Denote by $H(\Omega)$ the space of divergence free and equal to zero on $\partial\Omega$ vector fields with the finite Dirichlet integral:

$$H(\Omega) = \left\{\mathbf{u}\colon\ \mathbf{u}|_{\partial\Omega} = 0,\ \operatorname{div}\mathbf{u} = 0,\ \|\mathbf{u}\|_{H(\Omega)} = \|\nabla\mathbf{u}\|_{L_2(\Omega)} < \infty\right\},$$

where

$$\|\nabla\mathbf{u}\|^2_{L_2(\Omega)} = \int\limits_{\Omega} \sum_{i,j=1}^{2} \left|\frac{\partial u_i(x)}{\partial x_j}\right|^2 dx.$$

It is well known (see [10]) that each element $\mathbf{u} \in H(\Omega)$ can be approximated in the norm $\|\cdot\|_{H(\Omega)}$ by smooth divergence free vector fields $\mathbf{u}_n \in J_0^\infty(\Omega) = \{\mathbf{w} \in C_0^\infty(\Omega)\colon \operatorname{div}\mathbf{w} = 0\}$. This fact implies[6] $H_S(\Omega)$ the closure of $J_{0S}^\infty(\Omega)$ in the norm $\|\cdot\|_{H(\Omega)}$. Notice that functions from $H_S(\Omega)$ satisfy relations (2.1) in the sense of traces.

3. Construction of the extension

In this section we will construct an extension of the boundary value which satisfies Leray–Hopf's inequality (3.49).

Let $\psi \in C^\infty(\mathbb{R})$ be a nonnegative function such that $0 \leqslant \psi(t) \leqslant 1$,

$$\psi(t) = \begin{cases} 1, & t \geqslant 1, \\ 0, & t \leqslant 0, \end{cases} \tag{3.1}$$

and let $\gamma \in C^\infty(\mathbb{R})$ be a monotone function on $\mathbb{R}_+$ with $\gamma(t) \geqslant \gamma_0 > 0$,

$$\gamma(t) = \begin{cases} |t|^\alpha, & |t| \geqslant 3R_0, \\ 1, & |t| \leqslant 2R_0, \end{cases} \tag{3.2}$$

where $\alpha \in (0, 1)$.

Let $\Omega_+ = \{x \in \Omega\colon x_2 > 0\}$ and $\Omega_- = \{x \in \Omega\colon x_2 < 0\}$. Set

$$\Delta_+(x) = x_2\big(\chi(x_1) + \big(1 - \chi(x_1)\big)\delta(x)\big), \quad x \in \Omega_+, \tag{3.3}$$

where $\chi \in C^\infty(\mathbb{R})$ is a monotone function with

[6] Indeed, if $\mathbf{u}^n = (u_1^n, u_2^n) \in J_0^\infty(\Omega)$ and $\|\mathbf{u}^n - \mathbf{u}\|_{H(\Omega)} \to 0$ with $\mathbf{u} = (u_1, u_2) \in H_S(\Omega)$, then the vector field $\tilde{\mathbf{u}}^n$ defined by the formulas

$$\tilde{\mathbf{u}}^n(x_1, x_2) = \frac{1}{2}\big(u_1^n(x_1, x_2) + u_1^n(x_1, -x_2), u_2^n(x_1, x_2) - u_2^n(x_1, -x_2)\big)$$

belongs to $J_{0S}^\infty(\Omega)$ and also $\|\tilde{\mathbf{u}}^n - \mathbf{u}\|_{H(\Omega)} \to 0$.

$$\chi(t)=\begin{cases}1, & |t|\geqslant 2R_0,\\ 0, & |t|\leqslant \frac{3}{2}R_0,\end{cases}$$

and $\delta(x)$ is the regularized distance from the point $x\in\Omega$ to $\partial\Omega=\bigcup_{j=1}^N\Gamma_j$. Notice that $\delta(x)$ is infinitely differentiable function in $\mathbb{R}^2\setminus\partial\Omega$ and the following inequalities

$$a_1 d(x)\leqslant\delta(x)\leqslant a_2 d(x),\qquad \left|D^\alpha\delta(x)\right|\leqslant a_3 d^{1-|\alpha|}(x) \tag{3.4}$$

hold. Here $d(x)=dist(x,\partial\Omega)$ is the Euclidean distance from x to $\partial\Omega$ (see [29]).

Let $\varepsilon\in(0,1)$ be an arbitrary number. In the domain Ω_+ we define the cut-off function

$$\zeta_+(x,\varepsilon)=\psi\left(\varepsilon\ln\left(\frac{\varepsilon\gamma(x_1)}{\Delta_+(x)}\right)\right). \tag{3.5}$$

Obviously,

$$\zeta_+(x,\varepsilon)=\begin{cases}0, & \varepsilon\gamma(x_1)<\Delta_+(x),\\ 1, & \Delta_+(x)<\varepsilon e^{-\frac{1}{\varepsilon}}\gamma(x_1).\end{cases} \tag{3.6}$$

Lemma 3.1. *For the derivatives of $\zeta_+(x,\varepsilon)$ the following estimates*

$$\left|\frac{\partial\zeta_+(x,\varepsilon)}{\partial x_k}\right|\leqslant\frac{c_1\varepsilon}{\Delta_+(x)}, \tag{3.7}$$

$$\left|\frac{\partial\zeta_+(x,\varepsilon)}{\partial x_k}\right|\leqslant\frac{c(\varepsilon)}{\gamma(x_1)},\qquad \left|\frac{\partial^2\zeta_+(x,\varepsilon)}{\partial x_k\partial x_r}\right|\leqslant\frac{c(\varepsilon)}{\gamma^2(x_1)} \tag{3.8}$$

hold. The constant c_1 in (3.7) is independent of ε, while $c(\varepsilon)$ in (3.8) tends to ∞ as $\varepsilon\to 0$.

