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The existence theorem for the steady Navier–Stokes problem in exterior axially symmetric 3D domains

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Abstract We study the nonhomogeneous boundary value problem for the Navier–Stokes equations of steady motion of a viscous incompressible fluid in a three-dimensional exterior domain with multiply connected boundary. We prove that this problem has a solution for axially symmetric domains and data (without any smallness restrictions on the fluxes). Our main tool is a recent version of the Morse–Sard theorem for Sobolev functions obtained by Bourgain et al. (Rev Mat Iberoam 29(1):1–23, 2013).

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1 Introduction

In this paper we shall consider the Navier–Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega, \\ \operatorname{div}\, \mathbf{u} = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{a} & \text{on} \quad \partial\Omega, \\ \lim\limits_{|x|\to+\infty} \mathbf{u}(x) = \mathbf{u}_0 & \end{cases} \tag{1.1}$$

in the exterior domain of $\mathbb{R}^3$

$$\Omega = \mathbb{R}^3 \setminus \big(\bigcup_{j=1}^{N} \bar{\Omega}_j\big), \tag{1.2}$$

where Ω_i are bounded domains with connected C^2-smooth boundaries Γ_i and $\bar{\Omega}_j \cap \bar{\Omega}_i = \emptyset$ for $i \neq j$. In (1.1) $\nu > 0$ is the viscosity coefficient, $\mathbf{u}$, p are the (unknown) velocity and pressure fields, $\mathbf{a}$ and $\mathbf{u}_0$ are the (assigned) boundary data and a constant vector respectively, $\mathbf{f}$ is the body force density.

Let

$$\mathcal{F}_i = \int\limits_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\, dS, \quad i = 1, \dots N, \tag{1.3}$$

where $\mathbf{n}$ is the exterior (with respect to Ω) normal to $\partial\Omega$. Under suitable regularity hypotheses on Ω and $\mathbf{a}$ and assuming that

$$\mathcal{F}_i = 0, \quad i = 1, \dots N, \tag{1.4}$$

in the celebrated paper [26] of 1933, J. Leray was able to show that (1.1) has a solution $\mathbf{u}$ such that

$$\int\limits_{\Omega} |\nabla\mathbf{u}|^2\, dx < +\infty, \tag{1.5}$$

and $\mathbf{u}$ satisfies (1.1_4) in a suitable sense for general $\mathbf{u}_0$ and uniformly for $\mathbf{u}_0 = \mathbf{0}$. In the fifties the problem was reconsidered by Finn [12] and Ladyzhenskaia [23,24]. They showed that the solution satisfies the condition at infinity uniformly. Moreover, the condition (1.4) and the regularity of $\mathbf{a}$ have been relaxed by requiring $\sum_{i=1}^{N} |\mathcal{F}_i|$ to be sufficiently small [12] and $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ [24].

In 1973, Babenko [3] proved that if $(\mathbf{u}, p)$ is a solution to (1.1), (1.5) with $\mathbf{u}_0 \neq \mathbf{0}$, then $(\mathbf{u} - \mathbf{u}_0, p)$ behaves at infinity as the solutions to the linear Oseen system. In particular,[1]

$$\mathbf{u}(x) - \mathbf{u}_0 = O(r^{-1}), \quad p(x) = O(r^{-2}). \tag{1.6}$$

[1] See also [13]. Here the symbol $f(x) = O(g(r))$ means that there is a positive constant c such that $|f(x)| \leq cg(r)$ for large r.

However, nothing is known, in general, on the rate of convergence at infinity for $\mathbf{u}_0 = \mathbf{0}$.[2]

One of the most important problems in the theory of the steady-state Navier–Stokes equations concerns the possibility to prove existence of a solution to (1.1) without any assumptions on the fluxes $\mathcal{F}_i$ (see, e.g. [13]). To the best of our knowledge, one of the most general assumptions assuring existence is expressed by

$$\sum_{i=1}^{N} |\mathcal{F}_i| \operatorname{cap}(\Omega_i) < \frac{1}{2}\nu, \tag{1.7}$$

where $\mathcal{F}_i$ is defined by (1.3) and $\operatorname{cap}(\Omega_i)$ is the harmonic capacity of the set Ω_i (see [31]). Another "condition of smallness" of the fluxes $\mathcal{F}_i$ has been used in the paper [21], where the fluxes $\mathcal{F}_i$ are related to the L^3-norms of certain functions, harmonic in Ω (see the condition (2.8) from [21]). More simpler assumption is expressed by

$$\sum_{i=1}^{N} \max_{\Gamma_i} \frac{|\mathcal{F}_i|}{|x - x_i|} < 8\pi\nu \tag{1.8}$$

(see [37]), where x_i is a fixed point of Ω_i. The assumptions (1.7) and one from [21] could be weaker than (1.8) in some special geometries of Ω, but (1.8) is much easier to check (see also [5] for analogous conditions in bounded domains).

The present paper is devoted to the above question in the axially symmetric case. To introduce the problem we have to specify some notations. Let O_{x_1}, O_{x_2}, O_{x_3} be coordinate axes in $\mathbb{R}^3$ and $\theta = \operatorname{arctg}(x_2/x_1)$, $r = (x_1^2 + x_2^2)^{1/2}$, $z = x_3$ be cylindrical coordinates. Denote by v_θ, v_r, v_z the projections of the vector $\mathbf{v}$ on the axes θ, r, z.

A function f is said to be *axially symmetric* if it does not depend on θ. A vector-valued function $\mathbf{h} = (h_\theta, h_r, h_z)$ is called *axially symmetric* if h_θ, h_r and h_z do not depend on θ. A vector-valued function $\mathbf{h} = (h_\theta, h_r, h_z)$ is called *axially symmetric with no swirl* if $h_\theta = 0$, while h_r and h_z do not depend on θ.

Note that for axially-symmetric solutions $\mathbf{u}$ of (1.1) the vector $\mathbf{u}_0$ has to be parallel to the symmetry axis. The main result of the paper is the following.

Theorem 1.1 *Assume that $\Omega \subset \mathbb{R}^3$ is an exterior axially symmetric domain* (1.2) *with C^2-smooth boundary $\partial\Omega$, $\mathbf{u}_0 \in \mathbb{R}^3$ is a constant vector parallel to the symmetry axis, and $\mathbf{f} \in W^{1,2}(\Omega) \cap L^{6/5}(\Omega)$, $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ are axially symmetric. Then* (1.1) *admits at least one weak axially symmetric solution $\mathbf{u}$ satisfying* (1.5). *Moreover, if $\mathbf{a}$ and $\mathbf{f}$ are axially symmetric with no swirl, then* (1.1) *admits at least one weak axially symmetric solution satisfying* (1.5) *with no swirl.*

Remark 1.1 It is well known (see, e.g., [24]) that under hypothesis of Theorem 1.1, every weak solution $\mathbf{u}$ of problem (1.1) is more regular, i.e, $\mathbf{u} \in W^{2,2}_{\mathrm{loc}}(\overline{\Omega}) \cap W^{3,2}_{\mathrm{loc}}(\Omega)$.

[2] For small $\|\mathbf{a}\|_{L^\infty(\partial\Omega)}$ existence of a solution $(\mathbf{u}, p)$ to (1.1) such that $\mathbf{u} = O(r^{-1})$ is a simple consequence of Banach contractions theorem [36]. Moreover, one can show that $p = O(r^{-2})$ and the derivatives of order k of $\mathbf{u}$ and p behave at infinity as r^{-k-1}, r^{-k-2}, respectively [41] (see also [13,30]).

Let us emphasize that Theorem 1.1 furnishes the first existence result without any assumption on the fluxes for the stationary Navier–Stokes problem in an exterior domains.

Note that in the papers [19,20] existence of a solution to problem (1.1_1)–(1.1_3) in arbitrary C^2-smooth **bounded** plane or axially symmetric spatial domain Ω has been proved under the sole condition of zero total flux through the boundary (for a historical review in the case of bounded domains see, e.g., [19,34,35]).

The proof of the existence theorem is based on an a priori estimate which we derive using the classical *reductio ad absurdum* argument of Leray [26], see Sect. 3. As well-known, after applying Leray's argument one comes along to a solution of Euler system satisfying zero boundary conditions. Such solutions are studied in Sect. 4. The essentially new part here is the use of Bernoulli's law obtained in [16] for Sobolev solutions to the Euler equations (the detailed proofs are presented in [17,19] for the plane and the axially symmetric bounded domains, respectively). In Sect. 4 we present the proof of the Bernoulli Law for unbounded domains (Theorem 4.2, see also Lemma 4.5 and Remark 4.1). Furthermore, we prove here that the value of the pressure on the boundary components intersecting the symmetry axis coincides with the value of the pressure at infinity (see Corollary 4.1). This phenomenon is connected with the fact that the symmetry axis can be approximated by streamlines, where the total head pressure is constant (see Theorem 4.4).

The results concerning Bernoulli's law are based on the recent version of the Morse–Sard theorem proved by Bourgain et al. [6], see also Sect. 2.3. This theorem implies, in particular, that almost all level sets of a function $\psi \in W^{2,1}(\mathbb{R}^2)$ are finite unions of C^1-curves homeomorphic to a circle (see also [7] for a multidimensional case).

We obtain the required contradiction in Sect. 5 for the case $\mathbf{u}_0 = 0$. The above mentioned results allow to construct suitable subdomains (bounded by smooth streamlines) and to estimate the L^2-norm of the gradient of the total head pressure. We use here some ideas which are close (on a heuristic level) to the Hopf maximum principle for the solutions of elliptic PDEs (for a more detailed explanation see Sect. 5). Finally, a contradiction is obtained using the Coarea formula, isoperimetric inequality (see Lemmas 5.7–5.9), and some elementary facts from real analysis (see "Appendix"). In Sect. 6 we show how to modify our arguments for the case $\mathbf{u}_0 \neq \mathbf{0}$.

The analogous result (by quite different methods) for symmetric exterior **plane** domains was established in [18] under the additional assumption that all connected components of the boundary intersects the symmetry axis.

2 Notations and preliminary results

By *a domain* we mean an open connected set. We use standard notations for function spaces: $W^{k,q}(\Omega)$, $W^{\alpha,q}(\partial\Omega)$, where $\alpha \in (0,1)$, $k \in \mathbb{N}_0$, $q \in [1,+\infty]$. In our notation we do not distinguish function spaces for scalar and vector valued functions; it is clear from the context whether we use scalar or vector (or tensor) valued function spaces.

For $q \geq 1$ denote by $D^{k,q}(\Omega)$ the set of functions $f \in W^{k,q}_{\mathrm{loc}}$ such that $\|f\|_{D^{k,q}(\Omega)} = \|\nabla^k f\|_{L^q(\Omega)} < \infty$. Further, $D_0^{1,2}(\Omega)$ is the closure of the set of all smooth functions having compact supports in Ω with respect to the norm $\|\cdot\|_{D^{1,2}(\Omega)}$, and $H(\Omega) = \{\mathbf{v} \in D_0^{1,2}(\Omega) : \operatorname{div} \mathbf{v} = 0\}$.

Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^2-smooth boundary $\partial\Omega$, defined by (1.2). It is well known that functions $u \in D_0^{1,2}(\Omega)$ belong to $L^6(\Omega)$ and, hence tend (in some sense) to zero at infinity (see, e.g., [24]). Moreover, in exterior domains Ω with C^2-smooth boundaries any vector-field $\mathbf{v} \in H(\Omega)$ can be approximated in the norm $\|\cdot\|_{D^{1,2}(\Omega)}$ by solenoidal smooth vector-fields with compact supports (see [25]).

Working with Sobolev functions we always assume that the "best representatives" are chosen. If $w \in L^1_{\text{loc}}(\Omega)$, then the best representative w^* is defined by

$$w^*(x) = \begin{cases} \lim\limits_{R\to 0} \fint_{B_R(x)} w(z)dz, & \text{if the finite limit exists;} \\ 0 & \text{otherwise,} \end{cases}$$

where $\fint_{B_R(x)} w(z)dz = \dfrac{1}{\text{meas}(B_R(x))} \int\limits_{B_R(x)} w(z)dz$, $B_R(x) = \{y \in \mathbb{R}^3 : |y - x| < R\}$ is a ball of radius R centered at x. Also we use the notation $B_R = B_R(0)$, $S_R = \partial B_R$.

2.1 Extension of the boundary values

The next lemma concerns the existence of a solenoidal extensions of boundary values.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^3$ be an exterior axially symmetric domain* (1.2)*. If $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$, then there exists a solenoidal extension $\mathbf{A} \in W^{2,2}(\Omega)$ of $\mathbf{a}$ such that $\mathbf{A}(x) = \boldsymbol{\sigma}(x)$ for sufficiently large $|x|$, where*

$$\boldsymbol{\sigma}(x) = -\frac{x}{4\pi|x|^3} \sum_{i=1}^{N} \mathcal{F}_i \tag{2.1}$$

and $\mathcal{F}_i$ are defined by (1.3)*. Moreover, the following estimate*

$$\|\mathbf{A}\|_{W^{2,2}(\Omega)} \le c\|\mathbf{a}\|_{W^{3/2,2}(\partial\Omega)} \tag{2.2}$$

holds. Furthermore, if $\mathbf{a}$ is axially symmetric (axially symmetric with no swirl), then $\mathbf{A}$ is axially symmetric (axially symmetric with no swirl) too.

Proof The proof is based on a standard technique. Let $(\mathbf{u}_s, p_s) \in \big(H(\Omega) \cap W^{2,2}_{loc}(\bar\Omega)\big) \times \big(L^2(\Omega) \cap W^{1,2}_{loc}(\bar\Omega)\big)$ be the solution (see, e.g., [24]) to the Stokes system

$$\begin{aligned} \Delta\mathbf{u_s} - \nabla p_s &= \mathbf{0} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u_s} &= 0 \quad \text{in } \Omega, \\ \mathbf{u}_s &= \mathbf{a} \quad \text{on } \partial\Omega, \\ \lim_{r\to+\infty} \mathbf{u}_s(x) &= \mathbf{0}. \end{aligned} \tag{2.3}$$

The solution $\mathbf{u_s}$ satisfies the estimate

$$\|\mathbf{u_s}\|_{H(\Omega)} + \|\mathbf{u_s}\|_{W^{2,2}(\Omega')} \le c\|\mathbf{a}\|_{W^{3/2,2}(\partial\Omega)} \tag{2.4}$$

for any bounded Ω' with $\bar{\Omega}' \subset \Omega$ (see, e.g., [24]). Define

$$\mathbf{v}(x) = \mathbf{u}_s(x) - \boldsymbol{\sigma}(x).$$

Take $R > 0$ such that $\partial\Omega \Subset B_R$. Then

$$\int\limits_{\partial B_R} \mathbf{v} \cdot \mathbf{n}\, dS = 0 \tag{2.5}$$

Let $\zeta = \zeta(|x|)$ be a smooth cut-off function, equal to 1 in B_R and vanishing outside B_{2R}. Then $\operatorname{div}(\zeta \mathbf{v}) = \nabla\zeta \cdot \mathbf{v} \in \mathring{W}^{1,2}(B_{2R}\setminus B_R)$. Because of (2.5) the equation

$$\operatorname{div} \boldsymbol{\xi} = -\operatorname{div}(\zeta \mathbf{v}), \quad \boldsymbol{\xi}|_{\partial(B_{2R}\setminus B_R)} = 0 \tag{2.6}$$

has a solution $\boldsymbol{\xi} \in \mathring{W}^{2,2}(B_{2R}\setminus B_R)$ (see [4]) and

$$\|\boldsymbol{\xi}\|_{\mathring{W}^{2,2}(B_{2R}\setminus B_R)} \le c\|\mathbf{v}\|_{\mathring{W}^{1,2}(B_{2R}\setminus B_R)} \le c\|\mathbf{a}\|_{W^{3/2,2}(\partial\Omega)}. \tag{2.7}$$

The field

$$\mathbf{A} = \begin{cases} \mathbf{u}_s, & x \in B_R \cap \Omega, \\ \boldsymbol{\sigma} + \boldsymbol{\xi} + \zeta \mathbf{v}, & x \in B_{2R}\setminus B_R, \\ \boldsymbol{\sigma}, & x \in \Omega\setminus B_{2R} \end{cases}$$

is the desired solenoidal extension of $\mathbf{a}$ in Ω. If $\mathbf{a}$ is axially symmetric (axially symmetric with no swirl), then, according to results of Section 2.1 in [19], the extension $\mathbf{u}_s$ and the solution $\boldsymbol{\xi}$ to the divergence equation (2.6) are axially symmetric (axially symmetric with no swirl). Thus, in this case the extension $\mathbf{A}$ is axially symmetric (axially symmetric with no swirl) too. □

2.2 A uniform estimate of the pressure in the Stokes system

This subsection is rather technical: the estimates below are not considered as new or sharp, but they are sufficient for our purposes.

We use here the technique that is usual in the theory of elliptic problems in domains with corner points [15,29] and which is based on classical local estimates for elliptic in the sense of Agmon et al. [1] problems, see also [38,39] (the considered Stokes system is an important particular case of such problems).

Lemma 2.2 *Let* $\Omega \subset \mathbb{R}^3$ *be an exterior axially symmetric domain* (1.2), $\mathbf{f} \in L^{3/2}(\Omega_R)$ *and* $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$. *Take* $R_0 > 0$ *such that* $\partial\Omega \Subset \frac{1}{2}B_{R_0}$. *Suppose that* $R \geq R_0$ *and* $\mathbf{u} \in W^{1,2}(\Omega_R)$ *is a solution to the Stokes system*

$$\begin{aligned} \nu\Delta\mathbf{u} - \nabla p = \mathbf{f} \quad &\text{in } \Omega_R, \\ \operatorname{div}\mathbf{u} = 0 \quad &\text{in } \Omega_R, \\ \mathbf{u} = \mathbf{a} \quad &\text{on } \partial\Omega, \\ \mathbf{u} = \boldsymbol{\sigma} \quad &\text{on } \partial B_R, \end{aligned} \tag{2.8}$$

where $\Omega_R = \Omega \cap B_R$ *and* $\boldsymbol{\sigma}(x)$ *is defined by* (2.1). *Then the estimates*

$$\|\mathbf{u}\|_{L^6(\Omega_R)} \leq c\big(\|\nabla\mathbf{u}\|_{L^2(\Omega_R)} + \|\mathbf{a}\|_{W^{3/2,2}(\partial\Omega)}\big), \tag{2.9}$$

$$\begin{aligned} &\|\nabla^2\mathbf{u}\|_{L^{3/2}(\Omega_R)} + \|\nabla p\|_{L^{3/2}(\Omega_R)} \\ &\quad\leq c\big(\|\nabla\mathbf{u}\|_{L^2(\Omega_R)} + \|\mathbf{f}\|_{L^{3/2}(\Omega_R)} + \|\mathbf{a}\|_{W^{3/2,2}(\partial\Omega)}\big) \end{aligned} \tag{2.10}$$

hold with the constant c *independent of* $\mathbf{a}$, $\mathbf{u}$, $\mathbf{f}$, *and* R.

Proof Rewriting the Stokes system (2.8) for the new function $\mathbf{u}' = \mathbf{u} - \mathbf{A}$, where $\mathbf{A}$ is the solenoidal extension of $\mathbf{a}$ from Lemma 2.1, we can assume, without loss of generality, that $\mathbf{a} = 0$, i.e.,

$$\begin{aligned} \nu\Delta\mathbf{u} - \nabla p = \mathbf{f} \quad &\text{in } \Omega_R, \\ \operatorname{div}\mathbf{u} = 0 \quad &\text{in } \Omega_R, \\ \mathbf{u} = \mathbf{0} \quad &\text{on } \partial\Omega_R. \end{aligned} \tag{2.11}$$

Then the first estimate (2.9) follows easily from the well known inequality

$$\|g\|_{L^6(\mathbb{R}^3)} \leq c\|\nabla g\|_{L^2(\mathbb{R}^3)} \quad \forall g \in C_0^\infty(\mathbb{R}^3)$$

and (2.2).