Proof. We have

$$\frac{\partial\zeta_+(x,\varepsilon)}{\partial x_k}=\varepsilon\psi'\left(\varepsilon\ln\left(\frac{\varepsilon\gamma(x_1)}{\Delta_+(x)}\right)\right)R_k(x), \tag{3.9}$$

$$\frac{\partial^2\zeta_+(x,\varepsilon)}{\partial x_k\partial x_l}=\varepsilon^2\psi''\left(\varepsilon\ln\left(\frac{\varepsilon\gamma(x_1)}{\Delta_+(x)}\right)\right)R_k(x)R_l(x)+\varepsilon\psi'\left(\varepsilon\ln\left(\frac{\varepsilon\gamma(x_1)}{\Delta_+(x)}\right)\right)\frac{\partial R_k(x)}{\partial x_l}, \tag{3.10}$$

where

$$R_k(x)=\frac{1}{\gamma(x_1)}\frac{\partial\gamma(x_1)}{\partial x_k}-\frac{1}{\Delta_+(x)}\frac{\partial\Delta_+(x)}{\partial x_k}, \tag{3.11}$$

so that

$$\begin{aligned}\frac{\partial R_k(x)}{\partial x_l}&=\left\{\frac{1}{\gamma(x_1)}\frac{\partial^2\gamma(x_1)}{\partial x_k\partial x_l}-\frac{1}{\gamma^2(x_1)}\frac{\partial\gamma(x_1)}{\partial x_k}\frac{\partial\gamma(x_1)}{\partial x_l}\right\}\\ &\quad+\left\{\frac{1}{\Delta_+(x)}\frac{\partial^2\Delta_+(x)}{\partial x_k\partial x_l}-\frac{1}{\Delta_+^2(x)}\frac{\partial\Delta_+(x)}{\partial x_k}\frac{\partial\Delta_+(x)}{\partial x_l}\right\}.\end{aligned} \tag{3.12}$$

By construction

$$\operatorname{supp}\nabla\zeta_+\subset\left\{x:\ \varepsilon e^{-\frac{1}{\varepsilon}}\gamma(x_1)\leqslant\Delta_+(x)\leqslant\varepsilon\gamma(x_1)\right\}. \tag{3.13}$$

Using (3.13), the properties of the regularized distance $\delta(x)$ (see (3.4)) and the definition (3.3) of the function $\Delta_+(x)$ we conclude that

$$\left|\nabla\Delta_+(x)\right|\leqslant\text{const}\quad\text{for } x\in\operatorname{supp}\nabla\zeta_+;$$
$$\operatorname{supp}\left(\nabla^2\Delta_+\right)\subset\left\{x:\ |x_1|\leqslant 2R_0\right\}.$$

Therefore, (3.11), (3.12) and (3.2) yield

$$\left|R_k(x)\right| \leqslant C\left(\frac{1}{\gamma(x_1)} + \frac{1}{\Delta_+(x)}\right), \qquad \left|\frac{\partial R_k(x)}{\partial x_l}\right| \leqslant C\left(\frac{1}{\gamma^2(x_1)} + \frac{1}{\Delta_+^2(x)}\right). \tag{3.14}$$

Estimates (3.7) and (3.8) follow now from (3.13) and inequalities (3.9), (3.10), (3.14). □

Remark 3.1. For $|x_1| \leqslant \frac{3}{2}R_0$ the following equalities $\Delta_+(x) = x_2\delta(x)$, $\gamma(x_1) = 1$ hold. Then

$$R_1(x) = -\frac{1}{\Delta_+(x)}\frac{\partial\Delta_+(x)}{\partial x_1} = -\frac{1}{\delta(x)}\frac{\partial\delta(x)}{\partial x_1}, \tag{3.15}$$

$$R_2(x) = -\frac{1}{\Delta_+(x)}\frac{\partial\Delta_+(x)}{\partial x_2} = -\frac{1}{x_2} - \frac{1}{\delta(x)}\frac{\partial\delta(x)}{\partial x_2}. \tag{3.16}$$

For $|x_1| \geqslant 2R_0$ we have $\Delta_+(x) = x_2$ and

$$R_1(x) = \frac{\gamma'(x_1)}{\gamma(x_1)}, \qquad R_2(x) = -\frac{1}{x_2}. \tag{3.17}$$

Moreover, if $|x_1| \geqslant 3R_0$, then $\gamma(x_1) = |x_1|^\alpha$ and

$$R_1(x) = \frac{\alpha}{|x_1|}. \tag{3.18}$$

Therefore, it follows from (3.9) that

$$\left|\frac{\partial\zeta_+(x,\varepsilon)}{\partial x_1}\right| \leqslant \frac{c\varepsilon}{\delta(x)}, \qquad \left|\frac{\partial\zeta_+(x,\varepsilon)}{\partial x_2}\right| \leqslant c\varepsilon\left(\frac{1}{|x_2|} + \frac{1}{\delta(x)}\right) \quad \text{for } |x_1| \leqslant \frac{3}{2}R_0, \tag{3.19}$$

$$\left|\frac{\partial\zeta_+(x,\varepsilon)}{\partial x_1}\right| \leqslant \frac{c\varepsilon}{|x_1|}, \qquad \left|\frac{\partial\zeta_+(x,\varepsilon)}{\partial x_2}\right| \leqslant \frac{c\varepsilon}{|x_2|} \quad \text{for } |x_1| \geqslant 3R_0. \tag{3.20}$$

Finally, for $x \in \operatorname{supp}\zeta_+ \cap \{x\colon \frac{3}{2}R_0 \leqslant |x_1| \leqslant 2R_0\}$ we have

$$\frac{R_0}{2} \leqslant \delta(x) \leqslant \text{const}, \qquad \chi(x_1) + \bigl(1-\chi(x_1)\bigr)\delta(x) \geqslant \min\left\{1, \frac{R_0}{2}\right\}$$

and

$$\left|\frac{\partial\zeta_+(x,\varepsilon)}{\partial x_1}\right| \leqslant c\varepsilon\left(\frac{|\gamma'(x_1)|}{\gamma(x_1)} + \frac{|\chi'(x_1)|(1+\delta(x)) + (1-\chi(x_1))|\frac{\partial\delta(x)}{\partial x_1}|}{\chi(x_1) + (1-\chi(x_1))\delta(x)}\right) \leqslant c\varepsilon. \tag{3.21}$$

Define

$$\mathbf{b}(x) = \frac{1}{2\pi}\nabla\ln|x| = \frac{1}{2\pi}\left(\frac{x_1}{|x|^2}, \frac{x_2}{|x|^2}\right). \tag{3.22}$$

The vector field $\mathbf{b}(x)$ satisfies the symmetry conditions (1.13). Moreover, it is well known that the flux of $\mathbf{b}(x)$ over a curve γ is equal to 1:

$$\int_\gamma \mathbf{b}(x)\cdot\boldsymbol{\nu}(x)\,d\gamma = 1,$$

if and only if γ is a closed curve and the domain bounded by γ contains the point $x = 0$. Here $\boldsymbol{\nu}$ is unit vector of outward (with respect to the domain bounded by γ) normal to γ. Otherwise the flux is equal to zero.