Let us prove (2.10). Let $R = 2^l R_0$ with $l \in \mathbb{N}$. Denote

$$\omega_k = \{x \in \mathbb{R}^3 : R_0 2^{k-1} \leq |x| \leq R_0 2^k\},\ k = 1, 2, \dots, l;\ \omega_0 = \{x \in \Omega : |x| \leq R_0\},$$

$$\widehat{\omega}_k = \{x \in \mathbb{R}^3 : R_0 2^{k-2} < |x| < R_0 2^{k+1}\},\ k = 1, 2, \dots;\ \widehat{\omega}_0 = \{x \in \Omega : |x| < 2R_0\}.$$

Obviously, $\omega_k \subset \widehat{\omega}_k$ and $\Omega_R = \bigcup_{k=0}^{l} \omega_k$. Consider the system (2.11) in $\widehat{\omega}_k$, $k \leq l-1$. After the scaling $y = \dfrac{x}{R_0 2^k}$ we get the Stokes equations in the domain $\widehat{\sigma}_0 = \{y : \frac{1}{4} < |y| < 2\}$:

$$-\nu\Delta_y\mathbf{u} + \nabla_y\tilde{p} = \tilde{\mathbf{f}}, \quad \operatorname{div}_y\mathbf{u} = 0,$$

where $\tilde{p} = 2^k R_0 p$, $\tilde{\mathbf{f}} = 2^{2k} R_0^2 \mathbf{f}$. Let $\sigma_0 = \{y : \frac{1}{2} < |y| < 1\}$. By the local estimate for Agmon et al. elliptic type problems (see [1,38,39]) we have

$$\begin{aligned} &\|\mathbf{u}\|_{W^{2,\frac{3}{2}}(\sigma_0)} + \|\nabla_y \tilde{p}\|_{L^{\frac{3}{2}}(\sigma_0)} \\ &\quad \le c\Big(\|\tilde{\mathbf{f}}\|_{L^{\frac{3}{2}}(\widehat{\sigma}_0)} + \|\mathbf{u}\|_{L^{\frac{3}{2}}(\widehat{\sigma}_0)} + \|\tilde{p} - \tilde{p}_0\|_{L^{\frac{3}{2}}(\widehat{\sigma}_0)}\Big), \end{aligned} \tag{2.12}$$

where $\tilde{p}_0 = \frac{1}{|\widehat{\sigma}_0|} \int\limits_{\widehat{\sigma}_0} \tilde{p}(y)dy$.

Consider the functional

$$H(\mathbf{w}) = \int\limits_{\widehat{\sigma}_0} (\tilde{p}(y) - \tilde{p}_0) \operatorname{div} \mathbf{w} \, dy, \quad \forall \mathbf{w} \in \mathring{W}^{1,3}(\widehat{\sigma}_0).$$

Using Stokes equations and integrating by parts we obtain

$$\begin{aligned} |H(\mathbf{w})| = \Big|\int\limits_{\widehat{\sigma}_0} \nabla_y \tilde{p} \cdot \mathbf{w} \, dy\Big| &\le \nu \Big|\int\limits_{\widehat{\sigma}_0} \nabla_y \mathbf{u} \cdot \nabla_y \mathbf{w} \, dy\Big| \\ &+ \Big|\int\limits_{\widehat{\sigma}_0} \tilde{\mathbf{f}} \cdot \mathbf{w} \, dy\Big| \le c\Big(\|\tilde{\mathbf{f}}\|_{L^{\frac{3}{2}}(\widehat{\sigma}_0)} + \|\nabla_y \mathbf{u}\|_{L^{\frac{3}{2}}(\widehat{\sigma}_0)}\Big)\|\nabla_y \mathbf{w}\|_{L^3(\widehat{\sigma}_0)}. \end{aligned}$$

The norm of the functional H is equivalent to $\|\tilde{p} - \tilde{p}_0\|_{L^{\frac{3}{2}}(\widehat{\sigma}_0)}$ (see, e.g., [32,33]). Hence, the last estimate gives

$$\|\tilde{p} - \tilde{p}_0\|_{L^{\frac{3}{2}}(\widehat{\sigma}_0)} \le c\Big(\|\tilde{\mathbf{f}}\|_{L^{\frac{3}{2}}(\widehat{\sigma}_0)} + \|\nabla_y \mathbf{u}\|_{L^{\frac{3}{2}}(\widehat{\sigma}_0)}\Big)$$

and inequality (2.12) takes the form

$$\|\mathbf{u}\|_{W^{2,\frac{3}{2}}(\sigma_0)} + \|\nabla_y \tilde{p}\|_{L_{\frac{3}{2}}(\sigma_0)} \le c\Big(\|\tilde{\mathbf{f}}\|_{L_{\frac{3}{2}}(\widehat{\sigma}_0)} + \|\mathbf{u}\|_{W^{1,\frac{3}{2}}(\widehat{\sigma}_0)}\Big).$$

In particular,

$$\|\nabla_y^2 \mathbf{u}\|_{L^{\frac{3}{2}}(\sigma_0)} + \|\nabla_y \tilde{p}\|_{L_{\frac{3}{2}}(\sigma_0)} \le c\Big(\|\tilde{f}\|_{L_{\frac{3}{2}}(\widehat{\sigma}_0)} + \|\mathbf{u}\|_{W^{1,\frac{3}{2}}(\widehat{\sigma}_0)}\Big).$$

Returning to coordinates x, we get

$$\begin{aligned} &\|\nabla_x^2 \mathbf{u}\|_{L^{\frac{3}{2}}(\omega_k)} + \|\nabla_x p\|_{L^{\frac{3}{2}}(\omega_k)} \\ &\quad \le c\Big(\|\mathbf{f}\|_{L^{\frac{3}{2}}(\widehat{\omega}_k)} + 2^{-2k}\|\mathbf{u}\|_{L^{\frac{3}{2}}(\widehat{\omega}_k)} + 2^{-k}\|\nabla_x \mathbf{u}\|_{L^{\frac{3}{2}}(\widehat{\omega}_k)}\Big). \end{aligned} \tag{2.13}$$

Estimates (2.13) hold for $1 \leq k \leq l-1$. For $k=l$ we obtain, by the same argument, the following estimate

$$\begin{aligned}&\|\nabla_x^2\mathbf{u}\|_{L^{\frac{3}{2}}(\omega_l)}+\|\nabla_x p\|_{L^{\frac{3}{2}}(\omega_l)}\\&\leq c\Big(\|\mathbf{f}\|_{L^{\frac{3}{2}}(\omega_{l-1}\cup\omega_l)}+2^{-2l}\|\mathbf{u}\|_{L^{\frac{3}{2}}(\omega_{l-1}\cup\omega_l)}+2^{-l}\|\nabla_x\mathbf{u}\|_{L^{\frac{3}{2}}(\omega_{l-1}\cup\omega_l)}\Big).\end{aligned} \tag{2.14}$$

Furthermore, by local estimates (see [1,38,39]) for the solution to (2.11) we have

$$\begin{aligned}&\|\nabla_x^2\mathbf{u}\|_{L^{\frac{3}{2}}(\omega_0)}+\|\nabla_x p\|_{L^{\frac{3}{2}}(\omega_0)}\\&\leq c\Big(\|\mathbf{f}\|_{L^{\frac{3}{2}}(\omega_0\cup\omega_1)}+\|\mathbf{u}\|_{L^{\frac{3}{2}}(\omega_0\cup\omega_1)}+\|\nabla_x\mathbf{u}\|_{L^{\frac{3}{2}}(\omega_0\cup\omega_1)}\Big).\end{aligned} \tag{2.15}$$

(recall that $\mathbf{u}|_{\partial\Omega}=0$). Summing estimates (2.13)–(2.15) by k and taking into account that $r \sim 2^k$ for $x \in \omega_k$, we derive

$$\begin{aligned}&\|\nabla_x^2\mathbf{u}\|_{L^{\frac{3}{2}}(\Omega_R)}+\|\nabla_x p\|_{L^{\frac{3}{2}}(\Omega_R)}\\&\leq c\Big(\|\mathbf{f}\|_{L^{\frac{3}{2}}(\Omega_R)}+\||x|^{-2}\mathbf{u}\|_{L^{\frac{3}{2}}(\Omega_R)}+\||x|^{-1}\nabla_x\mathbf{u}\|_{L^{\frac{3}{2}}(\Omega_R)}\Big)\\&\leq c'\Big(\|\mathbf{f}\|_{L^{\frac{3}{2}}(\Omega_R)}+\||x|^{-1}\mathbf{u}\|_{L^2(\Omega_R)}+\|\nabla_x\mathbf{u}\|_{L^2(\Omega_R)}\Big)\\&\leq c''\Big(\|\mathbf{f}\|_{L^{\frac{3}{2}}(\Omega_R)}+\|\nabla_x\mathbf{u}\|_{L^2(\Omega_R)}\Big),\end{aligned} \tag{2.16}$$

where the constant c'' is independent of R. Here we have used the Hölder inequality and the well known estimate

$$\int\limits_{\mathbb{R}^3}\frac{|g(x)|^2}{|x|^2}dx \leq c\int\limits_{\mathbb{R}^3}|\nabla g(x)|^2dx \quad \forall\, g \in C_0^\infty(\mathbb{R}^3).$$

The required estimate (2.10) for the solution of the nonhomogeneous boundary value problem (2.8) follows from (2.16) and the estimate (2.2) for the extension $\mathbf{A}$ constructed in Sect. 2.1. □

Corollary 2.1 *Let $\Omega \subset \mathbb{R}^3$ be an exterior axially symmetric domain (1.2), $\mathbf{f} \in L^{3/2}(\Omega_R)$ and $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$. Take $R_0 > 0$ such that $\partial\Omega \Subset \frac{1}{2}B_{R_0}$ and $R \geq R_0$. Let $\mathbf{u} \in W^{1,2}(\Omega_R)$ be a solution to the Navier–Stokes system*

$$\begin{aligned}-\nu\Delta\mathbf{u}+\big(\mathbf{u}\cdot\nabla\big)\mathbf{u}+\nabla p=\mathbf{f} \quad &\text{in } \Omega_R,\\ \operatorname{div}\mathbf{u}=0 \quad &\text{in } \Omega_R,\\ \mathbf{u}=\mathbf{a} \quad &\text{on } \partial\Omega,\\ \mathbf{u}=\boldsymbol{\sigma} \quad &\text{on } \partial B_R,\end{aligned} \tag{2.17}$$

where $\boldsymbol{\sigma}(x)$ *is defined by* (2.1). *Then the estimates*

$$\|\mathbf{u}\|_{L^6(\Omega_R)} \le c\big(\|\nabla\mathbf{u}\|_{L^2(\Omega_R)} + \|\mathbf{a}\|_{W^{3/2,2}(\partial\Omega)}\big), \tag{2.18}$$

$$\|\nabla^2\mathbf{u}\|_{L^{3/2}(\Omega_R)} + \|\nabla p\|_{L^{3/2}(\Omega_R)} \le c\Big(\|\mathbf{f}\|_{L^{3/2}(\Omega_R)} + \|\nabla\mathbf{u}\|_{L^2(\Omega_R)} + \|\mathbf{a}\|_{W^{3/2,2}(\partial\Omega)} + \big(\|\nabla\mathbf{u}\|_{L^2(\Omega_R)} + \|\mathbf{a}\|_{W^{3/2,2}(\partial\Omega)}\big)^2\Big) \tag{2.19}$$

hold. Here the constant c is independent of $\mathbf{a}$, $\mathbf{u}$, $\mathbf{f}$, R.

Proof Estimate (2.18) follows by the same argument as (2.9). In order to prove (2.19), we consider the Navier–Stokes system (2.17) as the Stokes one with the right-hand side $\mathbf{f}' = \mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u}$. Since

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L^{3/2}(\Omega_R)} \le \|\mathbf{u}\|_{L^6(\Omega_R)}\|\nabla\mathbf{u}\|_{L^2(\Omega_R)} \le c\|\nabla\mathbf{u}\|^2_{L^2(\Omega_R)},$$

estimate (2.19) follows from (2.10). □

2.3 On Morse–Sard and Luzin N-properties of Sobolev functions from $W^{2,1}$

Let us recall some classical differentiability properties of Sobolev functions.

Lemma 2.3 (see Proposition 1 in [10]) *Let* $\psi \in W^{2,1}(\mathbb{R}^2)$. *Then the function* ψ *is continuous and there exists a set* A_ψ *such that* $\mathfrak{H}^1(A_\psi) = 0$, *and the function* ψ *is differentiable (in the classical sense) at each* $x \in \mathbb{R}^2\setminus A_\psi$. *Furthermore, the classical derivative at such points* x *coincides with* $\nabla\psi(x) = \lim_{r\to 0} \fint_{B_r(x)} \nabla\psi(z)dz$, *and* $\lim_{r\to 0} \fint_{B_r(x)} |\nabla\psi(z) - \nabla\psi(x)|^2 dz = 0$.

Here and henceforth we denote by $\mathfrak{H}^1$ the one-dimensional Hausdorff measure, i.e., $\mathfrak{H}^1(F) = \lim_{t\to 0+}\mathfrak{H}^1_t(F)$, where $\mathfrak{H}^1_t(F) = \inf\{\sum_{i=1}^\infty \operatorname{diam} F_i : \operatorname{diam} F_i \le t, F \subset \bigcup_{i=1}^\infty F_i\}$.

The next theorem have been proved recently by Bourgain et al. [6] (see also [7] for a multidimensional case).

Theorem 2.1 *Let* $\mathcal{D} \subset \mathbb{R}^2$ *be a bounded domain with Lipschitz boundary and* $\psi \in W^{2,1}(\mathcal{D})$. *Then*

- (i) $\mathfrak{H}^1(\{\psi(x) : x \in \bar{\mathcal{D}}\setminus A_\psi \;\&\; \nabla\psi(x) = 0\}) = 0$;
- (ii) *for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any set* $U \subset \bar{\mathcal{D}}$ *with* $\mathfrak{H}^1_\infty(U) < \delta$ *the inequality* $\mathfrak{H}^1(\psi(U)) < \varepsilon$ *holds;*
- (iii) *for* $\mathfrak{H}^1$*—almost all* $y \in \psi(\bar{\mathcal{D}}) \subset \mathbb{R}$ *the preimage* $\psi^{-1}(y)$ *is a finite disjoint family of* C^1*–curves* S_j, $j = 1, 2, \dots, N(y)$. *Each* S_j *is either a cycle in* $\mathcal{D}$ *(i.e.,* $S_j \subset \mathcal{D}$ *is homeomorphic to the unit circle* $\mathbb{S}^1$*) or it is a simple arc with endpoints on* $\partial\mathcal{D}$ *(in this case* S_j *is transversal to* $\partial\mathcal{D}$*).*

2.4 Some facts from topology

We shall need some topological definitions and results. By *continuum* we mean a compact connected set. We understand connectedness in the sense of general topology. A subset of a topological space is called *an arc* if it is homeomorphic to the unit interval [0, 1].

Let us shortly present some results from the classical paper of A.S. Kronrod [22] concerning level sets of continuous functions. Let $Q = [0, 1] \times [0, 1]$ be a square in $\mathbb{R}^2$ and let f be a continuous function on Q. Denote by E_t a level set of the function f, i.e., $E_t = \{x \in Q : f(x) = t\}$. A component K of the level set E_t containing a point x_0 is a maximal connected subset of E_t containing x_0. By T_f denote a family of all connected components of level sets of f. It was established in [22] that T_f equipped by a natural topology[3] is a one-dimensional topological tree.[4] Endpoints of this tree[5] are the components $C \in T_f$ which do not separate Q, i.e., $Q\setminus C$ is a connected set. Branching points of the tree are the components $C \in T_f$ such that $Q\setminus C$ has more than two connected components (see [22, Theorem 5]). By results of [22, Lemma 1], the set of all branching points of T_f is at most countable. The main property of a tree is that any two points could be joined by a unique arc. Therefore, the same is true for T_f.

Lemma 2.4 (see Lemma 13 in [22]) *If $f \in C(Q)$, then for any two different points $A \in T_f$ and $B \in T_f$, there exists a unique arc $J = J(A, B) \subset T_f$ joining A to B. Moreover, for every inner point C of this arc the points A, B lie in different connected components of the set $T_f\setminus\{C\}$.*

We can reformulate the above Lemma in the following equivalent form.

Lemma 2.5 *If $f \in C(Q)$, then for any two different points $A, B \in T_f$, there exists a continuous injective function $\varphi : [0, 1] \to T_f$ with the properties*

(i) $\varphi(0) = A$, $\varphi(1) = B$;
(ii) *for any* $t_0 \in [0, 1]$,

$$\lim_{[0,1]\ni t\to t_0} \sup_{x\in\varphi(t)} \operatorname{dist}(x, \varphi(t_0)) \to 0;$$

(iii) *for any $t \in (0, 1)$ the sets A, B lie in different connected components of the set $Q\setminus\varphi(t)$.*

Remark 2.1 If in Lemma 2.5 $f \in W^{2,1}(Q)$, then by Theorem 2.1 (iii), there exists a dense subset E of (0, 1) such that $\varphi(t)$ is a C^1– curve for every $t \in E$. Moreover, $\varphi(t)$ is either a cycle or a simple arc with endpoints on ∂Q.

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[3] The convergence in T_f is defined as follows: $T_f \ni C_i \to C$ iff $\sup_{x\in C_i} \operatorname{dist}(x, C) \to 0$.

[4] A locally connected continuum T is called *a topological tree*, if it does not contain a curve homeomorphic to a circle, or, equivalently, if any two different points of T can be joined by a unique arc. This definition implies that T has topological dimension 1.

[5] A point of a continuum K is called an *endpoint of K* (resp., *a branching point of K*) if its topological index equals 1 (more or equal to 3 resp.). For a topological tree T this definition is equivalent to the following: a point $C \in T$ is an endpoint of T (resp., a branching point of T), if the set $T\setminus\{C\}$ is connected (resp., if $T\setminus\{C\}$ has more than two connected components).

Remark 2.2 All results of Lemmas 2.4–2.5 remain valid for level sets of continuous functions $f : \overline{\mathcal{D}}_0 \to \mathbb{R}$, where $\overline{\mathcal{D}}_0 \subset \mathbb{R}^2$ is a compact set homeomorphic to the unit square $Q = [0, 1]^2$.

3 Leray's argument "reductio ad absurdum"

Let us consider the Navier–Stokes problem (1.1) with $\mathbf{f} \in W^{1,2}(\Omega) \cap L^{6/5}(\Omega)$ in the C^2-smooth axially symmetric exterior domain $\Omega \subset \mathbb{R}^3$ defined by (1.2). Without loss of generality, we may assume that $\mathbf{f} = \operatorname{curl} \mathbf{b} \in W^{1,2}(\Omega) \cap L^{6/5}(\Omega)$.[6]

We shall prove Theorem 1.1 for

$$\mathbf{u}_0 = \mathbf{0}.$$

The proof for $\mathbf{u}_0 \neq \mathbf{0}$ follows the same steps with minor standard modification, see Sect. 6.

By a *weak solution* of problem (1.1) we mean a function $\mathbf{u}$ such that $\mathbf{w} = \mathbf{u} - \mathbf{A} \in H(\Omega)$ and the integral identity

$$\begin{aligned} \nu \int_\Omega \nabla \mathbf{w} \cdot \nabla \boldsymbol{\eta}\, dx &= -\nu \int_\Omega \nabla \mathbf{A} \cdot \nabla \boldsymbol{\eta}\, dx - \int_\Omega \big(\mathbf{A} \cdot \nabla\big)\mathbf{A} \cdot \boldsymbol{\eta}\, dx \\ &- \int_\Omega \big(\mathbf{A} \cdot \nabla\big)\mathbf{w} \cdot \boldsymbol{\eta}\, dx - \int_\Omega \big(\mathbf{w} \cdot \nabla\big)\mathbf{w} \cdot \boldsymbol{\eta}\, dx \\ &- \int_\Omega \big(\mathbf{w} \cdot \nabla\big)\mathbf{A} \cdot \boldsymbol{\eta}\, dx + \int_\Omega \mathbf{f} \cdot \boldsymbol{\eta}\, dx \end{aligned} \tag{3.1}$$

holds for any $\boldsymbol{\eta} \in J_0^\infty(\Omega)$, where $J_0^\infty(\Omega)$ is a set of all infinitely smooth solenoidal vector-fields with compact support in Ω. Here $\mathbf{A}$ is the extension of the boundary data constructed in Lemma 2.1. We shall look for the axially symmetric (axially symmetric with no swirl) weak solution of problem (1.1). We find this solution as a limit of weak solution to the Navier–Stokes problem in a sequence of bounded domain Ω_{bk} that in the limit exhaust the unbounded domain Ω. The following result concerning the solvability of the Navier–Stokes problem in axially symmetric bounded domains was proved in [20].

Theorem 3.1 *Let $\Omega' = \Omega_0 \backslash \big(\bigcup_{j=1}^N \bar{\Omega}_j\big)$ be an axially symmetric bounded domain in $\mathbb{R}^3$ with multiply connected C^2-smooth boundary $\partial\Omega'$ consisting of $N + 1$ disjoint components $\Gamma_j = \partial\Omega_j$, $j = 0, \ldots, N$. If $\mathbf{f} \in W^{1,2}(\Omega')$ and $\mathbf{a} \in W^{3/2,2}(\partial\Omega')$ are axially symmetric and $\mathbf{a}$ satisfies*

$$\int_{\partial\Omega'} \mathbf{a} \cdot \mathbf{n}\, dS = 0,$$

[6] By the Helmholtz–Weyl decomposition, $\mathbf{f}$ can be represented as the sum $\mathbf{f} = \operatorname{curl} \mathbf{b} + \nabla\varphi$ with $\operatorname{curl} \mathbf{b} \in W^{1,2}(\Omega) \cap L^{6/5}(\Omega)$, and the gradient part is included then into the pressure term (see, e.g., [13,24]).

then (1.1$_1$)–(1.1$_3$) *with* $\Omega = \Omega'$ *admits at least one weak axially symmetric solution* $\mathbf{u} \in W^{1,2}(\Omega')$. *Moreover, if* $\mathbf{f}$ *and* $\mathbf{a}$ *are axially symmetric with no swirl, then the problem* (1.1$_1$)–(1.1$_3$) *with* $\Omega = \Omega'$ *admits at least one weak axially symmetric solution with no swirl.*

Consider the sequence of boundary value problems

$$\begin{aligned} \nu\Delta\widehat{\mathbf{w}}_k - (\widehat{\mathbf{w}}_k + \mathbf{A})\cdot\nabla(\widehat{\mathbf{w}}_k + \mathbf{A}) + \nu\Delta\mathbf{A} - \nabla\widehat{p}_k = \mathbf{f} \quad & \text{in } \Omega_{bk}, \\ \operatorname{div}\widehat{\mathbf{w}}_k = 0 \quad & \text{in } \Omega_{bk}, \\ \widehat{\mathbf{w}}_k = \mathbf{0} \quad & \text{on } \partial\Omega_{bk}, \end{aligned} \tag{3.2}$$

where $\Omega_{bk} = B_k \cap \Omega$ for $k \geq k_0$, $B_k = \{x : \ |x| < k\}$, $\frac{1}{2}B_{k_0} \supset \bigcup_{i=1}^N \bar{\Omega}_i$. By Theorem 3.1, each problem (3.2) has an axially symmetric solution $\mathbf{w}_k \in H(\Omega_{bk})$ satisfying the integral identity

$$\begin{aligned} \nu\int\limits_\Omega \nabla\widehat{\mathbf{w}}_k\cdot\nabla\boldsymbol{\eta}\,dx = &\int\limits_\Omega \mathbf{f}\cdot\boldsymbol{\eta}\,dx - \nu\int\limits_\Omega \nabla\mathbf{A}\cdot\nabla\boldsymbol{\eta}\,dx - \int\limits_\Omega \big(\mathbf{A}\cdot\nabla\big)\mathbf{A}\cdot\boldsymbol{\eta}\,dx \\ -\int\limits_\Omega \big(\mathbf{A}\cdot\nabla\big)\widehat{\mathbf{w}}_k\cdot\boldsymbol{\eta}\,dx &- \int\limits_\Omega \big(\widehat{\mathbf{w}}_k\cdot\nabla\big)\widehat{\mathbf{w}}_k\cdot\boldsymbol{\eta}\,dx - \int\limits_\Omega \big(\widehat{\mathbf{w}}_k\cdot\nabla\big)\mathbf{A}\cdot\boldsymbol{\eta}\,dx \end{aligned} \tag{3.3}$$

for all $\boldsymbol{\eta} \in H(\Omega_{bk})$. Here we have assumed that $\widehat{\mathbf{w}}_k$ and $\boldsymbol{\eta}$ are extended by zero to the whole domain Ω.

Assume that there is a positive constant c independent of k such that

$$\int\limits_\Omega |\nabla\widehat{\mathbf{w}}_k|^2 \leq c \tag{3.4}$$

(possibly along a subsequence of $\{\widehat{\mathbf{w}}_k\}_{k\in\mathbb{N}}$). The estimate (3.4) implies the existence of a solution to problem (1.1). Indeed, the sequence $\widehat{\mathbf{w}}_k$ is bounded in $H(\Omega)$. Hence, $\widehat{\mathbf{w}}_k$ converges weakly (modulo a subsequence) in $H(\Omega)$ and strongly in $L^q_{\text{loc}}(\overline{\Omega})$ $(q < 6)$ to a function $\widehat{\mathbf{w}} \in H(\Omega)$. Taking any test function $\boldsymbol{\eta}$ with compact support, we can find k such that $\operatorname{supp}\boldsymbol{\eta} \subset \Omega_{bk}$. Thus, we can pass to a limit as $k \to \infty$ in (3.3) and we obtain for the limit function $\widehat{\mathbf{w}}$ the integral identity (3.1). Then, by definition, $\mathbf{u} = \widehat{\mathbf{w}} + \mathbf{A}$ is a weak solution to the Navier–Stokes problem (1.1). Thus, to prove the assertion of Theorem 1.1, it is sufficient to establish the uniform estimate (3.4).

We shall prove (3.4) following a classical *reductio ad absurdum* argument of Leray [26] and Ladyzhenskaia [24]. Indeed, if (3.4) is not true, then there exists a sequence $\{\widehat{\mathbf{w}}_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty} J_k^2 = +\infty, \quad J_k^2 = \int\limits_\Omega |\nabla\widehat{\mathbf{w}}_k|^2.$$

The sequence $\mathbf{w}_k = \widehat{\mathbf{w}}_k/J_k$ is bounded in $H(\Omega)$ and it holds

$$\frac{\nu}{J_k}\int_\Omega \nabla\mathbf{w}_k\cdot\nabla\boldsymbol{\eta}\,dx = -\int_\Omega(\mathbf{w}_k\cdot\nabla)\mathbf{w}_k\cdot\boldsymbol{\eta}\,dx + \frac{1}{J_k}\int_\Omega(\mathbf{w}_k\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{A}\,dx$$
$$+\frac{1}{J_k}\int_\Omega(\mathbf{A}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{w}_k\,dx + \frac{1}{J_k^2}\int_\Omega(\mathbf{A}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{A}\,dx \tag{3.5}$$
$$-\frac{\nu}{J_k^2}\int_\Omega\nabla\mathbf{A}\cdot\nabla\boldsymbol{\eta}\,dx + \frac{1}{J_k^2}\int_\Omega\mathbf{f}\cdot\boldsymbol{\eta}\,dx$$

for all $\boldsymbol{\eta}\in H(\Omega_{bk})$. Extracting a subsequence (if necessary) we can assume that $\mathbf{w}_k$ converges weakly in $H(\Omega)$ and strongly in $L^q_{\rm loc}(\overline{\Omega})$ ($q<6$) to a vector field $\mathbf{v}\in H(\Omega)$ with

$$\int_\Omega |\nabla\mathbf{v}|^2\le 1. \tag{3.6}$$

Fixing in (3.5) a solenoidal smooth $\boldsymbol{\eta}$ with compact support and letting $k\to+\infty$ we get

$$\int_\Omega(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\eta}\,dx = 0 \quad \forall\,\boldsymbol{\eta}\in J_0^\infty(\Omega), \tag{3.7}$$

Hence, $\mathbf{v}\in H(\Omega)$ is a weak solution to the Euler equations, and for some $p\in D^{1,3/2}(\Omega)$ the pair $(\mathbf{v},p)$ satisfies the Euler equations almost everywhere:

$$\begin{cases} \big(\mathbf{v}\cdot\nabla\big)\mathbf{v}+\nabla p = 0 & \text{in}\quad \Omega,\\ \operatorname{div}\mathbf{v}=0 & \text{in}\quad\Omega,\\ \mathbf{v}=0 & \text{on}\quad\partial\Omega. \end{cases} \tag{3.8}$$

Adding some constants to p (if necessary) by virtue of the Sobolev inequality (see, e.g., [13] II.6) we may assume without loss of generality that

$$\|p\|_{L^3(\Omega)}\le \text{const}. \tag{3.9}$$

Put $\nu_k=(J_k)^{-1}\nu$. Multiplying equations (3.2) by $\frac{1}{J_k^2}=\frac{\nu_k^2}{\nu^2}$, we see that the pair $\big(\mathbf{u}_k=\frac{1}{J_k}\widehat{\mathbf{w}}_k+\frac{1}{J_k}\mathbf{A},\ p_k=\frac{1}{J_k^2}\widehat{p}_k\big)$ satisfies the following system

$$\begin{cases} -\nu_k\Delta\mathbf{u}_k+\big(\mathbf{u}_k\cdot\nabla\big)\mathbf{u}_k+\nabla p_k=\mathbf{f}_k & \text{in}\quad\Omega_{bk},\\ \operatorname{div}\mathbf{u}_k=0 & \text{in}\quad\Omega_{bk},\\ \mathbf{u}_k=\mathbf{a}_k & \text{on}\quad\partial\Omega_{bk}, \end{cases} \tag{3.10}$$

where $\mathbf{f}_k = \frac{\nu_k^2}{\nu^2}\mathbf{f}$, $\mathbf{a}_k = \frac{\nu_k}{\nu}\mathbf{A}$, $\mathbf{u}_k \in W^{3,2}_{\text{loc}}(\Omega)$, $p_k \in W^{2,2}_{\text{loc}}(\Omega)$.[7] From Corollary 2.1 we have

$$\|\mathbf{u}_k\|_{L^6(\Omega_{bk})} \leq \text{const.} \tag{3.11}$$

For $R > R_0$ denote $\Omega_R = \Omega \cap B(0, R)$. Since by Hölder inequality $\|\mathbf{f}\|_{L^{3/2}(\Omega_R)} \leq \|\mathbf{f}\|_{L^2(\Omega_R)}\sqrt{R}$, Corollary 2.1 implies also

$$\|\nabla p_k\|_{L^{3/2}(\Omega_R)} \leq C\sqrt{R} \qquad \forall R \in [R_0, R_k], \tag{3.12}$$

where C does not depend[8] on k, R. By construction, we have the weak convergences $\mathbf{u}_k \rightharpoonup \mathbf{v}$ in $W^{1,2}_{\text{loc}}(\overline{\Omega})$, $\quad p_k \rightharpoonup p$ in $W^{1,3/2}_{\text{loc}}(\overline{\Omega})$.[9]

In conclusion, we can prove the following lemma.

Lemma 3.1 *Assume that $\Omega \subset \mathbb{R}^3$ is an exterior axially symmetric domain of type (1.2) with C^2-smooth boundary $\partial\Omega$, and $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$, $\mathbf{f} = \operatorname{curl}\mathbf{b} \in W^{1,2}(\Omega)$ are axially symmetric. If the assertion of Theorem 1.1 is false, then there exist $\mathbf{v}$, p with the following properties.*

(E) The axially symmetric functions $\mathbf{v} \in H(\Omega)$, $p \in D^{1,3/2}(\Omega)$ satisfy the Euler system (3.8) and $\|p\|_{L^3(\Omega)} < \infty$.

(E-NS) Condition (E) is satisfied and there exist a sequences of axially symmetric functions $\mathbf{u}_k \in W^{1,2}(\Omega_{bk})$, $p_k \in W^{1,3/2}(\Omega_{bk})$, $\Omega_{bk} = \Omega \cap B_{R_k}$, $R_k \to \infty$ as $k \to \infty$, and numbers $\nu_k \to 0+$, such that estimates (3.11)–(3.12) hold, the pair $(\mathbf{u}_k, p_k)$ satisfies (3.10) with $\mathbf{f}_k = \frac{\nu_k^2}{\nu^2}\mathbf{f}$, $\mathbf{a}_k = \frac{\nu_k}{\nu}\mathbf{A}$ (here $\mathbf{A}$ is solenoidal extension of $\mathbf{a}$ from Lemma 2.1), and

$$\|\nabla\mathbf{u}_k\|_{L^2(\Omega_{bk})} \to 1, \quad \mathbf{u}_k \rightharpoonup \mathbf{v} \text{ in } W^{1,2}_{\text{loc}}(\overline{\Omega}), \quad p_k \rightharpoonup p \text{ in } W^{1,3/2}_{\text{loc}}(\overline{\Omega}), \tag{3.13}$$

$$\nu = \int\limits_{\Omega} (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{A}\,dx \tag{3.14}$$

Moreover, $\mathbf{u}_k \in W^{3,2}_{\text{loc}}(\Omega)$ and $p_k \in W^{2,2}_{\text{loc}}(\Omega)$.

Proof We need to prove only the identity (3.14), all other properties are already established above. Choosing $\boldsymbol{\eta} = \mathbf{w}_k$ in (3.5) yields

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[7] The interior regularity of the solution depends on the regularity of $\mathbf{f} \in W^{1,2}(\Omega)$, but not on the regularity of the boundary value $\mathbf{a}$, see [24].

[8] Of course, the above assumptions $\mathbf{f} \in W^{1,2}(\Omega) \cap L^{6/5}(\Omega)$ imply $\|\mathbf{f}\|_{L^{3/2}(\Omega_R)} \leq C$ and $\|\nabla p_k\|_{L^{3/2}(\Omega_{bk})} \leq C$, but we prefer to use here weaker estimate (3.12) which holds also in the general case $\mathbf{u}_0 \neq \mathbf{0}$ to make our arguments more universal.

[9] The weak convergence in $W^{1,2}_{\text{loc}}(\overline{\Omega})$ means the weak convergence in $W^{1,2}(\Omega')$ for every bounded subdomain $\Omega' \subset \Omega$.

$$\nu = \int_\Omega (\mathbf{w}_k \cdot \nabla)\mathbf{w}_k \cdot \mathbf{A}\, dx + \frac{1}{J_k}\int_\Omega \mathbf{A}\cdot\nabla\mathbf{w}_k\cdot\mathbf{A}\, dx$$
$$- \frac{\nu}{J_k}\int_\Omega \nabla\mathbf{A}\cdot\nabla\mathbf{w}_k\, dx + \frac{1}{J_k}\int_\Omega \mathbf{f}\cdot\mathbf{w}_k\, dx. \tag{3.15}$$

By the Hölder inequality

$$\left|\int_\Omega \mathbf{A}\cdot\nabla\mathbf{w}_k\cdot\mathbf{A}\, dx\right| \le \|\mathbf{A}\|^2_{L^4(\Omega)}\|\nabla\mathbf{w}_k\|_{L^2(\Omega)} \le \|\mathbf{A}\|^2_{L^4(\Omega)},$$
$$\left|\int_\Omega \nabla\mathbf{A}\cdot\nabla\mathbf{w}_k\, dx\right| \le \|\nabla\mathbf{A}\|_{L^2(\Omega)}\|\nabla\mathbf{w}_k\|_{L^2(\Omega)} \le \|\nabla\mathbf{A}\|_{L^2(\Omega)},$$
$$\left|\int_\Omega \mathbf{f}\cdot\mathbf{w}_k\, dx\right| \le \|\mathbf{f}\|_{L^{6/5}(\Omega)}\|\mathbf{w}_k\|_{L^6(\Omega)}.$$

Note that

$$\int_\Omega (\mathbf{w}_k\cdot\nabla)\mathbf{A}\cdot\mathbf{w}_k\, dx - \int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{A}\cdot\mathbf{v}\, dx = \int_\Omega \big((\mathbf{w}_k-\mathbf{v})\cdot\nabla\big)\mathbf{A}\cdot\mathbf{w}_k\, dx$$
$$+ \int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{A}\cdot(\mathbf{w}_k-\mathbf{v})\, dx = \mathcal{J}_k^{(1)} + \mathcal{J}_k^{(2)} \to 0 \quad \text{as} \quad k\to\infty.$$

Indeed, by Cauchy's and Hardy's inequalities

$$|\mathcal{J}_k^{(1)}| \le \left|\int_{\Omega_R} \big((\mathbf{w}_k-\mathbf{v})\cdot\nabla\big)\mathbf{A}\cdot\mathbf{w}_k\, dx\right| + \frac{c}{R}\int_{\mathbb{R}^3\setminus B_R} \frac{|\mathbf{w}_k-\mathbf{v}||\mathbf{w}_k|}{|x|^2}\, dx$$
$$\le \|\mathbf{w}_k-\mathbf{v}\|_{L^4(\Omega_R)}\|\mathbf{w}_k\|_{L^4(\Omega_R)}\|\nabla\mathbf{A}\|_{L^2(\Omega_R)}$$
$$+\frac{c_1}{R}\|\nabla(\mathbf{w}_k-\mathbf{v})\|_{L^2(\mathbb{R}^3\setminus B_R)}\|\nabla\mathbf{w}_k\|_{L^2(\mathbb{R}^3\setminus B_R)} \le c(R)\|\mathbf{w}_k-\mathbf{v}\|_{L^2(\Omega_R)} + \frac{2c_1}{R},$$

where $\Omega_R = B_R\cap\Omega$ and c_1 does not depend on R. Hence, letting first $k\to+\infty$ and then $R\to+\infty$, we conclude that

$$\lim_{k\to+\infty}\mathcal{J}_k^{(1)} = 0.$$

Analogously, it can be shown that $\lim_{k\to+\infty}\mathcal{J}_k^{(2)} = 0$. Consequently, we can let $k\to+\infty$ in (3.15) to get the required identity (3.14). □

Notice that because of (3.14) the limiting solution $\mathbf{v}$ of the Euler system (3.8) is nontrivial. Now, to finish the proof of the existence Theorem 1.1, we need to show that conditions (E-NS) lead to a contradiction. The next two sections are devoted to this purpose.

4 Euler equation

In this section we assume that the assumptions (E) (from Lemma 3.1) are satisfied. For definiteness, we assume that

(SO) Ω is the domain (1.2) symmetric with respect to the axis O_{x_3} and

$$\Gamma_j \cap O_{x_3} \neq \emptyset, \quad j = 1, \ldots, M',$$
$$\Gamma_j \cap O_{x_3} = \emptyset, \quad j = M'+1, \ldots, N.$$

(We allow also the cases $M' = N$ or $M' = 0$, i.e., when all components (resp., no components) of the boundary intersect the axis of symmetry.)

Denote $P_+ = \{(0, x_2, x_3) : x_2 > 0,\ x_3 \in \mathbb{R}\}$, $\mathcal{D} = \Omega \cap P_+$, $\mathcal{D}_j = \Omega_j \cap P_+$. Of course, on P_+ the coordinates x_2, x_3 coincides with coordinates r, z.

For a set $A \subset \mathbb{R}^3$ put $\breve{A} := A \cap P_+$, and for $B \subset P_+$ denote by $\widetilde{B}$ the set in $\mathbb{R}^3$ obtained by rotation of B around O_z-axis. From the conditions (SO) one can easily see that

(S_1) $\mathcal{D}$ is an unbounded plane domain with Lipschitz boundary. Moreover, $\breve{\Gamma}_j$ is a connected set for each $j = 1, \ldots, N$. In other words, the family $\{\breve{\Gamma}_j : j = 1, \ldots, N\}$ coincides with the family of all connected components of the set $P_+ \cap \partial\mathcal{D}$.

Then $\mathbf{v}$ and p satisfy the following system in the plane domain $\mathcal{D}$:

$$\begin{cases} \dfrac{\partial p}{\partial z} + v_r \dfrac{\partial v_z}{\partial r} + v_z \dfrac{\partial v_z}{\partial z} = 0, \\ \dfrac{\partial p}{\partial r} - \dfrac{(v_\theta)^2}{r} + v_r \dfrac{\partial v_r}{\partial r} + v_z \dfrac{\partial v_r}{\partial z} = 0, \\ \dfrac{v_\theta v_r}{r} + v_r \dfrac{\partial v_\theta}{\partial r} + v_z \dfrac{\partial v_\theta}{\partial z} = 0, \\ \dfrac{\partial (r v_r)}{\partial r} + \dfrac{\partial (r v_z)}{\partial z} = 0 \end{cases} \tag{4.1}$$

(these equations are fulfilled for almost all $x \in \mathcal{D}$).