Let $x^{(j)} = (x_1^{(j)}, 0) \in \Omega_j$, $j = 1, \ldots, N$. Put

$$\mathbf{b}^{(j)}(x) = -F_j\mathbf{b}\bigl(x - x^{(j)}\bigr).$$

Then

$$\int_{\Gamma_j} \mathbf{b}^{(j)}(x)\cdot\mathbf{n}(x)\,dS = F_j, \qquad \int_{\Gamma_i} \mathbf{b}^{(j)}(x)\cdot\mathbf{n}(x)\,dS = 0, \quad i \neq j.$$

In the domain Ω_+ the functions $\mathbf{b}^{(j)}(x)$ could be represented in the form

$$\mathbf{b}^{(j)}(x) = \frac{F_j}{2\pi}\nabla^\perp\varphi_+^{(j)}(x), \qquad \varphi_+^{(j)}(x) = \operatorname{arctg}\frac{x_1 - x_1^{(j)}}{x_2}, \quad x \in \Omega_+,\ j = 1, \dots, N,$$

where $\nabla^\perp = (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$. Notice that $|\varphi_+^{(j)}(x)| \leqslant \pi/2$ for $x \in \bar{\Omega}_+$ and $j = 1, \dots, N$. Define

$$\mathbf{B}_+^{(j)}(x,\varepsilon) = \frac{F_j}{2\pi}\nabla^\perp\big(\zeta_+(x,\varepsilon)\varphi_+^{(j)}(x)\big) = \frac{F_j}{2\pi}\big(\nabla^\perp\zeta_+(x,\varepsilon)\varphi_+^{(j)}(x) + \zeta_+(x,\varepsilon)\nabla^\perp\varphi_+^{(j)}(x)\big). \tag{3.23}$$

Then $\operatorname{div}\mathbf{B}_+^{(j)}(x,\varepsilon) = 0$ and, since $\zeta_+(x,\varepsilon) = 1$ in the neighborhood of $\partial\Omega_+$, we have

$$\mathbf{B}_+^{(j)}(x,\varepsilon)\big|_{\partial\Omega_+} = \frac{F_j}{2\pi}\nabla^\perp\varphi_+^{(j)}(x)\bigg|_{\partial\Omega_+}.$$

Lemma 3.2. *Let $j = 1, \dots, N$. Then for every $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta)$ such that the following inequality*

$$\bigg|\int\limits_{\Omega_+}\big(\mathbf{u}(x)\cdot\nabla\big)\mathbf{u}(x)\cdot\mathbf{B}_+^{(j)}(x,\varepsilon)\,dx\bigg| \leqslant \delta\int\limits_{\Omega_+}\big|\nabla\mathbf{u}(x)\big|^2\,dx \quad \forall\mathbf{u} \in H_S(\Omega) \tag{3.24}$$

holds.

Proof. Since

$$\big(\mathbf{u}(x)\cdot\nabla\big)\mathbf{u}(x) = \frac{1}{2}\nabla\big|\mathbf{u}(x)\big|^2 - \operatorname{curl}\mathbf{u}(x)\mathbf{u}^\perp(x),$$

where $\operatorname{curl}\mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$, $\mathbf{u}^\perp = (u_2, -u_1)$, we obtain

$$\begin{aligned}\int\limits_{\Omega_+}\big(\mathbf{u}(x)\cdot\nabla\big)\mathbf{u}(x)\cdot\mathbf{B}_+^{(j)}(x,\varepsilon)\,dx &= \int\limits_{\Omega_+}\frac{1}{2}\nabla\big|\mathbf{u}(x)\big|^2\cdot\mathbf{B}_+^{(j)}(x,\varepsilon)\,dx \\ &\quad - \frac{F_j}{2\pi}\int\limits_{\Omega_+}\operatorname{curl}\mathbf{u}(x)\mathbf{u}^\perp(x)\cdot\nabla^\perp\big(\zeta_+(x,\varepsilon)\varphi_+^{(j)}(x)\big)\,dx.\end{aligned} \tag{3.25}$$

We have $\mathbf{u}(x)|_{\partial\Omega_+\cap\partial\Omega} = 0$ and $B_{+,2}^{(j)}(x,\varepsilon)|_{x_2=0} = \mathbf{B}_+^{(j)}\cdot\mathbf{n}|_{x_2=0} = 0$. Hence, $|\mathbf{u}|^2(\mathbf{B}_+^{(j)}\cdot\mathbf{n})|_{\partial\Omega_+} = 0$ and integrating by parts yields

$$\int\limits_{\Omega_+}\frac{1}{2}\nabla\big|\mathbf{u}(x)\big|^2\cdot\mathbf{B}_+^{(j)}(x,\varepsilon)\,dx = 0.$$

Let us estimate the second integral at the right-hand side of (3.25):

$$\begin{aligned}&\int\limits_{\Omega_+}\operatorname{curl}\mathbf{u}(x)\mathbf{u}^\perp(x)\cdot\nabla^\perp\big(\zeta_+(x,\varepsilon)\varphi_+^{(j)}(x)\big)\,dx \\ &\quad = \int\limits_{\Omega_+}\operatorname{curl}\mathbf{u}(x)\mathbf{u}^\perp(x)\cdot\nabla^\perp\zeta_+(x,\varepsilon)\varphi_+^{(j)}(x)\,dx \\ &\qquad + \int\limits_{\Omega_+}\operatorname{curl}\mathbf{u}(x)\mathbf{u}^\perp(x)\cdot\nabla^\perp\varphi_+^{(j)}(x)\zeta_+(x,\varepsilon)\,dx = J_1^{(j)}(\varepsilon) + J_2^{(j)}(\varepsilon).\end{aligned} \tag{3.26}$$

For $J_2^{(j)}(\varepsilon)$ we have

$$J_2^{(j)}(\varepsilon)=\int\limits_{\Omega_+}\operatorname{curl}\mathbf{u}(x)\zeta_+(x,\varepsilon)\left(-u_2(x)\frac{x_1-x_1^{(j)}}{|x-x^{(j)}|^2}+u_1(x)\frac{x_2}{|x-x^{(j)}|^2}\right)dx$$
$$\leqslant c\|\nabla\mathbf{u}\|_{L_2(\Omega_+)}\left(\int\limits_{\Omega_+\cap\operatorname{supp}\zeta_+}\left(\left|u_2(x)\right|^2\frac{|x_1-x_1^{(j)}|^2}{|x-x^{(j)}|^4}+\left|u_1(x)\right|^2\frac{|x_2|^2}{|x-x^{(j)}|^4}\right)dx\right)^{1/2}. \tag{3.27}$$

Obviously,

$$\Omega_+\cap\operatorname{supp}\zeta_+\subset\Omega_{+,\varepsilon}=\left\{x\in\Omega_+:\ \Delta_+(x)\leqslant\varepsilon\gamma(x_1)\right\}.$$

Set

$$\Omega_{+,\varepsilon}^{(1)}=\left\{x\in\Omega_{+,\varepsilon}:\ \delta(x)\leqslant\sqrt{\varepsilon}\right\},$$
$$\Omega_{+,\varepsilon}^{(2)}=\left\{x\in\Omega_{+,\varepsilon}:\ \delta(x)\geqslant\sqrt{\varepsilon}\right\}.$$

Since $\mathbf{u}(x)|_{\partial\Omega\cap\partial\Omega_+}=0$, by the Poincaré inequality

$$\int\limits_{\Omega_{+,\varepsilon}^{(1)}}\left(\left|u_2(x)\right|^2\frac{|x_1-x_1^{(j)}|^2}{|x-x^{(j)}|^4}+\left|u_1(x)\right|^2\frac{|x_2|^2}{|x-x^{(j)}|^4}\right)dx$$
$$\leqslant\int\limits_{\Omega_{+,\varepsilon}^{(1)}}\left(\left|u_1(x)\right|^2+\left|u_2(x)\right|^2\right)dx\leqslant c\varepsilon\int\limits_{\Omega_+}\left|\nabla\mathbf{u}(x)\right|^2dx. \tag{3.28}$$

Let $\Omega_{+,\varepsilon}^{(2,1)}=\Omega_{+,\varepsilon}^{(2)}\cap\{x:\ |x_1|\leqslant\frac{3}{2}R_0\}$. If $x\in\Omega_{+,\varepsilon}^{(2,1)}$, then $x_2\delta(x)\leqslant\varepsilon$, while $\delta(x)\geqslant\sqrt{\varepsilon}$, and, hence, $x_2\leqslant\sqrt{\varepsilon}$. Therefore, by the Poincaré inequality we get

$$\int\limits_{\Omega_{+,\varepsilon}^{(2,1)}}\left|u_1(x)\right|^2\frac{|x_2|^2}{|x-x^{(j)}|^4}dx\leqslant c\varepsilon\int\limits_{\Omega_{+,\varepsilon}\cap\{x:\ |x_1|\leqslant\frac{3}{2}R_0\}}\left|u_1(x)\right|^2dx\leqslant c\varepsilon\int\limits_{\Omega_+}\left|\nabla\mathbf{u}(x)\right|^2dx. \tag{3.29}$$

In order to estimate the integral $\int_{\Omega_{+,\varepsilon}^{(2)}\setminus\Omega_{+,\varepsilon}^{(2,1)}}|u_1(x)\frac{x_2}{|x-x^{(j)}|^2}|^2\,dx$, we shall use the fact that $(u_1(x)\frac{x_2}{|x-x^{(j)}|^2})|_{x_2=0}=0$. Notice that $\Delta_+(x)=x_2$ for $|x_1|\geqslant 2R_0$ and, since $\delta(x)\geqslant\frac{R_0}{2}$ for $|x_1|\geqslant\frac{3}{2}R_0$, it is easy to verify that $\Delta_+(x)\geqslant\mu_0x_2$ for $x\in\Omega_{+,\varepsilon}\cap\{x:\ |x_1|\geqslant\frac{3}{2}R_0\}$, where $\mu_0>0$ depends on R_0 only. For simplicity and without loss of generality for sufficiently small ε we may take $\mu_0=1/2$. Thus, applying again the Poincaré inequality we obtain

$$J_{21}^{(j)}(\varepsilon)=\int\limits_{\Omega_{+,\varepsilon}^{(2)}\setminus\Omega_{+,\varepsilon}^{(2,1)}}\left|u_1(x)\frac{x_2}{|x-x^{(j)}|^2}\right|^2dx$$
$$\leqslant\int\limits_{\mathbb{R}\setminus(-\frac{3}{2}R_0,\frac{3}{2}R_0)}dx_1\int\limits_0^{2\varepsilon\gamma(x_1)}\left|u_1(x)\frac{x_2}{|x-x^{(j)}|^2}\right|^2dx_2$$
$$\leqslant c\varepsilon^2\int\limits_{\mathbb{R}\setminus(-\frac{3}{2}R_0,\frac{3}{2}R_0)}\gamma(x_1)^2\,dx_1\int\limits_0^{2\varepsilon\gamma(x_1)}\left|\frac{\partial}{\partial x_2}\left(u_1(x)\frac{x_2}{|x-x^{(j)}|^2}\right)\right|^2dx_2$$
$$\leqslant c\varepsilon^2\left(\int\limits_{\Omega_{+,\varepsilon}^{(2)}\setminus\Omega_{+,\varepsilon}^{(2,1)}}\frac{\gamma^2(x_1)}{x_1^2}\left|\nabla u_1(x)\right|^2dx+\int\limits_{\Omega_{+,\varepsilon}^{(2)}\setminus\Omega_{+,\varepsilon}^{(2,1)}}\frac{\gamma^2(x_1)}{x_1^{2\alpha}}\frac{|u_1(x)|^2}{|x|^2|x|^{2-2\alpha}}dx\right)$$

$$\leqslant c\varepsilon^2\biggl(\int\limits_{\Omega_+}\bigl|\nabla u_1(x)\bigr|^2\,dx+\int\limits_{\Omega_+}\frac{|u_1(x)|^2}{|x|^2\log^2|x|}\,dx\biggr)\leqslant c\varepsilon^2\int\limits_{\Omega}\bigl|\nabla u_1(x)\bigr|^2\,dx. \tag{3.30}$$

In the last estimate we have applied the inequality (2.2).

According to (2.1), $u_2(x)|_{x_2=0}=0$. Therefore,

$$\int\limits_{\Omega^{(2,1)}_{+,\varepsilon}}\bigl|u_2(x)\bigr|^2\frac{|x_1-x_1^{(j)}|^2}{|x-x^{(j)}|^4}\,dx\leqslant\int\limits_{\Omega_{+,\varepsilon}\cap\{x:\,x_2\leqslant\sqrt{\varepsilon}\}}\bigl|u_2(x)\bigr|^2\,dx$$
$$\leqslant c\varepsilon\int\limits_{\Omega_+}\bigl|\nabla u_2(x)\bigr|^2\,dx, \tag{3.31}$$

$$\int\limits_{\Omega^{(2)}_{+,\varepsilon}\setminus\Omega^{(2,1)}_{+,\varepsilon}}\bigl|u_2(x)\bigr|^2\frac{|x_1-x_1^{(j)}|^2}{|x-x^{(j)}|^4}\,dx\leqslant\int\limits_{\mathbb{R}\setminus(-\frac{3}{2}R_0,\frac{3}{2}R_0)}\frac{dx_1}{x_1^2}\int\limits_0^{2\varepsilon\gamma(x_1)}\bigl|u_2(x)\bigr|^2\,dx_2$$
$$\leqslant c\varepsilon^2\int\limits_{\mathbb{R}\setminus(-\frac{3}{2}R_0,\frac{3}{2}R_0)}\frac{\gamma(x_1)^2\,dx_1}{x_1^2}\int\limits_0^{2\varepsilon\gamma(x_1)}\biggl|\frac{\partial u_2(x)}{\partial x_2}\biggr|^2\,dx_2$$
$$\leqslant c\varepsilon^2\int\limits_{\Omega_+}\bigl|\nabla u_2(x)\bigr|^2\,dx. \tag{3.32}$$