We have the following integral estimates: $\mathbf{v} \in W^{1,2}_{\mathrm{loc}}(\mathcal{D})$,

$$\int_{\mathcal{D}} r |\nabla \mathbf{v}(r, z)|^2 \, dr dz < \infty. \tag{4.2}$$

$$\int_{\mathcal{D}} r |\mathbf{v}(r, z)|^6 \, dr dz < \infty. \tag{4.3}$$

Also, the inclusions $\nabla p \in L^{3/2}(\Omega)$, $p \in L^3(\Omega)$ can be rewritten in the following two-dimensional form:

$$\int_{\mathcal{D}} r|\nabla p(r,z)|^{3/2}\,drdz < \infty, \tag{4.4}$$

$$\int_{\mathcal{D}} r|p(r,z)|^{3}\,drdz < \infty. \tag{4.5}$$

The next statement was proved in [14, Lemma 4] and in [2, Theorem 2.2].

Theorem 4.1 *Let the conditions* (E) *be fulfilled. Then*

$$\forall j \in \{1,\dots,N\}\ \exists\, \widehat{p}_j \in \mathbb{R}: \quad p(x) \equiv \widehat{p}_j \quad \textit{for } \mathfrak{H}^2 - \textit{almost all } x \in \Gamma_j. \tag{4.6}$$

In particular, by axial symmetry,

$$p(x) \equiv \widehat{p}_j \quad \textit{for } \mathfrak{H}^1 - \textit{almost all } x \in \breve{\Gamma}_j. \tag{4.7}$$

We shall prove (see Corollary 4.1), that in the axially symmetric case $\widehat{p}_1 = \dots = \widehat{p}_{M'} = 0$, i.e., these values coincide with the constant limit of the pressure function p at infinity. To formulate this result, we need some preparation. Below without loss of generality we assume that the functions $\mathbf{v}$, p are extended to the whole half-plane P_+ as follows:

$$\mathbf{v}(x) := 0, \quad x \in P_+ \setminus \mathcal{D}, \tag{4.8}$$

$$p(x) := \widehat{p}_j, \ \ x \in P_+ \cap \bar{\mathcal{D}}_j, \ \ j = 1,\dots,N. \tag{4.9}$$

Obviously, the extended functions inherit the properties of the previous ones. Namely, $\mathbf{v} \in W^{1,2}_{\mathrm{loc}}(P_+)$, $p \in W^{1,3/2}_{\mathrm{loc}}(P_+)$, and the Euler equation (4.1) are fulfilled almost everywhere in P_+. Of course, for the corresponding axial-symmetric functions of three variables we have $\mathbf{v} \in H(\mathbb{R}^3)$, $p \in D^{1,3/2}(\mathbb{R}^3)$, and the Euler equation (3.8) are fulfilled almost everywhere in $\mathbb{R}^3$.

Lemma 4.1 *For almost all* $r_0 > 0$

$$|p(r_0,z)| + |\mathbf{v}(r_0,z)| \to 0 \quad \textit{as } |z| \to \infty. \tag{4.10}$$

Proof This Lemma follows from the fact that for almost all straight lines $L_{r_0} = \{(r,z) \in \mathbb{R}^2 : r = r_0\}$ we have the inclusion $|p(r_0,\cdot)| + |\mathbf{v}(r_0,\cdot)|^2 \in L^3(\mathbb{R}) \cap D^{1,3/2}(\mathbb{R})$, see (4.2)-(4.5). □

The main result of this section is a weak version of Bernoulli Law for the Sobolev solution $(\mathbf{v}, p)$ of Euler equation (4.1) (see Theorem 4.2 below). To formulate and to prove it, we need some preparation.

The last equality in (4.1) (which is fulfilled, after the above extension agreement, see (4.8)–(4.9), in the whole half-plane P_+) implies the existence of a stream function $\psi \in W^{2,2}_{\rm loc}(P_+)$ such that

$$\frac{\partial\psi}{\partial r} = -r v_z, \quad \frac{\partial\psi}{\partial z} = r v_r. \tag{4.11}$$

By (4.11), formula (4.3) can be rewritten in the following form:

$$\int\limits_{P_+} \frac{|\nabla\psi(r,z)|^6}{r^5}\, drdz < \infty. \tag{4.12}$$

By Sobolev Embedding Theorem, $\psi \in C(P_+)$ (recall that P_+ is an open half-plane, so here we do not assert the continuity at the points of singularity axis O_z). By virtue of (4.8), we have $\nabla\psi(x) = 0$ for almost all $x \in \mathcal{D}_j$. Then

$$\forall j \in \{1,\ldots,N\}\ \exists \xi_j \in \mathbb{R}: \quad \psi(x) \equiv \xi_j \quad \forall x \in P_+ \cap \bar{\mathcal{D}}_j. \tag{4.13}$$

Denote by $\Phi = p + \dfrac{|\mathbf{v}|^2}{2}$ the total head pressure corresponding to the solution $(\mathbf{v}, p)$. From (4.2)–(4.5) we get

$$\int\limits_{P_+} r|\Phi(r,z)|^3\, drdz + \int\limits_{P_+} r|\nabla\Phi(r,z)|^{3/2}\, drdz < \infty. \tag{4.14}$$

By direct calculations one easily gets the identity

$$v_r\Phi_r + v_z\Phi_z = 0 \tag{4.15}$$

for almost all $x \in P_+$. Identities (4.8)–(4.9) mean that

$$\Phi(x) \equiv \widehat{p}_j \quad \forall x \in P_+ \cap \bar{\mathcal{D}}_j, \quad j = 1,\ldots,N. \tag{4.16}$$

Theorem 4.2 (Bernoulli Law for Sobolev solutions) *Let the conditions* (E) *be valid. Then there exists a set $A_{\mathbf{v}} \subset P_+$ with $\mathfrak{H}^1(A_{\mathbf{v}}) = 0$, such that for any compact connected*[10] *set $K \subset P_+$ the following property holds : if*

$$\psi\big|_K = \text{const}, \tag{4.17}$$

then

$$\Phi(x_1) = \Phi(x_2) \quad \textit{for all } x_1, x_2 \in K \setminus A_{\mathbf{v}}. \tag{4.18}$$

Theorem 4.2 was obtained for bounded plane domains in [16, Theorem 1] (see also [17] for detailed proof). For the axially symmetric bounded domains the result was proved in [19, Theorem 3.3]. The proof for exterior axially symmetric domains

[10] We understand the connectedness in the sense of general topology.

is similar. To prove Theorem 4.2, we have to overcome two obstacles. First difficulty is the lack of the classical regularity, and here the results of [6] have a decisive role (according to these results, almost all level sets of plane $W^{2,1}$-functions are C^1-curves, see Sect. 2.3). The second obstacle is the set where $\nabla\psi(x) = 0 \neq \nabla\Phi(x)$, i.e., where $v_r(x) = v_z(x) = 0$, but $v_\theta(x) \neq 0$. Namely, if we do not assume the boundary conditions (3.8_3), then in general even in smooth case (4.17) does not imply (4.18). For example, if $v_r = v_z = 0$ in the whole domain, $v_\theta = r$, then $\psi \equiv$ const on the whole domain, while $p = \frac{r^2}{2}$ and $\Phi = r^2 \neq$ const. Without boundary assumptions one can prove only the assertion similar to Lemma 4.5 (see below). But Lemma 4.5 together with boundary conditions (3.8_3) imply Theorem 4.2.

First, we prove some auxiliary results.

Lemma 4.2 *Let the conditions* (E) *be fulfilled. Then*

$$p \in W^{2,1}_{\mathrm{loc}}(P_+). \tag{4.19}$$

Moreover, for any $\varepsilon > 0$ *the inclusion*

$$p \in D^{2,1}(P_\varepsilon) \tag{4.20}$$

holds, where $P_\varepsilon = \{(r, z) \in P_+ : r > \varepsilon\}$.

Proof Clearly, $p \in D^{1,3/2}(\mathbb{R}^3)$ is the weak solution to the Poisson equation

$$\Delta p = -\nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top \quad \text{in } \mathbb{R}^3 \tag{4.21}$$

(recall that after our agreement about extension of $\mathbf{v}$ and p, see (4.8)–(4.9), the Euler equation (3.8) are fulfilled in the whole $\mathbb{R}^3$). Let

$$G(x) = \frac{1}{4\pi} \int\limits_{\Omega} \frac{(\nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top)(y)}{|x-y|} dy.$$

By the results of [9], $\nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^3)$. Hence by Calderón–Zygmund theorem for Hardy's spaces [40] $G \in D^{2,1}(\mathbb{R}^3) \cap D^{1,3/2}(\mathbb{R}^3)$. Consider the function $p_* = p - G$. By construction $p_* \in D^{1,3/2}(\mathbb{R}^3)$ and $\Delta p_* = 0$ in $\mathbb{R}^3$. In particular, $\nabla p_* \in L^{3/2}(\mathbb{R}^3)$ is a harmonic (in the sense of distributions) function. From the mean-value property it follows that $p_* \equiv$ const. Consequently, $p \in D^{2,1}(\mathbb{R}^3)$. □

From inclusion (4.19) it follows that $\dfrac{\partial^2 p}{\partial r \partial z} \equiv \dfrac{\partial^2 p}{\partial z \partial r}$ for almost all $x \in P_+$. Denote $Z = \{x \in P_+ : v_r(x) = v_z(x) = 0\}$. Equation (4.1) yield the equality

$$\frac{\partial p}{\partial z} = 0, \quad \frac{\partial p}{\partial r} = \frac{(v_\theta)^2}{r} \quad \text{for almost all } x \in Z,$$

and, using identity $\frac{\partial^2 p}{\partial r \partial z} \equiv \frac{\partial^2 p}{\partial z \partial r}$ and Eq. (4.1), it is easy to deduce that

$$\frac{\partial \Phi}{\partial z}(x) = 0 \quad \text{for almost all } x \in P_+ \text{ such that } v_r(x) = v_z(x) = 0. \tag{4.22}$$

In the assertion below we collect the basic properties of Sobolev functions applied to $\mathbf{v}$ and Φ. Here and in the next statements we assume that the conditions (E) are fulfilled and that all functions are extended to the whole half-plane P_+ [see (4.8)–(4.9), (4.13)–(4.16)].

Theorem 4.3 *There exists a set $A_\mathbf{v} \subset P_+$ such that:*

(i) $\mathfrak{H}^1(A_\mathbf{v}) = 0$;

(ii) *For all $x \in P_+ \setminus A_\mathbf{v}$*

$$\lim_{R\to 0} \fint_{B_R(x)} |\mathbf{v}(y) - \mathbf{v}(x)|^2 dy = \lim_{R\to 0} \fint_{B_R(x)} |\Phi(y) - \Phi(x)|^{3/2} dy = 0,$$
$$\lim_{R\to 0} \frac{1}{R} \int_{B_R(x)} |\nabla \Phi(y)|^{3/2} dy = 0.$$

Moreover, the function ψ is differentiable at $x \in P_+ \setminus A_\mathbf{v}$ and $\nabla \psi(x) = (-r v_z(x), r v_r(x))$;

(iii) *For all $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^2$ such that $\mathfrak{H}^1_\infty(U) < \varepsilon$, $A_\mathbf{v} \subset U$, and the functions $\mathbf{v}$, Φ are continuous in $P_+ \setminus U$;*

(iv) *For each $x_0 = (r_0, z_0) \in P_+ \setminus A_\mathbf{v}$ and for any $\varepsilon > 0$ the convergence*

$$\lim_{\rho \to 0+} \frac{1}{2\rho} \mathfrak{H}^1(E(x_0, \varepsilon, \rho)) \to 1 \tag{4.23}$$

holds, where

$$E(x_0, \varepsilon, \rho) := \Bigg\{ t \in (-\rho, \rho) : \int_{r_0-\rho}^{r_0+\rho} \left| \frac{\partial \Phi}{\partial r}(r, z_0 + t) \right| dr + \int_{z_0-\rho}^{z_0+\rho} \left| \frac{\partial \Phi}{\partial z}(r_0 + t, z) \right| dz$$
$$+ \sup_{r \in [r_0-\rho, r_0+\rho]} |\Phi(r, z_0 + t) - \Phi(x_0)|$$
$$+ \sup_{z \in [z_0-\rho, z_0+\rho]} |\Phi(r_0 + t, z) - \Phi(x_0)| < \varepsilon \Bigg\}.$$

(v) *Take any function $g \in C^1(\mathbb{R}^2)$ and a closed set $F \subset P_+$ such that $\nabla g \neq 0$ on F. Then for almost all $y \in g(F)$ and for all connected components K of the set $F \cap g^{-1}(y)$ the equality $K \cap A_\mathbf{v} = \emptyset$ holds, the restriction $\Phi|_K$ is an absolutely continuous function, and formulas (4.1), (4.15) are fulfilled $\mathfrak{H}^1$-almost everywhere on K.*

The above theorem coincides with Theorem 3.7 from [19] (see also [17, Theorem 3.1] for more detailed explanations).

Really most of these properties are classical and could be found, e.g., in [11]. Note, that the property (iv) follows directly from the second convergence formula in (ii). The last property (v) follows (by coordinate transformation, cf. [28, §1.1.7]) from the well-known fact that any function $f \in W^{1,1}$ is absolutely continuous along almost all coordinate lines. The same fact together with (4.22), (4.15) imply

Lemma 4.3 *Denote $L_{r_0} = \{(r, z) \in P_+ : r = r_0\}$. Then for almost all $r_0 > 0$ the equality $L_{r_0} \cap A_{\mathbf{v}} = \emptyset$ holds. Moreover, $p(r_0, \cdot)$, $\mathbf{v}(r_0, \cdot)$ are absolutely continuous functions (locally) and*

$$\frac{\partial \Phi}{\partial z}(r_0, z) = 0 \quad \textit{for almost all } z \in \mathbb{R} \textit{ such that } v_r(r_0, z) = 0. \tag{4.24}$$

We need also the following assertion which was proved in [19] (see Lemma 3.6).

Lemma 4.4 *Suppose for $r_0 > 0$ the assertion of Lemma 4.3 is fulfilled, i.e., the equality $L_{r_0} \cap A_{\mathbf{v}} = \emptyset$ holds, $p(r_0, \cdot)$, $\mathbf{v}(r_0, \cdot)$ are absolutely continuous functions, and formula (4.24) is valid. Let $F \subset \mathbb{R}$ be a compact set such that*

$$\psi(r_0, z) \equiv \mathrm{const} \quad \textit{for all } z \in F \tag{4.25}$$

and

$$\Phi(r_0, \alpha) = \Phi(r_0, \beta) \quad \textit{for any interval } (\alpha, \beta) \textit{ adjoining } F \tag{4.26}$$

(recall that (α, β) is called an interval adjoining F if $\alpha, \beta \in F$ and $(\alpha, \beta) \cap F = \emptyset$). Then

$$\Phi(r_0, z) \equiv \mathrm{const} \quad \textit{for all } z \in F. \tag{4.27}$$

The next lemma plays the key role in the proof of the Bernoulli Law.

Lemma 4.5 *Let $P \subset P_+$ be a rectangle $P := \{(r, z) : r \in [r_1, r_2],\ z \in [z_1, z_2]\}$, $r_1 > 0$, and $K \subset P$ be a connected component of the set $\{x \in P : \psi(x) = y_0\}$, where $y_0 \in \mathbb{R}$. Then there exists an absolute continuous function $f : [r_1, r_2] \to \mathbb{R}$ such that*

$$\Phi(r, z) = f(r) \quad \textit{for all } (r, z) \in K \setminus A_{\mathbf{v}}. \tag{4.28}$$

Moreover, for each $r_0 \in [r_1, r_2]$ if $f(r) \not\equiv \mathrm{const}$ locally in any neighborhood of r_0, then

$$(r_0, z) \in K \quad \forall z \in [z_1, z_2]. \tag{4.29}$$

In other words, if for $r_0 \in [r_1, r_2]$ the inclusion (4.29) is not valid, then there exists a neighborhood $U(r_0)$ such that $f|_{U(r_0) \cap [r_1, r_2]} = \mathrm{const}$.

Remark 4.1 Notice that the above lemma is valid without assumption (3.8_3) (that $\mathbf{v}|_{\partial\Omega} = 0$). It is enough to suppose that $\mathbf{v} \in W^{1,2}(P)$, $p \in W^{1,3/2}(P)$ satisfy Euler system (4.1) almost everywhere in P.

Proof of Lemma 4.5 splits into six steps (see below). Steps 1–4 coincides (in essential) with the corresponding steps (having the same numbers) from the proof of Theorem 3.3 in [19], where we assumed additionally that

$$\mathbf{v}(x) = 0 \quad \forall x \in \partial P \setminus \{(r_1, z) : z \in (z_1, z_2)\}. \tag{4.30}$$

The new arguments appear in Steps 5 and 6, where we cannot simply repeat the arguments from [19] because of absence of boundary conditions (4.30).

STEP 1. Put $P^\circ = \text{Int}\, P = (r_1, r_2) \times (z_1, z_2)$. Let U_i be the connected components of the open set $U = P^\circ \setminus K$. In [19, Step 1 of the proof of Theorem 3.3] it was proved that for any U_i there exists a number β_i such that

$$\Phi(x) = \beta_i \quad \text{for all } x \in K \cap \partial U_i \setminus A_{\mathbf{v}}. \tag{4.31}$$

STEP 2. Using (4.31) and Lemma 4.4, in [19, Step 2 of the proof of Theorem 3.3] it was proved that for almost all $r \in (r_1, r_2)$ the identities

$$\Phi(r, z') = \Phi(r, z'') \quad \forall (r, z'), (r, z'') \in K \tag{4.32}$$

hold.

STEP 3. Denote by $\text{Proj}_r\, E$ the projection of the set E onto the r-axis. From (4.31), (4.32) it follows that

$$\text{Proj}_r\, U_i \cap \text{Proj}_r\, U_j \neq \emptyset \Rightarrow \beta_i = \beta_j. \tag{4.33}$$

Denote $V = \text{Proj}_r(P^\circ \setminus K)$,[11] and let (u_k, v_k) be the family of intervals adjoining the compact set $[r_1, r_2] \setminus V$ (in other words, (u_k, v_k) are the maximal intervals of the open set $V = \bigcup_i \text{Proj}_r\, U_i$). Then, repeating the assertion (4.33), we have

$$\text{Proj}_r\, U_i \subset (u_k, v_k) \supset \text{Proj}_r\, U_j \Rightarrow \beta_i = \beta_j. \tag{4.34}$$

In other words, for any interval (u_k, v_k) adjoining the set $[r_1, r_2] \setminus V$ there exists a constant $\beta(k)$ such that

$$\text{Proj}_r\, U_i \subset (u_k, v_k) \Rightarrow \big(\Phi(x) = \beta(k) \quad \text{for all } x \in K \cap \partial U_i \setminus A_{\mathbf{v}}\big). \tag{4.35}$$

By construction, for any $r \in (u_k, v_k)$ there exists $z \in (z_1, z_2)$ such that $(r, z) \in U_i$ for some component U_i with $\text{Proj}_r\, U_i \subset (u_k, v_k)$. From this fact and from the identities (4.35) and (4.32) it follows that

$$\Phi(r, z) = \beta(k) \quad \text{for almost all } r \in (u_k, v_k) \text{ and for any } (r, z) \in K. \tag{4.36}$$

[11] By construction, the set $[r_1, r_2] \setminus V$ coincides with the set of values $r \in [r_1, r_2]$ such that the whole segment $\{r\} \times [z_1, z_2]$ lies in K. The set V can be empty, but it can also coincide with the whole $[r_1, r_2]$. For example, if K is a circle, then $V = [r_1, r_2]$. Further, $V = \emptyset$ iff $K = P$.

Step 4. It was proved in [19, Step 4 of the proof of Theorem 3.3] that the assertions (4.35) and (4.36) imply

$$r \in [u_k, v_k] \Rightarrow \Phi(r, z) = \beta(k) \quad \forall (r, z) \in K \setminus A_{\mathbf{v}}. \tag{4.37}$$

Note, that formulas (4.35) and (4.36) in our proof correspond to the formulas (3.38) and (3.39) from [19]. Further, the role of the rectangle P from [19, Step 4 of the proof of Theorem 3.3] is played in our proof by the rectangle $[u_k, v_k] \times [z_1, z_2]$.