It follows from (3.27)–(3.32) that

$$J_2^{(j)}(\varepsilon)\leqslant c\sqrt{\varepsilon}\int\limits_{\Omega_+}\bigl|\nabla\mathbf{u}(x)\bigr|^2\,dx. \tag{3.33}$$

Consider the integral

$$J_1^{(j)}(\varepsilon)=\int\limits_{\Omega_+}\operatorname{curl}\mathbf{u}(x)\mathbf{u}^{\perp}(x)\cdot\nabla^{\perp}\zeta_+(x,\varepsilon)\varphi_+(x)\,dx.$$

Since $|\varphi_+(x)|\leqslant\pi/2$ for $x\in\bar{\Omega}_+$ and $\mathbf{u}^{\perp}\cdot\nabla^{\perp}\zeta_+=\mathbf{u}\cdot\nabla\zeta_+$, we have

$$\bigl|J_1^{(j)}(\varepsilon)\bigr|\leqslant c\biggl(\int\limits_{\Omega_+}\bigl|\nabla\mathbf{u}(x)\bigr|^2\,dx\biggr)^{1/2}$$
$$\times\biggl(\int\limits_{\Omega_{+,\varepsilon}}\biggl(\bigl|u_1(x)\bigr|^2\biggl|\frac{\partial\zeta_+(x,\varepsilon)}{\partial x_1}\biggr|^2+\bigl|u_2(x)\bigr|^2\biggl|\frac{\partial\zeta_+(x,\varepsilon)}{\partial x_2}\biggr|^2\biggr)\,dx\biggr)^{1/2}. \tag{3.34}$$

Set $\widehat{\Omega}_{+,\varepsilon}=\Omega_{+,\varepsilon}\cap\{x:\ |x_1|\leqslant\frac{3}{2}R_0\}$. Since $u_1(x)|_{\partial\Omega\cap\partial\Omega_+}=0$, we can use estimates (3.7), (3.19) and Hardy's inequality (see [18] for details) to prove

$$\int\limits_{\widehat{\Omega}_{+,\varepsilon}}\bigl|u_1(x)\bigr|^2\biggl|\frac{\partial\zeta_+(x,\varepsilon)}{\partial x_1}\biggr|^2\,dx\leqslant c\varepsilon^2\int\limits_{\widehat{\Omega}_{+,\varepsilon}}\frac{|u_1(x)|^2}{\delta^2(x)}\,dx$$
$$\leqslant c\varepsilon^2\int\limits_{\widehat{\Omega}_{+,\varepsilon}}\frac{|u_1(x)|^2}{dist(x,\partial\Omega\cap\Omega_+)^2}\,dx\leqslant c\varepsilon^2\int\limits_{\Omega_+}\bigl|\nabla u_1(x)\bigr|^2\,dx. \tag{3.35}$$

The velocity component u_2 satisfies

$$u_2(x)\big|_{\partial\Omega\cap\partial\Omega_+}=0, \qquad u_2(x)\big|_{x_2=0}=0.$$

Since $\Delta_+(x)\geqslant \frac{1}{2}x_2$ for $x_1\in\Omega_{+,\varepsilon}\cap\{x\colon |x_1|\geqslant\frac{3}{2}R_0\}$, estimates (3.7), (3.19) and Hardy's inequality yield

$$\begin{aligned}
&\int\limits_{\Omega_{+,\varepsilon}}\big|u_2(x)\big|^2\left|\frac{\partial\zeta_+(x,\varepsilon)}{\partial x_2}\right|^2dx\\
&\quad\leqslant c\varepsilon^2\int\limits_{\Omega_{+,\varepsilon}\setminus\widehat{\Omega}_{+,\varepsilon}}\frac{|u_2(x)|^2}{\Delta_+^2(x)}\,dx+\int\limits_{\widehat{\Omega}_{+,\varepsilon}}\big|u_2(x)\big|^2\left|\frac{\partial\zeta_+(x,\varepsilon)}{\partial x_2}\right|^2dx\\
&\quad\leqslant c\varepsilon^2\int\limits_{\Omega_{+,\varepsilon}\setminus\widehat{\Omega}_{+,\varepsilon}}\frac{|u_2(x)|^2}{x_2^2}dx+c\varepsilon^2\int\limits_{\widehat{\Omega}_{+,\varepsilon}}\left(\frac{|u_2(x)|^2}{x_2^2}+\frac{|u_2(x)|^2}{\delta^2(x)}\right)dx\\
&\quad\leqslant c\varepsilon^2\left(\int\limits_{\Omega_{+,\varepsilon}}\left|\frac{\partial u_2(x)}{\partial x_2}\right|^2dx+\int\limits_{\Omega_{+,\varepsilon}}\big|\nabla u_2(x)\big|^2dx\right)\\
&\quad\leqslant c\varepsilon^2\int\limits_{\Omega_+}\big|\nabla u_2(x)\big|^2dx. \qquad (3.36)
\end{aligned}$$

Finally, from (3.17), (3.20), (3.21), Poincaré's inequality and (2.4) we obtain that

$$\begin{aligned}
\int\limits_{\Omega_{+,\varepsilon}\setminus\widehat{\Omega}_{+,\varepsilon}}\big|u_1(x)\big|^2\left|\frac{\partial\zeta_+(x,\varepsilon)}{\partial x_1}\right|^2dx&\leqslant c\varepsilon^2\int\limits_{\frac{3}{2}R_0<|x_1|<3R_0}dx_1\int\limits_0^{2\varepsilon\gamma(x_1)}\big|u_1(x_1,x_2)\big|^2dx_2\\
&\quad+c\varepsilon^2\int\limits_{\mathbb{R}\setminus(-3R_0,3R_0)}\frac{dx_1}{x_1^2}\int\limits_0^{2\varepsilon\gamma(x_1)}\big|u_1(x_1,x_2)\big|^2dx_2\leqslant c\varepsilon^2\int\limits_{\Omega}\big|\nabla u_1(x)\big|^2dx\\
&\quad+c\varepsilon^2\int\limits_{\mathbb{R}\setminus(-3R_0,3R_0)}\int\limits_0^{2\varepsilon\gamma(x_1)}\frac{|u_1(x_1,x_2)|^2}{|x|^2}dx_2\,dx_1\\
&\leqslant c\varepsilon^2\int\limits_{\Omega}\big|\nabla u_1(x)\big|^2dx. \qquad (3.37)
\end{aligned}$$

Estimates (3.34)–(3.37) yield

$$J_1^{(j)}(\varepsilon)\leqslant c\varepsilon\int\limits_{\Omega_+}\big|\nabla\mathbf{u}(x)\big|^2dx. \qquad (3.38)$$

The desired estimate (3.24) follows from (3.33) and (3.38) by taking $\varepsilon=\varepsilon(\delta)$ sufficiently small. □

Lemma 3.3. *Let $\alpha\in(\frac{1}{3},1)$. Then $\mathbf{B}_+^{(j)}\in L^4(\Omega_+)$, $\nabla\mathbf{B}_+^{(j)}\in L^2(\Omega_+)$ and*

$$\big\|\mathbf{B}_+^{(j)}\big\|_{L^4(\Omega_+)}+\big\|\nabla\mathbf{B}_+^{(j)}\big\|_{L^2(\Omega_+)}\leqslant c|F_j|, \quad j=1,\dots,N. \qquad (3.39)$$

Proof. By the definition of $\mathbf{B}_+^{(j)}(x,\varepsilon)$ (see (3.23)) and estimate (3.8) we derive

$$\begin{aligned}
\big|\mathbf{B}_+^{(j)}(x,\varepsilon)\big|&\leqslant c|F_j|\left(\big|\nabla\zeta_+(x,\varepsilon)\big|\big|\varphi_+^{(j)}(x)\big|+\frac{|\zeta_+(x,\varepsilon)|}{|x-x^{(j)}|}\right)\\
&\leqslant c|F_j|\left(\frac{1}{\gamma(x_1)}+\frac{1}{|x-x^{(j)}|}\right),
\end{aligned}$$

$$\left|\nabla \mathbf{B}_+^{(j)}(x,\varepsilon)\right| \leqslant c|F_j|\left(\frac{1}{\gamma^2(x_1)}+\frac{1}{|x-x^{(j)}|^2}\right).$$

Therefore,

$$\begin{aligned}\int\limits_{\Omega_+}\left|\mathbf{B}_+^{(j)}(x,\varepsilon)\right|^4 dx &\leqslant c|F_j|^4\int\limits_{\Omega_{+,\varepsilon}}\left(\frac{1}{\gamma^4(x_1)}+\frac{1}{|x-x^{(j)}|^4}\right)dx\\ &\leqslant c|F_j|^4\left(1+\int\limits_{3R_0}^{\infty}\frac{dx_1}{|x_1|^{4\alpha}}\int\limits_0^{2\varepsilon|x_1|^\alpha}dx_2+\int\limits_{-\infty}^{-3R_0}\frac{dx_1}{|x_1|^{4\alpha}}\int\limits_0^{2\varepsilon|x_1|^\alpha}dx_2\right)\\ &\leqslant c|F_j|^4\left(1+\int\limits_{3R_0}^{\infty}\frac{dx_1}{|x_1|^{3\alpha}}\right)\leqslant c|F_j|^4 \quad \text{if } \alpha>\frac{1}{3}.\end{aligned}$$

It can be proved analogously that

$$\int\limits_{\Omega_+}\left|\nabla \mathbf{B}_+^{(j)}(x,\varepsilon)\right|^2 dx \leqslant c|F_j|^2 \quad \text{if } \alpha>\frac{1}{3}. \qquad \square$$

Define

$$\mathbf{B}^{(j)}(x,\varepsilon)=\begin{cases}(B_{+,1}^{(j)}(x_1,x_2,\varepsilon),\,B_{+,2}^{(j)}(x_1,x_2,,\varepsilon)), & x\in\Omega_{+,\varepsilon},\\ (B_{+,1}^{(j)}(x_1,-x_2,\varepsilon),\,-B_{+,2}^{(j)}(x_1,-x_2,,\varepsilon)), & x\in\Omega_{-,\varepsilon},\end{cases} \tag{3.40}$$

and

$$\mathbf{B}(x,\varepsilon)=\sum_{j=1}^{N}\mathbf{B}^{(j)}(x,\varepsilon). \tag{3.41}$$

The vector field $\mathbf{B}$ is symmetric,

$$\operatorname{div}\mathbf{B}=0, \qquad \int\limits_{\Gamma_j}\mathbf{B}\cdot\mathbf{n}\,dS=F_j, \quad j=1,\dots,N. \tag{3.42}$$

Let $\mathbf{h}_1(x)=\mathbf{h}(x)-\mathbf{B}(x,\varepsilon)|_{\partial\Omega}$. We have

$$\int\limits_{\Gamma_j}\mathbf{h}_1(x)\cdot\mathbf{n}(x)\,dS=0, \quad j=1,\dots,N. \tag{3.43}$$

If $\mathbf{h}\in W^{1/2,2}(\partial\Omega)$, then obviously $\mathbf{h}_1\in W^{1/2,2}(\partial\Omega)$ and

$$\begin{aligned}\|\mathbf{h}_1\|_{W^{1/2,2}(\partial\Omega)} &\leqslant c\big(\|\mathbf{h}\|_{W^{1/2,2}(\partial\Omega)}+\|\mathbf{B}|_{\partial\Omega}\|_{W^{1/2,2}(\partial\Omega)}\big)\\ &\leqslant c\left[\|\mathbf{h}\|_{W^{1/2,2}(\partial\Omega)}+\left(\sum_{j=1}^{N}F_j^2\right)^{1/2}\right]\leqslant c\|\mathbf{h}\|_{W^{1/2,2}(\partial\Omega)}.\end{aligned}$$

Because of condition (3.43), there exists a function $\mathbf{H}\in H(\Omega)$ such that $\operatorname{supp}\mathbf{H}(x,\varepsilon)$ is contained in a small neighborhood of the boundary $\partial\Omega$,

$$\begin{gathered}\operatorname{div}\mathbf{H}=0, \qquad \mathbf{H}(x,\varepsilon)|_{\partial\Omega}=\mathbf{h}_1(x), \qquad \mathbf{H}\in L^4(\Omega), \qquad \nabla\mathbf{H}\in L^2(\Omega),\\ \|\mathbf{H}\|_{L^4(\Omega)}+\|\nabla\mathbf{H}\|_{L^2(\Omega)}\leqslant c\|\mathbf{h}_1\|_{W^{1/2,2}(\partial\Omega)}\leqslant c\|\mathbf{h}\|_{W^{1/2,2}(\partial\Omega)}.\end{gathered} \tag{3.44}$$

Moreover, $\mathbf{H}(x,\varepsilon)$ satisfies Leray–Hopf's inequality, i.e., for every $\delta>0$ there exists $\varepsilon=\varepsilon(\delta)$ such that

$$\left|\int_\Omega \big(\mathbf{u}(x)\cdot\nabla\big)\mathbf{u}(x)\cdot\mathbf{H}(x,\varepsilon)\,dx\right| \leqslant \delta\int_\Omega \big|\mathbf{u}(x)\big|^2\,dx \quad \forall\mathbf{u}\in H(\Omega) \tag{3.45}$$

holds (see [18]).