Step 5. By construction, for each $r \in [r_1, r_2] \setminus V$ the whole segment $\{r\} \times [z_1, z_2]$ is contained in K. On this step we shall prove that $\Phi(x)$ is constant along each of these segments, i.e.,

$$\forall r \in [r_1, r_2] \setminus V \quad \exists \beta(r) \quad \Phi(r, z) = \beta(r) \quad \forall (r, z) \in P \setminus A_{\mathbf{v}}. \tag{4.38}$$

Indeed, if r is an endpoint of an adjoining interval, i.e., if $r = u_k$ or $r = v_k$, then (4.38) immediately follows from (4.37). Further, by Step 2 there exists a set $\tilde{E} \subset [r_1, r_2] \setminus V$ such that $\mathfrak{H}^1([r_1, r_2] \setminus (V \cup \tilde{E})) = 0$ and the assertion (4.38) is true for any $r \in \tilde{E}$. But for any $r \in [r_1, r_2] \setminus V$ there exists a sequence of points $r_\mu \to r$, $\mu = 3, 4, \dots$, such that each r_μ is an endpoint of some adjoining interval (u_k, v_k) or $r_\mu \in \tilde{E}$, i.e., the assertion (4.38) is true for each r_μ. Finally, the assertion (4.38) follows for $r = \lim_{\mu \to \infty} r_\mu$ from the continuity properties of $\Phi(\cdot)$ [see Theorem 4.3 (ii)–(iv)].

Step 6. Define the target function $f : [r_1, r_2] \to \mathbb{R}$ as follows: $f(r) = \beta(k)$ for $r \in (u_k, v_k)$ (see Step 4) and $f(r) = \beta(r)$ for $r \in [r_1, r_2] \setminus V$ (see Step 5). By construction [see (4.37) and (4.38)] we have the identity (4.28). Also by construction,

$$f(r) \equiv \text{const} \quad \text{on each adjoining interval } (u_k, v_k). \tag{4.39}$$

Take an arbitrary $z_0 \in (z_1, z_2)$ such that the segment $[r_1, r_2] \times \{z_0\}$ does not contain points from $A_{\mathbf{v}}$ and $\Phi(\cdot, z_0)$ in an absolutely continuous function, i.e., $\Phi(\cdot, z_0) \in W^{1,1}([r_1, r_2])$. Then by construction, $\Phi(r, z_0) = f(r)$ for each $r \in [r_1, r_2] \setminus V$, in particular, $f(r)$ coincides with an absolute continuous function on the set $[r_1, r_2] \setminus V$. The last fact together with (4.39) implies the absolute continuity of $f(\cdot)$ on the whole interval $[r_1, r_2]$. The Lemma is proved completely. □

Proof of Theorem 4.2 Suppose the conditions (E) are fulfilled and $K \subset P_+$ is a compact connected set, $\psi|_K \equiv \text{const}$. Take the set $A_{\mathbf{v}}$ from Theorem 4.3. Let P be a rectangle $P := \{(r, z) : r \in [r_1, r_2],\ z \in [z_1, z_2]\}$, $r_1 > 0$, such that $K \subset P$, and

$$\operatorname{Proj}_r K = [r_1, r_2]. \tag{4.40}$$

We may assume without loss of generality that K is a connected component of the set $\{x \in P : \psi(x) = y_0\}$, where $y_0 \in \mathbb{R}$. Apply Lemma 4.5 to this situation, and take the corresponding function $f(r)$. Then the target identity (4.18) is equivalent to the identity $f(r) \equiv \text{const}$. We prove this fact getting a contradiction. Suppose the last identity is false. Consider the nonempty compact set $\mathcal{R} = \{r_0 \in [r_1, r_2] : f(r) \neq \text{const}$ in any neighborhood of $r_0\}$. By assumption, $\mathcal{R} \neq \emptyset$, thus $f(\mathcal{R}) = f([r_1, r_2])$ is an interval of positive length[12]. Since f is an absolute continuous function, it has Luzin N-property, i.e., it maps sets of measure zero into the sets of measure zero. In particular, the measure of $\mathcal{R}$ must be positive, moreover, there exists a set of positive measure $\mathcal{R}' \subset \mathcal{R}$ such that $f(r) \neq 0$ for each $r \in \mathcal{R}'$. So, by Lemma 4.1 there exists $r_0 \in \mathcal{R}'$ such that $L_{r_0} \cap A_{\mathbf{v}} = \emptyset$,

$$|p(r_0, z)| + |\mathbf{v}(r_0, z)| \to 0 \quad \text{as } |z| \to \infty,$$

and

$$f(r_0) \neq 0. \tag{4.41}$$

Take a sequence of rectangles $P_m = [r_1, r_2] \times [z_1^m, z_2^m]$ such that $z_1^m \to -\infty$, $z_2^m \to +\infty$. Let K_m be a connected component of the level set $\{x \in P_m : \psi(x) = y_0\}$ containing K. From (4.40) it follows that

$$\mathrm{Proj}_r\, K_m = [r_1, r_2]. \tag{4.42}$$

Apply Lemma 4.5 to K_m, P_m, and take the corresponding function $f_m(r)$. By construction [see, e.g., (4.40), (4.42)] functions f_m do not depend on m, i.e., $f(r) = f_m(r)$ for each $r \in [r_1, r_2]$ and for all $m = 1, 2, \ldots$. Then by the second assertion of Lemma 4.5, the whole segment $\{r_0\} \times [z_1^m, z_2^m]$ is contained in K_m for each m, and $\Phi(r_0, z) \equiv f(r_0)$ for all $z \in [z_1^m, z_2^m]$. Passing to a limit as $m \to \infty$, we get $\Phi(r_0, z) \equiv f(r_0) \neq 0$ for all $z \in \mathbb{R}$. The last identity contradicts convergence (4.10). The Theorem is proved. □

For $\varepsilon > 0$ and $R > 0$ denote by $S_{\varepsilon,R}$ the set $S_{\varepsilon,R} = \{(r, z) \in P_+ : r \geq \varepsilon,\ r^2 + z^2 = R^2\}$.

Lemma 4.6 *For any $\varepsilon > 0$ there exists a sequence $\rho_j > 0$, $\rho_j \to +\infty$, such that $S_{\varepsilon,\rho_j} \cap A_{\mathbf{v}} = \emptyset$ and*

$$\sup_{x \in S_{\varepsilon,\rho_j}} |\Phi(x)| \to 0 \quad \text{as } j \to \infty. \tag{4.43}$$

Proof Fix $\varepsilon > 0$. From the inclusion $\nabla\Phi \in L^{3/2}(\mathbb{R}^3)$ it follows that there exists a sequence $\rho_j \to \infty$ such that $S_{\varepsilon,\rho_j} \cap A_{\mathbf{v}} = \emptyset$ and

$$\int\limits_{S_{\varepsilon,\rho_j}} |\nabla\Phi(x)|^{3/2}\, d\mathfrak{H}^1 \leq \frac{1}{\rho_j} \quad \text{as } j \to \infty.$$

By Hölder inequality,

$$\int\limits_{S_{\varepsilon,\rho_j}} |\nabla\Phi(x)|\, d\mathfrak{H}^1 \leq \Big(\int\limits_{S_{\varepsilon,\rho_j}} |\nabla\Phi|^{3/2}\, d\mathfrak{H}^1 \Big)^{2/3} (\pi\rho_j)^{1/3} \leq \big(\frac{\pi}{\rho_j}\big)^{1/3},$$

[12] Notice that the set $\mathcal{R}$ itself may have empty interior, for example, if f is a Cantor staircase type function, i.e., if f is constant on each interval I_j, and the union of these intervals is everywhere dense set, then $\mathcal{R}$ coincides with corresponding Cantor set and has empty interior.

consequently,

$$\operatorname{diam} \Phi(S_{\varepsilon,\rho_j}) \to 0 \quad \text{as } j \to \infty,$$

that in virtue of Lemma 4.1 implies the assertion of Lemma 4.6. □

One of the main results of this section is the following.

Theorem 4.4 *Assume that conditions* (E) *are satisfied. Let* K_j *be a sequence of continuums with the following properties:* $K_j \subset \bar{\mathcal{D}}\setminus O_z$, $\psi|_{K_j} = \text{const}$, *and* $\lim_{j\to\infty} \inf_{(r,z)\in K_j} r = 0$, $\underline{\lim}_{j\to\infty} \sup_{(r,z)\in K_j} r > 0$. *Then* $\Phi(K_j) \to 0$ *as* $j \to \infty$. *Here we denote by* $\Phi(K_j)$ *the corresponding constant* $c_j \in \mathbb{R}$ *such that* $\Phi(x) = c_j$ *for all* $x \in K_j \setminus A_{\mathbf{v}}$ *(see Theorem 4.2).*

Proof Let the assumptions of the Theorem be fulfilled. We shall use the Bernoulli law and the fact that the axis O_z is "almost" a streamline. More precisely, O_z is a singularity line for $\mathbf{v}$, ψ, p, but it can be accurately approximated by usual streamline (on which $\Phi = \text{const}$). Recall that the functions $\mathbf{v}$, p, Φ, ψ are extended to the whole half-plane P_+ (see (4.8), (4.9), (4.16), (4.13)), and the assertion of the Bernoulli Law (Theorem 4.2) is true for these extended functions.

By Lemma 4.1, there exists a constant $r_0 > 0$ such that the straight line L_{r_0} satisfies the assertion of Lemma 4.3 and

$$\begin{aligned}
&L_{r_0} \cap A_{\mathbf{v}} = \emptyset, \quad p(r_0, z) \to 0, \quad |\mathbf{v}(r_0, z)| \to 0 \text{ as } |z| \to \infty, \\
&\mathbf{v}(r_0, \cdot) \in L^6(\mathbb{R}), \quad \mathbf{v}(r_0, \cdot) \in D^{1,2}(\mathbb{R}) \subset C(\mathbb{R}), \\
&p(r_0, \cdot) \in D^{1,3/2}(\mathbb{R}) \subset C(\mathbb{R}), \\
&r_0 < \underline{\lim}_{j\to\infty} \sup_{(r,z)\in K_j} r.
\end{aligned} \tag{4.44}$$

In particular, the function $\Phi(r_0, \cdot)$ is continuous on $\mathbb{R}$. In virtue of the last inequality, we can assume without loss of generality that

$$K_j \cap L_{r_0} \neq \emptyset \quad \forall j \in \mathbb{N}. \tag{4.45}$$

Suppose that the assertion of Theorem 4.4 is false, i.e.,

$$\Phi(K_j) \to p_* \neq 0 \quad \text{as } j \to \infty. \tag{4.46}$$

Then by (4.44_1) there exists a constant $C_1 > 0$ such that

$$\sup_{j\in\mathbb{N},\ (r_0,z)\in K_j} |z| \leq C_1, \tag{4.47}$$

consequently,

$$K_j \cap \{(r_0, z) : z \in [-C_1, C_1]\} \neq \emptyset \quad \forall j \in \mathbb{N}. \tag{4.48}$$

Take a sequence of numbers $\tilde{z}_i \in (C_1, \infty)$ with $\tilde{z}_i < \tilde{z}_{i+1} \to +\infty$ as $i \to \infty$. Since, by estimate (4.12),

$$\int_{-\tilde{z}_i}^{\tilde{z}_i} \int_0^1 \frac{|\nabla\psi(r,z)|^6}{r^5} dr\, dz < \infty,$$

we conclude that there exists a sequence of numbers $r_{ik} \to 0+$ such that

$$\int_{-\tilde{z}_i}^{\tilde{z}_i} \frac{|\nabla\psi(r_{ik},z)|^6}{r_{ik}^5} dz < \frac{1}{r_{ik}},$$

that is,

$$\int_{-\tilde{z}_i}^{\tilde{z}_i} |\nabla\psi(r_{ik},z)|^6\, dz < r_{ik}^4 \to 0 \quad \text{as } k \to \infty.$$

In particular, by Hölder inequality

$$\int_{-\tilde{z}_i}^{\tilde{z}_i} |\nabla\psi(r_{ik},z)|\, dz < \frac{1}{i}$$

for sufficiently large k. This implies the existence of sequences $r_i = r_{ik_i} \in (0, r_0)$ with the following properties

$$r_i > r_{i+1} \to 0 \quad \text{as } i \to \infty, \tag{4.49}$$

$$\int_{-\tilde{z}_i}^{\tilde{z}_i} |\nabla\psi(r_i,z)|\, dz \to 0 \quad \text{as } i \to \infty. \tag{4.50}$$

Further, by the same estimate (4.12) and similar arguments, there exists a sequence $z_i \in [\tilde{z}_i - 1, \tilde{z}_i]$ such that

$$\int_{r_i}^{r_0} |\nabla\psi(r,z_i)|\, dr + \int_{r_i}^{r_0} |\nabla\psi(r,-z_i)|\, dr \to 0 \quad \text{as } i \to \infty. \tag{4.51}$$

Thus,

$$\int_{-z_i}^{z_i} |\nabla\psi(r_i,z)|\, dz + \int_{r_i}^{r_0} |\nabla\psi(r,z_i)|\, dr + \int_{r_i}^{r_0} |\nabla\psi(r,-z_i)|\, dr \to 0 \tag{4.52}$$

as $i \to \infty$. Let P_i be the rectangle $P_i = [r_i, r_0] \times [-z_i, z_i]$. Denote by A_i, B_i the points $A_i = (r_0, -z_i) \in L_{r_0} \cap \partial P_i$, $B_i = (r_0, z_i) \in L_{r_0} \cap \partial P_i$. Denote by $[A_i, B_i] = \{(r_0, z) : z \in [-z_i, z_i]\}$ the closed segment and by $]A_i, B_i[= [A_i, B_i] \setminus \{A_i, B_i\}$ the corresponding open one. Put $T_i = (\partial P_i) \setminus]A_i, B_i[$. By construction [see (4.52)]

$$\operatorname{diam} \psi(T_i) \to 0 \quad \text{as } i \to \infty. \tag{4.53}$$

Take $R_* > \sqrt{r_0^2 + C_1^2}$ such that

$$S_{r_0, R_*} \cap K_j = \emptyset \quad \forall j \in \mathbb{N}. \tag{4.54}$$

The existence of such R_* follows from Lemma 4.6 and (4.46). Indeed, Lemma 4.6 gives us $\sup |\Phi(S_{r_0, R_k})| \to 0$ for some $R_k \to \infty$, and (4.46) means that $\Phi(K_j) \to p_* \neq 0$.

For $x \in P_i$ denote by K_x^i the connected component of the level set $\{y \in P_i : \psi(y) = \psi(x)\}$ containing x. Put $F_i = \{z \in [-z_i, z_i] : K^i_{(r_0, z)} \cap T_i \neq \emptyset\}$. Then for each $i \in \mathbb{N}$ there exists an index $j(i) \geq i$ such that

$$\forall j \geq j(i) \quad \{(r_0, z) \in K_j : z \in F_i\} \neq \emptyset. \tag{4.55}$$

Indeed, the connected set K_j intersects L_{r_0} and L_{r_i} for sufficiently large j, moreover, by (4.47) $K_j \cap L_{r_0} \subset \{r_0\} \times [-C_1, C_1] \subset [A_i, B_i] \subset P_i$. From the last assertions and (4.54), by obvious topological reasons, we derive the existence of a connected set $K_j^i \subset K_j \cap P_i$ which intersect both lines L_{r_0} and L_{r_i}. This means validity of (4.55).

Now take a point $z^i_{j(i)} \in F_i$ such that $(r_0, z^i_{j(i)}) \in K_j$. Since the sequence of points $z^i_{j(i)}$ is bounded [see (4.47)], we may assume without loss of generality that

$$z^i_{j(i)} \to z_* \quad \text{as } i \to \infty.$$

Then $\psi(K_{j(i)}) \to \psi(r_0, z_*)$ as $i \to \infty$. Denote $\xi_* = \psi(r_0, z_*)$. Since by construction $K_{j(i)} \cap T_i \neq \emptyset$ and the convergence (4.53) holds, we have

$$\sup_{x \in T_i} |\psi(x) - \xi_*| \to 0 \quad \text{as } i \to \infty.$$

By construction we have also the following properties of sets F_i.

(I$_\sim$) F_i is a compact set, $\pm z_i \in F_i$.

Indeed, the set $F_i \subset [z_i, z_0]$ is closed because of the following reason: if $F_i \ni z_\mu \to z$, then there exists a subsequence z_{μ_k} such that the components $K^i_{(r_0, z_{\mu_k})}$ converge with respect to the Hausdorff distance[13] to some set K. Of course, $K \ni (r_0, z)$ is a compact

[13] The Hausdorff distance d_H between two compact sets $A, B \subset \mathbb{R}^n$ is defined as follows: $d_H(A, B) = \max\big(\sup_{a \in A} \operatorname{dist}(a, B), \sup_{b \in B} \operatorname{dist}(b, A)\big)$ (see, e.g., § 7.3.1 in [8]). By Blaschke selection theorem [ibid], for any uniformly bounded sequence of compact sets $A_i \subset \mathbb{R}^n$ there exists a subsequence A_{i_j} which converges to some compact set A_0 with respect to the Hausdorff distance. Of course, if all A_i are compact connected sets, then the limit set A_0 is also connected.

connected set, $\psi|_K = \text{const}$, and $K \cap T_i \neq \emptyset$ (since by construction $K^i_{(r_0,z_\mu)} \cap T_i \neq \emptyset$). Therefore, $K \subset K^i_{(r_0,z)}$ (see the definition of the sets K^i_x), hence $z \in F_i$.

Put $U_i =]-z_i, z_i[\setminus F_i$. For $z \in U_i$ let $(\alpha_i(z), \beta_i(z))$ be the maximal open interval from U_i containing z. Of course, $\alpha_i(z), \beta_i(z) \in F_i$. The next two properties are evident.

(II$\sim$) U_i is an open set, $U_i \subset U_{i+1}$.

(III$\sim$) $\sup\limits_{x\in T_i} |\psi(x) - \xi_*| \geq \sup\limits_{z\in F_i} |\psi(r_0, z) - \xi_*| \to 0$ as $i \to \infty$.

(IV$\sim$) $\forall z \in U_i$ $\exists$ a compact connected set $K \subset P_i$ such that $(r_0, \alpha_i(z)) \in K$, $(r_0, \beta_i(z)) \in K$ and $\psi|_K = \text{const}$.

Indeed, if the components $K^i_{(r_0,\alpha_i(z))}$, $K^i_{(r_0,\beta_i(z))}$ do not coincide, then, by results of [22], there exists a compact connected set $K' \subset P_i$, $\psi|_{K'} = \text{const}$, which separates them, i.e., points $(r_0, \alpha_i(z))$, $(r_0, \beta_i(z))$ lie in the different connected components of the set $P_i\setminus K'$. Then, by topological obviousness, $K' \cap T_i \neq \emptyset$ and $K' \cap \{(r_0, z) : z \in (\alpha_i(z), \beta_i(z))\} \neq \emptyset$. But the last formulas contradict the condition $(\alpha_i(z), \beta_i(z)) \cap F_i = \emptyset$ and the definition of F_i. Thus the property (IV$\sim$) is proved.

The property (IV$\sim$) together with the Bernoulli Law (Theorem 4.2) imply the following identity:

(V$\sim$) $\forall z \in U_i$ $\Phi(r_0, \alpha_i(z)) = \Phi(r_0, \beta_i(z))$.

Put $U = \bigcup_i U_i$, $F = \mathbb{R}\setminus U$. Then we have

(VI$\sim$) F is a closed set, $z_* \in F$, U is an open set.

For $z \in U$ put $\alpha(z) = \lim_{i\to\infty} \alpha_i(z)$, $\beta(z) = \lim_{i\to\infty} \beta_i(z)$. Notice that the limits exist since the functions $\alpha_i(z)$, $\beta_i(z)$ are monotone in virtue of (II$\sim$). Moreover, if $z < z_*$, then $\beta(z)$ is finite because of inequalities $\beta_i(z) \leq z_*$. Analogously, if $U \ni z > z_*$, then $\alpha(z) \in [z_*, z)$. By construction, $(\alpha(z), \beta(z)) \subset U$, and, if $\alpha(z)$ or $\beta(z)$ is finite, then it belongs to F.

From (III$\sim$), (V$\sim$) and continuity of ψ and $\Phi(r_0, \cdot)$ we have

(VII$\sim$) $\forall z \in F$ $\psi(r_0, z) = \xi_*$.

(VIII$\sim$) $\forall z \in U$ if both values $\alpha(z)$ and $\beta(z)$ are finite, then $\Phi(r_0, \alpha(z)) = \Phi(r_0, \beta(z))$.