The function $\mathbf{H}(x,\varepsilon)$ is not necessary symmetric. However, its boundary value is symmetric and, therefore, $\mathbf{H}(x,\varepsilon)$ can be symmetrized defining the function $\widetilde{\mathbf{H}}(x,\varepsilon)$ as follows

$$\widetilde{H}_1(x,\varepsilon)=\frac{1}{2}\big[H_1(x_1,x_2,\varepsilon)+H_1(x_1,-x_2,\varepsilon)\big],$$
$$\widetilde{H}_2(x,\varepsilon)=\frac{1}{2}\big[H_2(x_1,x_2,\varepsilon)-H_2(x_1,-x_2,\varepsilon)\big].$$

Setting

$$\mathbf{A}(x,\varepsilon)=\mathbf{B}(x,\varepsilon)+\widetilde{\mathbf{H}}(x,\varepsilon), \tag{3.46}$$

we then have proved the:

Lemma 3.4.

(i) *The vector field $\mathbf{A}(x,\varepsilon)$ is symmetric,*

$$\operatorname{div}\mathbf{A}(x,\varepsilon)=0, \qquad \mathbf{A}(x,\varepsilon)|_{\partial\Omega}=\mathbf{h}(x). \tag{3.47}$$

(ii) $\mathbf{A}\in L^4(\Omega)$, $\nabla\mathbf{A}\in L^2(\Omega)$,

$$\|\mathbf{A}\|_{L^4(\Omega)}+\|\nabla\mathbf{A}\|_{L^2(\Omega)}\leqslant c\|\mathbf{h}\|_{W^{1/2,2}(\partial\Omega)}. \tag{3.48}$$

The constant c in this inequality depends on ε and tends to infinity as $\varepsilon\to 0$ (see Lemma 3.1). Below we use this inequality with sufficiently small but fixed ε.

(iii) *For every $\delta>0$ there exists $\varepsilon=\varepsilon(\delta)$ such that the inequality*

$$\left|\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{A}\,dx\right|\leqslant\delta\int_\Omega|\nabla\mathbf{u}|^2\,dx \quad \forall\mathbf{u}\in H_S(\Omega) \tag{3.49}$$

holds.

4. Existence theorem

Consider Navier–Stokes problem (1.2). Let $\mathbf{A}$ be the symmetric extension of the boundary value $\mathbf{h}$ constructed in the previous section. By a *weak solution* of problem (1.2) we understand a function $\mathbf{u}$ such that $\mathbf{w}=\mathbf{u}-\mathbf{A}\in H_S(\Omega)$ and the integral identity

$$\nu\int_\Omega\nabla\mathbf{w}\cdot\nabla\boldsymbol{\eta}\,dx=-\nu\int_\Omega\nabla\mathbf{A}\cdot\nabla\boldsymbol{\eta}\,dx-\int_\Omega(\mathbf{A}\cdot\nabla)\mathbf{A}\cdot\boldsymbol{\eta}\,dx$$
$$-\int_\Omega\big((\mathbf{w}+\mathbf{A})\cdot\nabla\big)\mathbf{w}\cdot\boldsymbol{\eta}\,dx-\int_\Omega(\mathbf{w}\cdot\nabla)\mathbf{A}\cdot\boldsymbol{\eta}\,dx+\int_\Omega\mathbf{f}\cdot\boldsymbol{\eta}\,dx \tag{4.1}$$

holds for any $\boldsymbol{\eta}\in J_{0S}^\infty(\Omega)$.[7]

[7] Note that for the symmetric weak solution the integral identity (4.1) remains valid for the nonsymmetric functions $\boldsymbol{\eta}\in J_0^\infty(\Omega)$. Indeed each test function $\boldsymbol{\eta}$ can be represented as a sum $\boldsymbol{\eta}=\boldsymbol{\eta}_S+\boldsymbol{\eta}_{AS}$, where $\boldsymbol{\eta}_S$ is symmetric and $\boldsymbol{\eta}_{AS}$ is antisymmetric, and it is easy to check that all integrals in (4.1) vanish for $\boldsymbol{\eta}=\boldsymbol{\eta}_{AS}$.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a symmetric exterior domain (1.1) with multiply connected Lipschitz boundary $\partial\tilde{\Omega}$ consisting of N disjoint components Γ_j, $j = 0, \dots, N$. Assume that $\mathbf{f}$ is a symmetric distribution such that the corresponding linear functional $H(\Omega) \ni \boldsymbol{\eta} \mapsto \int_\Omega \mathbf{f} \cdot \boldsymbol{\eta}$ is continuous (with respect to the norm $\|\cdot\|_{H(\Omega)}$), and $\mathbf{h}$ is a symmetric field in $W^{1/2,2}(\partial\Omega)$. Then problem (1.2) admits at least one symmetric weak solution $\mathbf{u} = \mathbf{w} + \mathbf{A}$, where $\mathbf{w} \in H_S(\Omega)$. The following estimate*

$$\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} \leqslant c\big(\|\mathbf{h}\|^2_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{h}\|^4_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{f}\|^2_*\big) \tag{4.2}$$

is valid.

Proof. We find the weak solution $\mathbf{w}$ of problem (1.2) in the unbounded domain Ω by the Leray invading domain method, i.e., as a limit of a sequence of weak solutions $\{\mathbf{w}_k\}$ to the Navier–Stokes problems in the bounded domains $\Omega_k = \Omega \cap B_k$, $k \geqslant R_0$, $B_k = \{x\colon |x| < k\}$. Obviously, $\lim_{k\to\infty} \Omega_k = \Omega$. Consider the following problems

$$\begin{aligned} -\nu\Delta\mathbf{w}_k + (\mathbf{w}_k + \mathbf{A})\cdot\nabla(\mathbf{w}_k + \mathbf{A}) - \nu\Delta\mathbf{A} + \nabla p_k = \mathbf{f} \quad &\text{in } \Omega_k, \\ \operatorname{div}\mathbf{w}_k = 0 \quad &\text{in } \Omega_k, \\ \mathbf{w}_k = 0 \quad &\text{on } \partial\Omega_k. \end{aligned} \tag{4.3}$$