Then Lemma 4.4 yields

$$\forall z \in F \quad \Phi(r_0, z) = \Phi(r_0, z_*), \tag{4.56}$$

and from (4.46), (4.56) and the choice of (r_0, z_*) we deduce that

$$\forall z \in F \quad \Phi(r_0, z) = p_*. \tag{4.57}$$

Now to finish the proof of the Theorem, i.e., to receive a desired contradiction, we need to deduce the identity

$$p_* = 0 \tag{4.58}$$

[it will contradict the assumption (4.46)]. For this purpose, consider two possible cases.

(i) Suppose that F is an unbounded set. Then by (4.44) $\lim_{F\ni z\to\pm\infty} \Phi(r_0, z) = 0$, and the target equality (4.58) follows from (4.57).

(ii) Suppose that F is bounded. Then there exists $z \in U$, $z < z_*$, such that $\alpha(z) = -\infty$. By definition, $\alpha_i(z) \to -\infty$ as $i \to \infty$. By (4.44), $\lim_{i\to\infty} \Phi(r_0, \alpha_i(z)) = 0$.

On the other hand, by (V$_\sim$) we have

$$\lim_{i\to\infty}\Phi(r_0,\alpha_i(z))=\lim_{i\to\infty}\Phi(r_0,\beta_i(z))=\Phi(r_0,\beta(z))=p_*$$

(the last two equalities follow from (4.57) and from the finiteness of $\beta(z)\in F\cap(-\infty,z_*])$. Thus, the equality (4.58) is proved, but it contradicts the assumption (4.46). The obtained contradiction finishes the proof of the Theorem. □

Corollary 4.1 *Assume that conditions* (E) *are satisfied. Then* $\Phi|_{\Gamma_j}\equiv 0$ *whenever* $\Gamma_j\cap O_z\neq\emptyset$*, i.e.,*

$$\widehat{p}_1=\cdots=\widehat{p}_{M'}=0,$$

where $\widehat{p}_j$ *are the constants from Theorem* 4.1.

5 Obtaining a contradiction

From now on we assume that assumptions (E-NS) (see Lemma 3.1) are satisfied. Our goal is to prove that they lead to a contradiction. This implies the validity of Theorem 1.1.

First, we introduce the main idea of the proof (which is taken from [20]) in a heuristic way. It is well known that every $\Phi_k=p_k+\frac{1}{2}|\mathbf{u}_k|^2$ satisfies the linear elliptic equation

$$\Delta\Phi_k=\omega_k^2+\frac{1}{\nu_k}\operatorname{div}(\Phi_k\mathbf{u}_k)-\frac{1}{\nu_k}\mathbf{f}_k\cdot\mathbf{u}_k, \tag{5.1}$$

where $\boldsymbol{\omega}_k=\operatorname{curl}\mathbf{u}_k$ and $\omega_k(x)=|\boldsymbol{\omega}_k(x)|$. If $\mathbf{f}_k=0$, then by Hopf's maximum principle, in a subdomain $\Omega'\Subset\Omega$ with C^2—smooth boundary $\partial\Omega'$ the maximum of Φ_k is attained at the boundary $\partial\Omega'$, and if $x_*\in\partial\Omega'$ is a maximum point, then the normal derivative of Φ_k at x_* is strictly positive. It is not sufficient to apply this property directly. Instead we will use some "integral analogs" that lead to a contradiction by using the Coarea formula (see Lemmas 5.7–7.1). For sufficiently large k we construct a set $E_k\subset\Omega$ (see the Proof of Lemma 5.9) consisting of level sets of Φ_k such that E_k separates the boundary components Γ_j where $\Phi\neq 0$ from the boundary components Γ_i where $\Phi=0$ and from infinity. On the one hand, the area of each of these level sets is bounded from below (since they separate the boundary components), and by the Coarea formula this implies the estimate from below for $\int_{E_k}|\nabla\Phi_k|$ (see the Proof of Lemma 5.9). On the other hand, elliptic equation (5.1) for Φ_k, the convergence $\mathbf{f}_k\to 0$, and boundary conditions (3.10$_3$) allow us to estimate $\int_{E_k}|\nabla\Phi_k|^2$ from above (see Lemma 5.7), and this asymptotically contradicts the previous estimate. (We use also isoperimetric inequality, see the Proof of Lemma 5.9, and some elementary Lemmas from real analysis, see "Appendix").

Recall that by assumptions (SO)

$$\begin{aligned}&\Gamma_j\cap O_z\neq\emptyset,\quad j=1,\ldots,M',\\&\Gamma_j\cap O_z=\emptyset,\quad j=M'+1,\ldots,N.\end{aligned} \tag{5.2}$$

Consider the constants $\widehat{p}_j$ from Theorem 4.1 (see also Corollary 4.1). We need the following fact.

Lemma 5.1 *The identity*

$$-\nu = \sum_{j=M'+1}^{N} \widehat{p}_j \mathcal{F}_j \tag{5.3}$$

holds.

Proof We calculate the integral $\int\limits_{\Omega} (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{A}\,dx$ in the equality (3.14) by using the Euler equation (3.8_1). In virtue of assumptions (E),

$$\|p\|_{L^3(\Omega)} + \|\nabla p\|_{L^{3/2}(\Omega)} < \infty, \tag{5.4}$$

and by Theorem 4.1 and Corollary 4.1 we have

$$p(x)|_{\Gamma_j} \equiv 0, \quad j = 1,\dots,M', \qquad p(x)|_{\Gamma_j} \equiv \widehat{p}_j, \quad j = M'+1,\dots,N.$$

From the inclusions (5.4) it is easy to deduce

$$\sup_{m\in\mathbb{N}} \int\limits_{|x|=m} |p|^2\,dS < \infty. \tag{5.5}$$

Indeed, for balls $B_m = B(0,m)$ we have the uniform estimates

$$\int\limits_{B_m\setminus\frac{1}{2}B_m} |p|^2\,dx \le \Bigg(\int\limits_{B_m\setminus\frac{1}{2}B_m} |p|^3\,dx\Bigg)^{\frac{2}{3}} \cdot \Big(\operatorname{meas}\big(B_m\setminus\frac{1}{2}B_m\big)\Big)^{\frac{1}{3}} \le C_1 m, \tag{5.6}$$

$$\int\limits_{B_m\setminus\frac{1}{2}B_m} |p\nabla p|\,dx \le \Bigg(\int\limits_{B_m\setminus\frac{1}{2}B_m} |p|^3\,dS\Bigg)^{\frac{1}{3}} \cdot \Bigg(\int\limits_{B_m\setminus\frac{1}{2}B_m} |\nabla p|^{3/2}\,dS\Bigg)^{\frac{2}{3}} \le C_2. \tag{5.7}$$

Denote $\sigma_1 = \min_{R\in[\frac{1}{2}m,m]} \int_{S_R} |p|^2\,dS$, $\sigma_2 = \max_{R\in[\frac{1}{2}m,m]} \int_{S_R} |p|^2\,dS - \sigma_1$. Then

$$\sigma_1 \le \frac{2}{m} \int\limits_{B_m\setminus\frac{1}{2}B_m} |p|^2\,dx \le 2C_1. \tag{5.8}$$

Analogously, since

$$\Bigg(\int\limits_{S_R} |p|^2\,dS\Bigg)'_R = \frac{2}{R}\int\limits_{S_R} |p|^2\,dS + 2\int\limits_{S_R} p\nabla p\cdot\mathbf{n}\,dS, \tag{5.9}$$

by (5.6)–(5.7) we have

$$\sigma_2 \le \frac{4}{m} \int\limits_{B_m \setminus \frac{1}{2} B_m} |p|^2\, dx + 2 \int\limits_{B_m \setminus \frac{1}{2} B_m} |p \nabla p|\, dx \le 4C_1 + 2C_2. \tag{5.10}$$

Because of inequality $\int\limits_{|x|=m} |p|^2\, dS \le \sigma_1 + \sigma_2$, we have proved the required uniform boundedness of these integrals.

Hence

$$\int\limits_{|x|=m} |p||\mathbf{A}|\, dS \le \Big(\int\limits_{|x|=m} |p|^2 dS \Big)^{\frac{1}{2}} \Big(\int\limits_{|x|=m} |\mathbf{A}|^2 dS \Big)^{\frac{1}{2}} \to 0 \quad \text{as } m \to \infty.$$

Multiplying equation (3.8_1) by $\mathbf{A}$, integrating by parts in $\Omega_m = \{x \in \Omega : |x| < m\}$ and passing to a limit as $m \to \infty$, we obtain

$$\begin{aligned}
\int\limits_{\Omega} (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \mathbf{A}\, dx &= -\int\limits_{\Omega} \nabla p \cdot \mathbf{A}\, dx = -\lim_{m \to \infty} \int\limits_{\Omega_m} \operatorname{div}(p\mathbf{A})\, dx \\
&= -\sum_{j=M'+1}^{N} \widehat{p}_j \mathcal{F}_j - \lim_{m \to \infty} \int\limits_{|x|=m} p\mathbf{A} \cdot \mathbf{n}\, dS \\
&= -\sum_{j=M'+1}^{N} \widehat{p}_j \mathcal{F}_j.
\end{aligned} \tag{5.11}$$

The required equality (5.3) follows from the last identity and (3.14). □

If $\widehat{p}_{M'+1} = \cdots = \widehat{p}_N = 0$, we get $\int\limits_{\Omega} (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \mathbf{A}\, dx = 0$. However, this contradicts the equality (5.3). Thus, there is $\widehat{p}_j \neq 0$ for some $j \in \{M'+1, \dots, N\}$.

Further we consider separately three possible cases.

(a) The maximum of Φ is attained at infinity, i.e., it is zero:

$$0 = \operatorname*{ess\,sup}_{x \in \Omega} \Phi(x). \tag{5.12}$$

(b) The maximum of Φ is attained on a boundary component which does not intersect the symmetry axis:

$$0 < \widehat{p}_N = \max_{j=M'+1,\dots,N} \widehat{p}_j = \operatorname*{ess\,sup}_{x \in \Omega} \Phi(x), \tag{5.13}$$

(c) The maximum of Φ is not zero and it is not attained[14] on $\partial\Omega$:

$$\max_{j=M'+1,\dots,N} \widehat{p}_j < \operatorname*{ess\,sup}_{x \in \Omega} \Phi(x) > 0. \tag{5.14}$$

[14] The case $\operatorname{ess\,sup}_{x \in \Omega} \Phi(x) = +\infty$ is not excluded.

5.1 The case $\mathrm{ess\,sup}_{x\in\Omega}\,\Phi(x) = 0$.

Let us consider case (5.12). By Corollary 4.1,

$$\widehat{p}_1 = \cdots = \widehat{p}_{M'} = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) = 0. \tag{5.15}$$

Since the identity $\widehat{p}_{M'+1} = \cdots = \widehat{p}_N = 0$ is impossible, we have that $\widehat{p}_j < 0$ for some $j \in \{M'+1, \ldots, N\}$. Change (if necessary) the numbering of the boundary components $\Gamma_{M'+1}$, …, Γ_N so that

$$\widehat{p}_0 = \widehat{p}_1 = \cdots = \widehat{p}_M = 0, \quad M \geq M', \tag{5.16}$$

$$\widehat{p}_j < 0, \quad j = M+1, \ldots, N. \tag{5.17}$$

Recall that in our notation $P_+ = \{(0, x_2, x_3) : x_2 > 0,\ x_3 \in \mathbb{R}\}$, $\mathcal{D} = \Omega \cap P_+$. Of course, on P_+ the coordinates x_2, x_3 coincides with coordinates r, z, and $O_z = O_{x_3}$ is a symmetry axis of Ω. For a set $A \subset \mathbb{R}^3$ put $\breve{A} := A \cap P_+$.

We receive a contradiction following the arguments of [19,20]. Take the positive constant $\delta_p = -\sup_{j=M+1,\ldots,N} \Phi(\Gamma_j)$. Our first goal is to separate the boundary components where $\Phi < 0$ from infinity and from the singularity axis O_z by level sets of Φ compactly supported in $\mathcal{D}$. More precisely, for any $t \in (0, \delta_p)$ and $j = M+1, \ldots, N$ we construct a continuum $A_j(t) \Subset P_+$ with the following properties:

(i) The set $\breve{\Gamma}_j$ lies in a bounded connected component of the open set $P_+ \setminus A_j(t)$;
(ii) $\psi|_{A_j(t)} \equiv \mathrm{const}$, $\Phi(A_j(t)) = -t$;
(iii) (monotonicity) If $0 < t_1 < t_2 < \delta_p$, then $A_j(t_1)$ lies in the unbounded connected component of the set $P_+ \setminus A_j(t_2)$ (in other words, the set $A_j(t_2) \cup \breve{\Gamma}_j$ lies in the bounded connected component of the set $P_+ \setminus A_j(t_1)$, see Fig. 1).

For this construction, we shall use the results of Sect. 2.4. To do it, we have to consider the restrictions of the stream function ψ to suitable compact subdomains of P_+.

Fix $j \in \{M+1, \ldots, N\}$. Take $r_j > 0$ such that $L_{r_j} \cap \Gamma_j \neq \emptyset$ and the conditions (4.44) are satisfied with r_j instead of r_0. In particular,

$$L_{r_j} \cap A_{\mathbf{v}} = \emptyset, \quad \Phi(r_j, \cdot) \in C(\mathbb{R}), \quad \text{and} \quad \Phi(r_j, z) \to 0 \text{ as } z \to \infty. \tag{5.18}$$

Take monotone sequences of positive numbers $\varepsilon_i \to 0+$, $\rho_i \to +\infty$ such that

$$\varepsilon_i < \inf_{(r,z)\in\breve{\Gamma}_j} r \quad \text{and} \quad S_{\varepsilon_i,\rho_i} \cap A_{\mathbf{v}} = \emptyset \quad \forall i \in \mathbb{N}, \tag{5.19}$$

$$\lim_{i\to\infty} \sup |\Phi(S_{\varepsilon_i,\rho_i})| = 0, \tag{5.20}$$

where $S_{\varepsilon,\rho} = \{(r, z) \in P_+ : r \geq \varepsilon,\ \sqrt{r^2+z^2} = \rho\}$ (the existence of such sequences follow from Lemma 4.6).

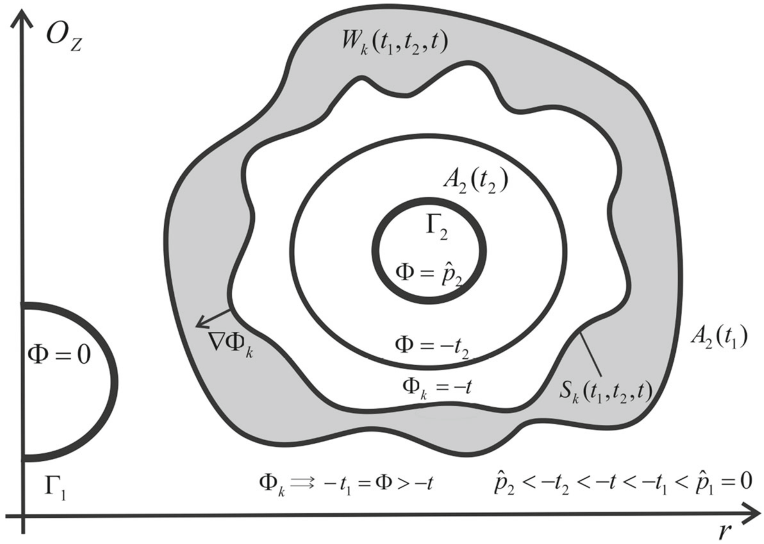

Fig. 1 The surface $S_k(t_1, t_2, t)$ for the case of $M = 1$, $N = 2$

By Remark 2.2, we can apply Kronrod's results to the stream function $\psi|_{\bar{\mathcal{D}}_{\varepsilon_i,\rho_i}}$, where $\mathcal{D}_{\varepsilon,\rho} = \{(r, z) \in P_+ : r > \varepsilon, \sqrt{r^2 + z^2} < \rho\}$. Notice that $\breve{\Gamma}_j \subset D_{\varepsilon_i,\rho_i}$. Accordingly, T^i_ψ means the corresponding Kronrod tree for the restriction $\psi|_{\bar{\mathcal{D}}_{\varepsilon_i,\rho_i}}$.

For any element $C \in T^i_\psi$ with $C \setminus A_{\mathbf{v}} \neq \emptyset$ we can define the value $\Phi(C)$ as $\Phi(C) = \Phi(x)$, where $x \in C \setminus A_{\mathbf{v}}$. This definition is correct because of the Bernoulli Law. (In particular, $\Phi(C)$ is well defined if diam $C > 0$.)

Take $x_i = (r_j, z_i) \in S_{\varepsilon_i,\rho_i}$. Denote by B_{x_i} and B^i_j the elements of T^i_ψ with $x_i \in B_{x_i}$ and $\breve{\Gamma}_j \subset B^i_j$. Consider the arc $[B^i_j, B_{x_i}] \subset T^i_\psi$. Recall that, by definition, a connected component C of a level set of $\psi|_{\bar{\mathcal{D}}_{\varepsilon_i,\rho_i}}$ belongs to the arc $[B^i_j, B_{x_i}]$ iff $C = B_{x_i}$, or $C = B^i_j$, or C separates B_{x_i} from B^i_j in $\bar{\mathcal{D}}_{\varepsilon_i,\rho_i}$, i.e., if B_{x_i} and B^i_j lie in different connected components of $\bar{\mathcal{D}}_{\varepsilon_i,\rho_i} \setminus C$. In particular, since $B_{x_i} \cap L_{r_j} \neq \emptyset \neq B^i_j \cap L_{r_j}$, we have

$$C \cap L_{r_j} \neq \emptyset \quad \forall C \in [B^i_j, B_{x_i}]. \tag{5.21}$$

Therefore, in view of equality

$$L_{r_j} \cap A_{\mathbf{v}} = \emptyset \tag{5.22}$$

the value $\Phi(C)$ is well defined for all $C \in [B^i_j, B_{x_i}]$. Moreover, we have

Lemma 5.2 *The restriction $\Phi|_{[B^i_j, B_{x_i}]}$ is a continuous function.*

Proof The assertion follows immediately[15] from the assumptions (5.21)–(5.22), from the continuity of $\Phi(r_j, \cdot)$, and from the definition of convergence in T^i_ψ (see Sect. 2.4). □

[15] See also the proof of Lemma 3.5 in [20].

Define the natural order[16] on the arc $[B_j^i, B_{x_i}]$. Namely, we say, that $A < C$ for some different elements $A, C \in [B_j^i, B_{x_i}]$ iff C closer to B_{x_i} than A, i.e., if the sets B_{x_i} and C lie in the same connected component of the set $\bar{\mathcal{D}}_{\varepsilon_i,\rho_i} \setminus A$.

Put

$$K_{\varepsilon_i} = \min\{C \in [B_j^i, B_{x_i}] : C \cap \partial\mathcal{D}_{\varepsilon_i,\rho_i} \neq \emptyset\}.$$

The next assertion is an analog of Lemma 4.6 from [20].