Weak solutions $\mathbf{w}_k \in H_S(\Omega_k)$ of (4.3) satisfy the following integral identities

$$\begin{aligned} \nu\int_\Omega \nabla\mathbf{w}_k \cdot \nabla\boldsymbol{\eta}\,dx = {}& -\nu\int_\Omega \nabla\mathbf{A}\cdot\nabla\boldsymbol{\eta}\,dx - \int_\Omega (\mathbf{A}\cdot\nabla)\mathbf{A}\cdot\boldsymbol{\eta}\,dx \\ & - \int_\Omega \big((\mathbf{A}+\mathbf{w}_k)\cdot\nabla\big)\mathbf{w}_k\cdot\boldsymbol{\eta}\,dx - \int_\Omega (\mathbf{w}_k\cdot\nabla)\mathbf{A}\cdot\boldsymbol{\eta}\,dx + \int_\Omega \mathbf{f}\cdot\boldsymbol{\eta}\,dx \end{aligned} \tag{4.4}$$

for any test function $\boldsymbol{\eta} \in H_S(\Omega_k)$. Here we have assumed that $\mathbf{w}_k$ and $\boldsymbol{\eta}$ are extended by zero to the whole domain Ω.

It is well known (e.g., [18]) that identities (4.4) are equivalent to the operator equations in the space $H_S(\Omega_k)$:

$$\mathbf{w}_k = \mathfrak{B}\mathbf{w}_k \tag{4.5}$$

with the compact operator $\mathfrak{B} : H_S(\Omega_k) \hookrightarrow H_S(\Omega_k)$. The solvability of (4.5) can be proved applying the Leray–Schauder Fixed Point Theorem. To do this we need only to show that all possible solutions of the equation

$$\mathbf{w}_k^{(\lambda)} = \lambda\mathfrak{B}\mathbf{w}_k^{(\lambda)}, \quad \lambda \in [0,1], \tag{4.6}$$

are uniformly bounded with respect to λ in the norm $\|\cdot\|_{H(\Omega_k)}$. Let us take in the integral identity corresponding to Eq. (4.6) $\boldsymbol{\eta} = \mathbf{w}_k^{(\lambda)}$. This gives

$$\begin{aligned} \nu\int_\Omega \big|\nabla\mathbf{w}_k^{(\lambda)}\big|^2\,dx = {}& -\lambda\nu\int_\Omega \nabla\mathbf{A}\cdot\nabla\mathbf{w}_k^{(\lambda)}\,dx + \lambda\int_\Omega (\mathbf{A}\cdot\nabla)\mathbf{w}_k^{(\lambda)}\cdot\mathbf{A}\,dx \\ & + \lambda\int_\Omega \mathbf{f}\cdot\mathbf{w}_k^{(\lambda)}\,dx - \lambda\int_\Omega \big(\mathbf{w}_k^{(\lambda)}\cdot\nabla\big)\mathbf{A}\cdot\mathbf{w}_k^{(\lambda)}\,dx. \end{aligned} \tag{4.7}$$

Estimating first three terms on the right-hand side of (4.7) by the Cauchy inequality and using (3.41), (3.46) we get

$$\begin{aligned} \nu\int_\Omega \big|\nabla\mathbf{w}_k^{(\lambda)}\big|^2\,dx \leqslant {}& \frac{\nu}{4}\int_\Omega \big|\nabla\mathbf{w}_k^{(\lambda)}\big|^2\,dx + c\Big(\int_\Omega |\nabla\mathbf{A}|^2\,dx + \int_\Omega |\mathbf{A}|^4\,dx + \|\mathbf{f}\|^2_*\Big) \\ & + \Big|\int_\Omega \big(\mathbf{w}_k^{(\lambda)}\cdot\nabla\big)\mathbf{A}\cdot\mathbf{w}_k^{(\lambda)}\,dx\Big|. \end{aligned} \tag{4.8}$$

We estimate the last integral on the right-hand side of (4.8) by the Leray–Hopf inequality (3.49), fixing ε in the definition of $\mathbf{A}(x,\varepsilon)$ so small that δ in (3.49) satisfies the inequality $\delta \leqslant \frac{\nu}{4}$, i.e.,

$$\left|\int_\Omega \big(\mathbf{w}_k^{(\lambda)}\cdot\nabla\big)\mathbf{A}\cdot\mathbf{w}_k^{(\lambda)}\,dx\right| \leqslant \frac{\nu}{4}\int_\Omega \big|\nabla\mathbf{w}_k^{(\lambda)}\big|^2\,dx. \tag{4.9}$$

From (4.8), (4.9) and (3.48) it follows that

$$\begin{aligned}\frac{\nu}{2}\int_\Omega \big|\nabla\mathbf{w}_k^{(\lambda)}\big|^2\,dx &\leqslant c\bigg(\int_\Omega |\nabla\mathbf{A}|^2\,dx+\int_\Omega |\mathbf{A}|^4\,dx+\|\mathbf{f}\|_*^2\bigg)\\ &\leqslant c\big(\|\mathbf{h}\|^2_{W^{1/2,2}(\partial\Omega)}+\|\mathbf{h}\|^4_{W^{1/2,2}(\partial\Omega)}+\|\mathbf{f}\|_*^2\big),\end{aligned} \tag{4.10}$$

where the constant c is independent of $\lambda\in[0,1]$ and k. Hence, by the Leray–Schauder Fixed Point Theorem each operator equation (4.5) has at least one weak symmetric solution $\mathbf{w}_k\in H_S(\Omega)$. This solutions satisfy integral identities (4.4) and estimates

$$\|\mathbf{w}_k\|^2_{H(\Omega)} \leqslant c\big(\|\mathbf{h}\|^2_{W^{1/2,2}(\partial\Omega)}+\|\mathbf{h}\|^4_{W^{1/2,2}(\partial\Omega)}+\|\mathbf{f}\|_*^2\big) \tag{4.11}$$

with the constant c independent of k. Hence $\{\mathbf{w}_k\}$ (modulo a subsequence) tends weakly in $H_S(\Omega)$ and strongly in $L^q_{\mathrm{loc}}(\bar\Omega)$ $(1\leqslant q<\infty)$ to a function $\mathbf{w}\in H_S(\Omega)$. Taking any test function $\boldsymbol{\eta}$ with compact support, we can find k such that $\operatorname{supp}\boldsymbol{\eta}\subset\Omega_k$. Thus, we can pass to a limit as $k\to\infty$ in (4.4) and we obtain for the limit function $\mathbf{w}$ the integral identity (4.1). Then, by definition, $\mathbf{u}=\mathbf{w}+\mathbf{A}$ is a weak solution to the Navier–Stokes problem (1.2). Obviously for the limit function $\mathbf{w}$ estimate (4.11) remains valid. Then estimate (4.2) follows from (4.11) and (3.48). The theorem is proved. □

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