Lemma 5.3 *$\Phi(K_{\varepsilon_i}) \to 0$ as $i \to \infty$. In particular, $|\Phi(K_{\varepsilon_i})| < |\widehat{p}_j| = |\Phi(\Gamma_j)|$, and, consequently, $B_j^i < K_{\varepsilon_i}$ for sufficiently large i.*

Proof By definition, $K_{\varepsilon_i} \cap \partial\mathcal{D}_{\varepsilon_i,\rho_i} \neq \emptyset$. By construction, $\partial\mathcal{D}_{\varepsilon_i,\rho_i} \subset S_{\varepsilon_i,\rho_i} \cup L_{\varepsilon_i}$. If $K_{\varepsilon_i} \cap S_{\varepsilon_i,\rho_i} \neq \emptyset$, then the smallness of $\Phi(K_{\varepsilon_i})$ follows immediately from the assumption (5.20). Now let

$$K_{\varepsilon_i} \cap L_{\varepsilon_i} \neq \emptyset. \tag{5.23}$$

Recall that, by (5.21), we have also

$$K_{\varepsilon_i} \cap L_{r_j} \neq \emptyset. \tag{5.24}$$

Now the smallness of $\Phi(K_{\varepsilon_i})$ follows from Theorem 4.4. □

In view of above Lemma we may assume without loss of generality that

$$|\Phi(K_{\varepsilon_i})| < |\widehat{p}_j| = |\Phi(\Gamma_j)| \quad \text{and} \quad B_j^i < K_{\varepsilon_i} \quad \forall i \in \mathbb{N}. \tag{5.25}$$

By construction,

$$C \cap \partial\mathcal{D}_{\varepsilon_i,\rho_i} = \emptyset \quad \forall C \in [B_j^i, K_{\varepsilon_i}). \tag{5.26}$$

In particular,

$$B_j^i \cap \partial\mathcal{D}_{\varepsilon_i,\rho_i} = \emptyset \quad \forall i \in \mathbb{N}. \tag{5.27}$$

Therefore, really B_j^i does not depend on i, so we have

$$B_j^i \equiv B_j \quad \forall i \in \mathbb{N} \tag{5.28}$$

for some continuum B_j. Also, by equality

$$K_{\varepsilon_i} = \sup\{C \in [B_j, B_{x_i}] : C \cap \partial\mathcal{D}_{\varepsilon_i,\rho_i} = \emptyset\}$$

and by inclusions $\mathcal{D}_{\varepsilon_i,\rho_i} \Subset \mathcal{D}_{\varepsilon_{i+1},\rho_{i+1}}$ we have

$$[B_j, K_{\varepsilon_i}) \subset [B_j, K_{\varepsilon_{i+1}}) \quad \forall i \in \mathbb{N}, \tag{5.29}$$

[16] Recall, that by Lemma 2.4, the set $[B_j^i, B_{x_i}]$ is homeomorphic to the segment of a real line, i.e. it is an arc. So we could define a natural order on this arc and take maxima, minima etc.—as for usual segment. There two symmetric possibilities to define a usual linear order on the arc; here by our choice $B_j^i < B_{x_i}$.

where, as usual, $[B_j, K_{\varepsilon_i}) = [B_j, K_{\varepsilon_i}]\setminus\{K_{\varepsilon_i}\}$. Denote

$$[B_j, \infty) = \bigcup_{i\in\mathbb{N}} [B_j, K_{\varepsilon_i}).$$

The set $[B_j, \infty)$ inherits the order and the topology from the arcs $[B_j, K_{\varepsilon_i})$. Obviously, $[B_j, \infty)$ is homeomorphic to the ray $[0, \infty) \subset \mathbb{R}$.

By construction, we have the following properties of the set $[B_j, \infty)$.

($*$1) each element $C \in [B_j, \infty)$ is a continuum, $C \Subset P_+$, and the set $\breve{\Gamma}_j$ lies in a bounded connected component of the open set $P_+\setminus C$ for $C \neq B_j$.

($*$2) $\psi|_C \equiv$ const for all $C \in [B_j, \infty)$.

($*$3) (monotonicity) If $C', C'' \in [B_j, \infty)$ and $C' < C''$, then C'' lies in the unbounded connected component of the set $P_+\setminus C'$; i.e., the set $C' \cup B_j$ lies in the bounded connected component of the set $P_+\setminus C''$.

($*$4) (continuity) If $C_m \to C_0 \in [B_j, \infty)$, then $\sup_{x\in C_m} \operatorname{dist}(x, C_0) \to 0$ and $\Phi(C_m) \to \Phi(C_0)$ as $m \to \infty$. In particular, $\Phi|_{[B_j,\infty)}$ is a continuous function.

($*$5) (range of values) $\Phi(C) < 0$ for every $C \in [B_j, \infty)$. Moreover, if $[B_j, \infty) \ni C_m \to \infty$ as $m \to \infty$, then $\Phi(C_m) \to 0$ as $m \to \infty$.

In the last property we use the following natural definition: for a sequence $C_m \in [B_j, \infty)$ we say that $C_m \to \infty$ as $m \to \infty$ if for any $i \in \mathbb{N}$ there exists $M(i)$ such that $C_m \notin [B_j, K_{\varepsilon_i})$ for all $m > M(i)$.

We say that a set $\mathcal{Z} \subset [B_j, \infty)$ has T-measure zero if $\mathfrak{H}^1(\{\psi(C) : C \in \mathcal{Z}\}) = 0$.

Lemma 5.4 *For every $j = M+1, \ldots, N$, T-almost all $C \in [B_j, \infty)$ are C^1-curves homeomorphic to the circle and $C \cap A_{\mathbf{v}} = \emptyset$. Moreover, there exists a subsequence Φ_{k_l} such that $\Phi_{k_l}|_C$ converges to $\Phi|_C$ uniformly $\Phi_{k_l}|_C \rightrightarrows \Phi|_C$ on T-almost all $C \in [B_j, \infty)$.*

Below we assume (without loss of generality) that the subsequence Φ_{k_l} coincides with the whole sequence Φ_k.

Proof The first assertion of the lemma follows from Theorem 2.1 (iii) and (5.26). The validity of the second one for T-almost all $C \in [B_j, \infty)$ was proved in [17, Lemma 3.3]. □

Below we will call *regular* the cycles C which satisfy the assertion of Lemma 5.4.

Since $\operatorname{diam} C > 0$ for every $C \in [B_j, \infty)$, by [20, Lemma 3.6] we obtain that the function $\Phi|_{[B_j,\infty)}$ has the following analog of Luzin's N-property.

Lemma 5.5 *For every $j = M+1, \ldots, N$, if $\mathcal{Z} \subset [B_j, \infty)$ has T-measure zero, then $\mathfrak{H}^1(\{\Phi(C) : C \in \mathcal{Z}\}) = 0$.*

Note that Lemma 5.5 is not tautological: in definition of T-zero measure we have stream function ψ, but Lemma 5.5 says about another function, total head pressure Φ. It looks like Luzin N-property: $\psi(E)$ has zero measure implies $\Phi(E)$ has zero measure.

From the last two assertions we get

Corollary 5.1 *For every* $j = M+1, \dots, N$ *and for almost all* $t \in (0, |\widehat{p}_j|)$ *we have*

$$\big(C \in [B_j, \infty) \text{ and } \Phi(C) = -t\big) \Rightarrow C \text{ is a regular cycle.}$$

Below we will say that a value $t \in (0, \delta_p)$ is *regular* if it satisfies the assertion of Corollary 5.1. Denote by $\mathscr{T}$ the set of all regular values. Then the set $(0, \delta_p)\setminus\mathscr{T}$ has zero measure.

For $t \in (0, \delta_p)$ and $j \in \{M+1, \dots, N\}$ denote

$$A_j(t) = \max\{C \in [B_j, \infty) : \Phi(C) = -t\}.$$

By construction, the function $A_j(t)$ is nonincreasing and satisfies the properties (i)–(iii) from the beginning of this subsection. Moreover, by definition of regular values we have the following additional property:

(iv) If $t \in \mathscr{T}$, then $A_j(t)$ is a regular cycle.[17]

For $t \in \mathscr{T}$ denote by $V(t)$ the unbounded connected component of the open set $\mathcal{D}\setminus\big(\cup_{j=M+1}^N A_j(t)\big)$. Since $A_{j_1}(t)$ can not separate $A_{j_2}(t)$ from infinity[18] for $A_{j_1}(t) \neq A_{j_2}(t)$, we have

$$\mathcal{D} \cap \partial V(t) = A_{M+1}(t) \cup \cdots \cup A_N(t).$$

By construction, the sequence of domains $V(t)$ is increasing, i.e., $V(t_1) \subset V(t_2)$ for $t_1 < t_2$. Hence, the sequence of sets $(\partial\mathcal{D}) \cap (\partial V(t))$ is nondecreasing:

$$(\partial\mathcal{D}) \cap \partial V(t_1) \subseteqq (\partial\mathcal{D}) \cap \partial V(t_2) \quad \text{if } t_1 < t_2. \tag{5.30}$$

Every set $(\partial\mathcal{D})\cap\partial V(t)\setminus O_z$ consists of several components $\breve{\Gamma}_l$ with $l \leq M$ (since cycles $\cup_{j=M+1}^N A_j(t)$ separate infinity from $\breve{\Gamma}_{M+1}, \dots, \breve{\Gamma}_N$, but not necessary from other $\breve{\Gamma}_l$). Since there are only finitely many components $\breve{\Gamma}_l$, using monotonicity property (5.30) we conclude that for sufficiently small t the set $(\partial\mathcal{D}) \cap (\partial V(t))$ is independent of t. So we may assume, without loss of generality, that $(\partial\mathcal{D}) \cap (\partial V(t))\setminus O_z = \breve{\Gamma}_1 \cup \cdots \cup \breve{\Gamma}_K$ for $t \in \mathscr{T}$, where $M' \leq K \leq M$. Therefore,

$$\partial V(t)\setminus O_z = \breve{\Gamma}_1 \cup \cdots \cup \breve{\Gamma}_K \cup A_{M+1}(t) \cup \cdots \cup A_N(t), \quad t \in \mathscr{T}. \tag{5.31}$$

Let $t_1, t_2 \in \mathscr{T}$ and $t_1 < t_2$. The next geometrical objects plays an important role in the estimates below: for $t \in (t_1, t_2)$ we define the level set $S_k(t, t_1, t_2) \subset \{x \in \mathcal{D} : \Phi_k(x) = -t\}$ separating cycles $\cup_{j=M+1}^N A_j(t_1)$ from $\cup_{j=M+1}^N A_j(t_2)$ as follows. Namely, take arbitrary $t', t'' \in \mathscr{T}$ such that $t_1 < t' < t'' < t_2$. From Properties (ii),(iv)

[17] Some of these cycles $A_j(t)$ could coincide, i.e., equalities of type $A_{j_1}(t) = A_{j_2}(t)$ are possible (if Kronrod arcs $[B_{j_1}, \infty)$ and $[B_{j_2}, \infty)$ have nontrivial intersection), but this a priori possibility has no influence on our arguments.

[18] Indeed, if $A_{j_2}(t)$ lies in a bounded component of $\mathcal{D}\setminus A_{j_1}(t)$, then by construction $A_{j_1}(t) \in [B_{j_2}, \infty)$ and $A_{j_1}(t) > A_{j_2}(t)$ with respect to the above defined order on $[B_{j_2}, \infty)$, but it contradicts the definition of $A_{j_1}(t) = \max\{C \in [B_{j_2}, \infty) : \Phi(C) = -t\}$.

we have the uniform convergence $\Phi_k|_{A_j(t_1)} \rightrightarrows -t_1$, $\Phi_k|_{A_j(t_2)} \rightrightarrows -t_2$ as $k \to \infty$ for every $j = M+1, \dots, N$. Thus there exists $k_\circ = k_\circ(t_1, t_2, t', t'') \in \mathbb{N}$ such that for all $k \ge k_\circ$

$$\Phi_k|_{A_j(t_1)} > -t', \quad \Phi_k|_{A_j(t_2)} < -t'' \quad \forall j = M+1, \dots, N. \tag{5.32}$$

In particular,

$$\begin{aligned} &\forall t \in [t', t''] \, \forall k \ge k_\circ \quad \Phi_k|_{A_j(t_1)} > -t, \quad \Phi_k|_{A_j(t_2)} < -t, \\ &\forall j = M+1, \dots, N. \end{aligned} \tag{5.33}$$

For $k \ge k_\circ$, $j = M+1, \dots, N$, and $t \in [t', t'']$ denote by $W_k^j(t_1, t_2; t)$ the connected component of the open set $\{x \in V(t_2) \setminus \overline{V}(t_1) : \Phi_k(x) > -t\}$ such that $\partial W_k^j(t_1, t_2; t) \supset A_j(t_1)$ (see Fig. 1) and put

$$W_k(t_1, t_2; t) = \bigcup_{j=M+1}^{N} W_k^j(t_1, t_2; t), \quad S_k(t_1, t_2; t) = (\partial W_k(t_1, t_2; t)) \cap V(t_2) \setminus \overline{V}(t_1).$$

Clearly, $\Phi_k \equiv -t$ on $S_k(t_1, t_2; t)$. By construction (see Fig. 1),

$$\partial W_k(t_1, t_2; t) = S_k(t_1, t_2; t) \cup A_{M+1}(t_1) \cup \dots \cup A_N(t_1). \tag{5.34}$$

(Note that $W_k(t_1, t_2; t))$ and $S_k(t_1, t_2; t)$ are well defined for all $t \in [t', t'']$ and $k \ge k_\circ = k_\circ(t_1, t_2, t', t'')$.)

Since by (E–NS) each Φ_k belongs to $W^{2,2}_{\rm loc}(\mathcal{D})$, by the Morse–Sard theorem for Sobolev functions (see Theorem 2.1) we have that for almost all $t \in [t', t'']$ the level set $S_k(t_1, t_2; t)$ consists of finitely many C^1-cycles and Φ_k is differentiable (in classical sense) at every point $x \in S_k(t_1, t_2; t)$ with $\nabla \Phi_k(x) \ne 0$. The values $t \in [t', t'']$ having the above property will be called *k-regular*.

Recall that for a set $A \subset P_+$ we denote by $\widetilde{A}$ the set in $\mathbb{R}^3$ obtained by rotation of A around O_z-axis. By construction, for every regular value $t \in [t', t'']$ the set $\widetilde{S}_k(t', t''; t)$ is a finite union of smooth surfaces (tori), and

$$\int\limits_{\widetilde{S}_k(t_1,t_2;t)} \nabla \Phi_k \cdot \mathbf{n} \, dS = - \int\limits_{\widetilde{S}_k(t_1,t_2;t)} |\nabla \Phi_k| \, dS < 0, \tag{5.35}$$

where $\mathbf{n}$ is the unit outward normal vector to $\partial \widetilde{W}_k(t_1, t_2; t)$.

For $h > 0$ denote $\Gamma_h = \{x \in \Omega : \operatorname{dist}(x, \Gamma_1 \cup \dots \cup \Gamma_K) = h)\}$, $\Omega_h = \{x \in \Omega : \operatorname{dist}(x, \Gamma_1 \cup \dots \cup \Gamma_K) < h)\}$. Since the distance function $\operatorname{dist}(x, \partial\Omega)$ is C^1-regular and the norm of its gradient is equal to one in the neighborhood of $\partial\Omega$, there is a constant $\delta_0 > 0$ such that for every $h \le \delta_0$ the set Γ_h is a union of K C^1-smooth surfaces homeomorphic to balls or torus, and

$$\mathfrak{H}^2(\Gamma_h) \le c_0 \quad \forall h \in (0, \delta_0], \tag{5.36}$$

where the constant $c_0 = 3\mathfrak{H}^2(\Gamma_1 \cup \cdots \cup \Gamma_K)$ is independent of h.

By direct calculations, (4.1) implies

$$\nabla\Phi = \mathbf{v} \times \boldsymbol{\omega} \quad \text{in } \Omega, \tag{5.37}$$

where $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$, i.e.,

$$\boldsymbol{\omega} = (\omega_r, \omega_\theta, \omega_z) = \Big(-\frac{\partial v_\theta}{\partial z}, \ \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \ \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}\Big).$$

Set $\omega(x) = |\boldsymbol{\omega}(x)|$. Since $\Phi \ne \text{const}$ on $V(t)$, (5.37) implies $\int_{\widetilde{V}(t)} \omega^2\, dx > 0$ for every $t \in \mathscr{T}$. Hence, from the weak convergence $\boldsymbol{\omega}_k \rightharpoonup \boldsymbol{\omega}$ in $L^2(\Omega)$ (recall that $\boldsymbol{\omega}_k = \operatorname{curl} \mathbf{u}_k$, $\omega_k(x) = |\boldsymbol{\omega}_k(x)|$) it follows

Lemma 5.6 *For any $t \in \mathscr{T}$ there exist constants $\varepsilon_t > 0$ and $k_t \in \mathbb{N}$ such that for all $k \ge k_t$*

$$A_j(t) \Subset \frac{1}{2}B_k, \quad j = M+1, \ldots, N,$$
$$\Gamma_j \Subset \frac{1}{2}B_k, \quad j = 1, \ldots, N,$$

and

$$\int\limits_{\widetilde{V}(t)\cap B_k} \omega_k^2\, dx > \varepsilon_t. \tag{5.38}$$

Here $B_k = \{x \in \mathbb{R}^3 : |x| < R_k\}$ are the balls where the solutions $\mathbf{u}_k \in W^{1,2}(\Omega \cap B_k)$ are defined.

Now we are ready to prove the key estimate.

Lemma 5.7 *For any $t_1, t_2, t', t'' \in \mathscr{T}$ with $t_1 < t' < t'' < t_2$ there exists $k_* = k_*(t_1, t_2, t', t'')$ such that for every $k \ge k_*$ and for almost all $t \in [t, t'']$ the inequality*

$$\int\limits_{\widetilde{S}_k(t_1,t_2;t)} |\nabla\Phi_k|\, dS < \mathcal{F}t, \tag{5.39}$$

holds with the constant $\mathcal{F}$ independent of t, t_1, t_2, t', t'' and k.

Proof Fix $t_1, t_2, t', t'' \in \mathscr{T}$ with $t_1 < t' < t'' < t_2$. Below we always assume that $k \ge k_\circ(t_1, t_2, t', t'')$ [see (5.32)–(5.33)], in particular, the set $S_k(t_1, t_2; t)$ is well defined for all $t \in [t', t'']$. We assume also that $R_k > 2$ and $k \ge k_{t_1}$ (see Lemma 5.6).

The main idea of the proof of (5.39) is quite simple: we will integrate the equation

$$\Delta\Phi_k = \omega_k^2 + \frac{1}{\nu_k}\,\mathrm{div}(\Phi_k \mathbf{u}_k) - \frac{1}{\nu_k}\mathbf{f}_k \cdot \mathbf{u}_k \tag{5.40}$$

over the suitable domain $\Omega_k(t)$ with $\partial\Omega_k(t) \supset \widetilde{S}_k(t_1, t_2; t)$ such that the corresponding boundary integrals

$$\Bigl| \int\limits_{\bigl(\partial\Omega_k(t)\bigr)\setminus \widetilde{S}_k(t_1,t_2;t)} \nabla\Phi_k \cdot \mathbf{n}\, dS \Bigr| \tag{5.41}$$

$$\frac{1}{\nu_k}\Bigl| \int\limits_{\bigl(\partial\Omega_k(t)\bigr)\setminus \widetilde{S}_k(t_1,t_2;t)} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, dS \Bigr| \tag{5.42}$$

are negligible. We split the construction of the domain $\Omega_k(t)$ into several steps.

STEP 1. We claim that for any $\varepsilon > 0$ the estimate

$$\frac{1}{R^2}\Bigl|\int\limits_{S_R} \Phi_k\, dS\Bigr| < \varepsilon \tag{5.43}$$

holds for sufficiently large R uniformly with respect to k. Indeed, the weak convergence $\Phi_k \rightharpoonup \Phi$ in $W^{1,3/2}_{\mathrm{loc}}(\overline{\Omega})$ implies

$$\frac{1}{R_*^2}\Bigl|\int\limits_{S_{R_*}} \Phi_k\, dS\Bigr| \to \frac{1}{R_*^2}\Bigl|\int\limits_{S_{R_*}} \Phi\, dS\Bigr| \tag{5.44}$$

for any fixed $R_* > R_0$. Take $R_\varepsilon > R_0$ sufficiently large such that

$$\frac{1}{R_\varepsilon^2}\Bigl|\int\limits_{S_{R_\varepsilon}} \Phi\, dS\Bigr| < \frac{\varepsilon}{2} \tag{5.45}$$

(this inequality holds for sufficiently large R_ε because of the inclusion $\Phi \in L^3(\Omega) \cap D^{1,3/2}(\Omega)$, see the proof of (5.5)). Take arbitrary $R \in [R_\varepsilon, R_k]$. Then $R \le 2^l R_\varepsilon$ for some $l \in \mathbb{N}$. We have

$$\begin{aligned}
\Bigl|\frac{1}{R^2}\int\limits_{S_R} \Phi_k\, dS - \frac{1}{R_\varepsilon^2}\int\limits_{S_{R_\varepsilon}} \Phi_k\, dS\Bigr| &= \Bigl|\int\limits_{|x|\in[R_\varepsilon, R]} \frac{1}{|x|^3} x \cdot \nabla\Phi_k\, dx\Bigr| \\
&\le \Bigl|\sum_{m=0}^{l-1} \int\limits_{|x|\in[2^m R_\varepsilon, 2^{m+1} R_\varepsilon]} \frac{1}{|x|^3} x \cdot \nabla\Phi_k\, dx\Bigr|
\end{aligned}$$

$$\leq \sum_{m=0}^{l-1} \Bigg(\int\limits_{|x|\in[2^m R_\varepsilon, 2^{m+1} R_\varepsilon]} |\nabla \Phi_k|^{\frac{3}{2}}\, dx \Bigg)^{\frac{2}{3}} \Bigg(\int\limits_{|x|\in[2^m R_\varepsilon, 2^{m+1} R_\varepsilon]} \frac{1}{|x|^6}\, dx \Bigg)^{\frac{1}{3}}$$
$$\leq C \sum_{m=0}^{l-1} \sqrt{2^m R_\varepsilon} \frac{1}{2^m R_\varepsilon} \leq C' \frac{1}{\sqrt{R_\varepsilon}}, \tag{5.46}$$

where the constants C, C' do not depend on l and k [here we have used the estimate (3.12)]. Consequently, if we take a sufficiently large R_ε, then

$$\Bigg| \frac{1}{R^2} \int\limits_{S_R} \Phi_k\, dS - \frac{1}{R_\varepsilon^2} \int\limits_{S_{R_\varepsilon}} \Phi_k\, dS \Bigg| < \frac{\varepsilon}{2}$$

for all $k \in \mathbb{N}$ and $R > R_\varepsilon$. Now the required estimate (5.43) follows from the last inequality and formulas (5.45), (5.44) (with $R_* = R_\varepsilon$).

STEP 2. By direct calculations, (3.10) implies

$$\nabla \Phi_k = -\nu_k \operatorname{curl} \boldsymbol{\omega}_k + \mathbf{u}_k \times \boldsymbol{\omega}_k + \mathbf{f}_k = -\nu_k \operatorname{curl} \boldsymbol{\omega}_k + \mathbf{u}_k \times \boldsymbol{\omega}_k + \frac{\nu_k^2}{\nu^2} \operatorname{curl} \mathbf{b}.$$

By the Stokes theorem, for any C^1-smooth closed surface $S \subset \Omega$ and $\mathbf{g} \in W^{2,2}(\Omega)$ we have

$$\int\limits_S \operatorname{curl} \mathbf{g} \cdot \mathbf{n}\, dS = 0.$$

So, in particular,

$$\int\limits_S \nabla \Phi_k \cdot \mathbf{n}\, dS = \int\limits_S (\mathbf{u}_k \times \boldsymbol{\omega}_k) \cdot \mathbf{n}\, dS. \tag{5.47}$$

Since by construction for every $x \in S_{R_k} = \{y \in \mathbb{R}^3 : |y| = R_k\}$ there holds the equality

$$\mathbf{u}_k(x) \equiv \frac{\nu_k}{\nu} \mathbf{A}(x) \equiv -\nu_k \frac{\sum_{i=1}^N F_i}{4\pi\nu} \frac{x}{|x|^3}, \tag{5.48}$$

we see that

$$\int\limits_{S_{R_k}} \nabla \Phi_k \cdot \mathbf{n}\, dS = 0. \tag{5.49}$$

Indeed, by (5.48) the vector $\mathbf{u}_k(x)$ is parallel to x for every $x \in S_{R_k}$, consequently, $\boldsymbol{\omega}_k(x) \times \mathbf{u}_k(x)$ is perpendicular to x for $x \in S_{R_k}$, and by virtue of (5.47) we obtain (5.49).

Furthermore, using (5.48) we get

$$\frac{1}{\nu_k}\Big|\int\limits_{S_{R_k}}\Phi_k\mathbf{u}_k\cdot\mathbf{n}\,dS\Big| = \frac{C}{R_k^2}\Big|\int\limits_{S_{R_k}}\Phi_k\,dS\Big|. \tag{5.50}$$

Thus applying (5.43) for sufficiently large k we have

$$\frac{1}{\nu_k}\Big|\int\limits_{S_{R_k}}\Phi_k\mathbf{u}_k\cdot\mathbf{n}\,dS\Big| < \varepsilon. \tag{5.51}$$

Step 3. Denote $\Gamma_0 = \Gamma_1\cup\cdots\cup\Gamma_K$. Recall that by the pressure normalization condition,

$$\Phi|_{\Gamma_0} = 0. \tag{5.52}$$

Our purpose on this step is as follows: for arbitrary $\varepsilon > 0$ and for sufficiently large k to prove the estimates

$$\Big|\int\limits_{\Gamma_0}\nabla\Phi_k\cdot\mathbf{n}\,dS\Big| = \Big|\int\limits_{\Gamma_0}(\mathbf{u}_k\times\boldsymbol{\omega}_k)\cdot\mathbf{n}\,dS\Big| < \varepsilon, \tag{5.53}$$

$$\frac{1}{\nu_k}\Big|\int\limits_{\Gamma_0}\Phi_k\mathbf{u}_k\,dS\Big| < \varepsilon. \tag{5.54}$$

Recall that in our notation $\mathbf{u}_k = \frac{\nu_k}{\nu}\mathbf{A} + \mathbf{w}_k$, where $\mathbf{w}_k \in H(\Omega)$, $\|\mathbf{w}_k\|_{H(\Omega)} = 1$, and $\mathbf{A}$ is a solenoidal extension of $\mathbf{a}$ from Lemma 2.1. In particular, we have

$$\mathbf{u}_k(x) \equiv \frac{\nu_k}{\nu}\mathbf{A}(x) \quad \forall x\in\Gamma_0. \tag{5.55}$$

To establish (5.54), we use the uniform boundedness

$$\|\Phi_k\|_{L^3(\Omega_{\delta_0})} + \|\nabla\Phi_k\|_{L^{3/2}(\Omega_{\delta_0})} \le C, \tag{5.56}$$

[recall that $\Omega_h = \{x\in\Omega : \operatorname{dist}(x,\Gamma_0)\le h\}$ and the positive parameter δ_0 was discussed before formula (5.36)]. From (5.56) and the weak convergence $\Phi_k \rightharpoonup \Phi$ in $W^{1,3/2}(\Omega_{\delta_0})$ we easily have

$$\Phi_k \to \Phi \quad \text{in } L^q(\Gamma_0) \quad \forall q\in[1,2). \tag{5.57}$$

Thus by virtue of (5.52),

$$\int\limits_{\Gamma_0}|\Phi_k|\,dS \to 0 \quad \text{as } k\to\infty. \tag{5.58}$$

Since

$$\|\mathbf{A}\|_{L^\infty(\Omega)} \le C\|\mathbf{A}\|_{W^{2,2}(\Omega)} < \infty, \tag{5.59}$$

by identity (5.55) we have

$$\frac{1}{\nu_k}\Big|\int\limits_{\Gamma_0} \Phi_k \mathbf{u}_k \, dS\Big| = \nu\Big|\int\limits_{\Gamma_0} \Phi_k \mathbf{A}\, dS\Big| \le C\int\limits_{\Gamma_0} |\Phi_k|\, dS \underset{k\to\infty}{\to} 0, \tag{5.60}$$

that implies the required estimate (5.54) for sufficiently large k.

To prove (5.53), we need also the uniform estimate

$$\|\nu_k \mathbf{u}_k\|_{W^{2,3/2}(\Omega_{\delta_0})} \le C, \tag{5.61}$$

where C is independent of k (this inequality follows from the construction [see (3.10)] by well-known estimates [1,38,39] for the solutions to the Stokes system, see also formula (2.12) of the present paper). Thus by Sobolev imbedding theorems

$$\|\nu_k \nabla \mathbf{u}_k\|_{L^1(\Gamma_h)} \le C \quad \forall h \in [0, \delta_0] \ \forall k \in \mathbb{N}, \tag{5.62}$$

where $C > 0$ is independent of k, h (recall that $\Omega_h = \{x \in \Omega : \operatorname{dist}(x, \Gamma_0) \le h\}$, $\Gamma_h = \{x \in \Omega : \operatorname{dist}(x, \Gamma_0) = h\}$). Moreover, by elementary calculations, (5.61) implies the uniform Hölder continuity of the function $[0, \delta_0] \ni h \mapsto \|\nu_k \nabla \mathbf{u}_k\|_{L^1(\Gamma_h)}$, i.e., there exists a constant $\sigma > 0$ (independent of k) such that

$$\Big|\int\limits_{\Gamma_{h'}} \big|\nu_k \nabla \mathbf{u}_k\big|\, dS - \int\limits_{\Gamma_{h''}} \big|\nu_k \nabla \mathbf{u}_k\big|\, dS\Big| \le \sigma\big|h' - h''\big|^{\frac{1}{3}} \quad \forall h', h'' \in [0, \delta_0] \quad \forall k \in \mathbb{N}. \tag{5.63}$$

From the last property and from the uniform boundedness of the Dirichlet integral

$$\|\nabla \mathbf{u}_k\|_{L^2(\Omega)} \le 2, \tag{5.64}$$

one can easily deduce that

$$\sup_{h\in[0,\delta_0]} \int\limits_{\Gamma_h} \big|\nu_k \nabla \mathbf{u}_k\big|\, dS \to 0 \quad \text{as } k \to \infty, \tag{5.65}$$

in particular,

$$\int\limits_{\Gamma_0} \big|\nu_k \nabla \mathbf{u}_k\big|\, dS \to 0 \quad \text{as } k \to \infty. \tag{5.66}$$

Then from the identity (5.55) and the estimate (5.59) we have

$$\Big|\int\limits_{\Gamma_0} (\mathbf{u}_k \times \boldsymbol{\omega}_k)\cdot \mathbf{n}\, dS\Big| = \frac{\nu_k}{\nu}\Big|\int\limits_{\Gamma_0} (\mathbf{A} \times \boldsymbol{\omega}_k)\cdot \mathbf{n}\, dS\Big| \le C\int\limits_{\Gamma_0} \big|\nu_k \nabla \mathbf{u}_k\big|\, dS \to 0 \quad \text{as } k \to \infty. \tag{5.67}$$

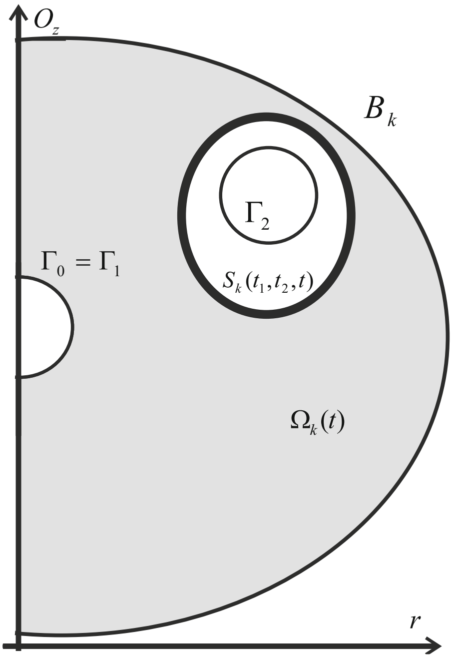

Fig. 2 The domain $\Omega_k(t)$ for the case $M = K = 1$, $N = 2$, $\Gamma_0 = \Gamma_1$

Hence the required estimate (5.53) is proved.

STEP 4. Take arbitrary $\varepsilon > 0$ and fix it (the precise value of ε will be specified below). Now, for $t \in \mathscr{T} \cap [t', t'']$ and sufficiently large k (in particular, such that the claims of previous Steps are fulfilled) denote by $\Omega_{S_k(t_1,t_2;t)}$ the bounded open set in $\mathbb{R}^3$ such that $\partial\Omega_{S_k(t_1,t_2;t)} = \widetilde{S}_k(t_1, t_2; t)$ (recall that for $A \subset P_+$ we denote by $\widetilde{A}$ the set in $\mathbb{R}^3$ obtained by rotation of A around O_z-axis). Then consider the domain (see Fig. 2)

$$\Omega_k(t) = \Omega \cap B_k \setminus \Omega_{S_k(t_1,t_2;t)}. \tag{5.68}$$

Here $B_k = \{x \in \mathbb{R}^3 : |x| < R_k\}$ are the balls where the solutions $\mathbf{u}_k \in W^{1,2}(\Omega \cap B_k)$ are defined.

By construction (see Fig. 2), $\partial\Omega_k(t) = \widetilde{S}_k(t_1, t_2; t) \cup S_{R_k} \cup \Gamma_0$. Integrating the equation

$$\Delta\Phi_k = \omega_k^2 + \frac{1}{\nu_k}\operatorname{div}(\Phi_k \mathbf{u}_k) - \frac{1}{\nu_k}\mathbf{f}_k \cdot \mathbf{u}_k \tag{5.69}$$

over the domain $\Omega_k(t)$, we have

$$\int\limits_{\widetilde{S}_k(t_1,t_2;t)} \nabla\Phi_k \cdot \mathbf{n}\, dS + \int\limits_{S_{R_k}\cup\Gamma_0} \nabla\Phi_k \cdot \mathbf{n}\, dS = \int\limits_{\Omega_k(t)} \omega_k^2\, dx - \frac{1}{\nu_k}\int\limits_{\Omega_k(t)} \mathbf{f}_k \cdot \mathbf{u}_k\, dx$$
$$+ \frac{1}{\nu_k}\int\limits_{\widetilde{S}_k(t_1,t_2;t)} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, dS + \frac{1}{\nu_k}\int\limits_{S_{R_k}\cup\Gamma_0} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, dS$$

$$= \int\limits_{\Omega_k(t)} \omega_k^2\,dx - \frac{1}{\nu_k}\int\limits_{\Omega_k(t)} \mathbf{f}_k\cdot\mathbf{u}_k\,dx - t\bar{\mathcal{F}} + \frac{1}{\nu_k}\int\limits_{S_{R_k}\cup\Gamma_0} \Phi_k\mathbf{u}_k\cdot\mathbf{n}\,dS, \qquad (5.70)$$

where $\bar{\mathcal{F}} = \frac{1}{\nu}(\mathcal{F}_{K+1}+\cdots+\mathcal{F}_N)$ (here we use the identity $\Phi_k \equiv -t$ on $\widetilde{S}_k(t_1,t_2;t)$). In view of (5.35), (5.49), (5.51) and (5.53)–(5.54) we can estimate

$$\int\limits_{\widetilde{S}_k(t_1,t_2;t)} |\nabla\Phi_k|\,dS \le t\mathcal{F} + 3\varepsilon + \frac{1}{\nu_k}\int\limits_{\Omega_k(t)} \mathbf{f}_k\cdot\mathbf{u}_k\,dx - \int\limits_{\Omega_k(t)} \omega_k^2\,dx \qquad (5.71)$$

with $\mathcal{F} = |\bar{\mathcal{F}}|$. By definition, $\frac{1}{\nu_k}\|\mathbf{f}_k\|_{L^{6/5}(\Omega)} = \frac{\nu_k}{\nu^2}\|\mathbf{f}\|_{L^{6/5}(\Omega)} \to 0$ as $k\to\infty$. Therefore, using the uniform estimate $\|\mathbf{u}_k\|_{L^6(\Omega_{bk})} \le \text{const}$, we have

$$\Big|\frac{1}{\nu_k}\int\limits_{\Omega_k(t)} \mathbf{f}_k\cdot\mathbf{u}_k\,dx\Big| < \varepsilon$$

for sufficiently large k. Then (5.71) yields

$$\int\limits_{\widetilde{S}_k(t_1,t_2;t)} |\nabla\Phi_k|\,dS < t\mathcal{F} + 4\varepsilon - \int\limits_{\Omega_k(t)} \omega_k^2\,dx. \qquad (5.72)$$

By construction, $\Omega_k(t) = \big[\overline{\widetilde{V}}(t_1) \cup \widetilde{W}_k(t_1,t_2;t)\big] \cap B_k \supset \widetilde{V}(t_1)\cap B_k$ (recall that the sets $V(t_1)$ and $W_k(t_1,t_2;t)$ were defined previously, see text after Corollary 5.1). Choosing ε sufficiently small so that $4\varepsilon < \varepsilon_t$ [see (5.38)] and a sufficiently large k, we deduce from (5.38) that

$$4\varepsilon - \int\limits_{\Omega_k(t)} \omega_k^2\,dx \le 4\varepsilon - \int\limits_{\widetilde{V}(t_1)\cap B_k} \omega_k^2\,dx < 0.$$

Estimate (5.39) is proved. □

We need the following technical fact from the one-dimensional real analysis.

Lemma 5.8 *Let $f:\mathscr{S}\to\mathbb{R}$ be a positive decreasing function defined on a measurable set $\mathscr{S}\subset(0,\delta)$ with* $\operatorname{meas}[(0,\delta)\setminus\mathscr{S}] = 0$. *Then*

$$\sup_{t_1,t_2\in\mathscr{S}} \frac{[f(t_2)]^{\frac{4}{3}}(t_2-t_1)}{(t_2+t_1)(f(t_1)-f(t_2))} = \infty. \qquad (5.73)$$

The proof of this fact is elementary, see "Appendix".

Below we will use the key estimate (5.39) to prove some geometrical relations that contradict Lemma 5.8.

For $t\in\mathscr{T}$ denote by $U(t)$ the union of bounded connected components (tori) of the set $\mathbb{R}^3\setminus\big(\cup_{j=M+1}^N \widetilde{A}_j(t)\big)$. By construction, $U(t_2) \Subset U(t_1)$ for $t_1 < t_2$.

Lemma 5.9 *For any $t_1, t_2 \in \mathscr{T}$ with $t_1 < t_2$ the estimate*

$$\operatorname{meas} U(t_2)^{\frac{4}{3}} \leq C\frac{t_2+t_1}{t_2-t_1}\big[\operatorname{meas} U(t_1) - \operatorname{meas} U(t_2)\big] \tag{5.74}$$

holds with the constant C independent of t_1, t_2.

Proof Fix $t_1, t_2 \in \mathscr{T}$ with $t_1 < t_2$. Take a pair $t', t'' \in \mathscr{T}$ such that $t_1 < t' < t'' < t_2$. For $k \geq k_*(t_1, t_2, t', t'')$ (see Lemma 5.7) put

$$E_k = \bigcup_{t\in[t',t'']} \widetilde{S}_k(t_1, t_2; t).$$

By the Coarea formula (see, e.g, [27]), for any integrable function $g : E_k \to \mathbb{R}$ the equality

$$\int\limits_{E_k} g|\nabla\Phi_k|\,dx = \int\limits_{t'}^{t''} \int\limits_{\widetilde{S}_k(t_1,t_2;t)} g(x)\,d\mathfrak{H}^2(x)\,dt \tag{5.75}$$

holds. In particular, taking $g = |\nabla\Phi_k|$ and using (5.39), we obtain

$$\begin{aligned} \int\limits_{E_k} |\nabla\Phi_k|^2\,dx &= \int\limits_{t'}^{t''} \int_{\widetilde{S}_k(t_1,t_2;t)} |\nabla\Phi_k|(x)\,d\mathfrak{H}^2(x)\,dt \\ &\leq \int\limits_{t'}^{t''} \mathcal{F}t\,dt = \frac{\mathcal{F}}{2}\big((t'')^2 - (t')^2\big). \end{aligned} \tag{5.76}$$

Now, taking $g = 1$ in (5.75) and using the Hölder inequality we have

$$\begin{aligned} &\int\limits_{t'}^{t''} \mathfrak{H}^2\big(\widetilde{S}_k(t_1, t_2; t)\big)\,dt = \int\limits_{E_k} |\nabla\Phi_k|\,dx \\ &\leq \Big(\int\limits_{E_k} |\nabla\Phi_k|^2\,dx\Big)^{\frac{1}{2}} \big(\operatorname{meas}(E_k)\big)^{\frac{1}{2}} \leq \sqrt{\frac{\mathcal{F}}{2}\big((t'')^2 - (t')^2\big)\operatorname{meas}(E_k)}. \end{aligned} \tag{5.77}$$

By construction (see property (iii) in the beginning of Sect. 5.1), each of the sets $A_j(t_1)$ and $A_j(t_2)$ is a smooth cycle surrounding the component $\breve{\Gamma}_j$ for $j = M+1, \ldots, N$, moreover, the cycle $A_j(t_1)$ lies in the unbounded connected component of the open set $P_+ \setminus A_j(t_2)$. Furthermore, for almost all $t \in [t', t'']$ the set $S_k(t_1, t_2; t)$ is a finite union of smooth cycles in P_+ and $S_k(t_1, t_2; t)$ separates $A_j(t_1)$ from $A_j(t_2)$ for all $j = M+1, \ldots, N$. In particular, the set $U(t_2)$ is contained in the union of bounded connected components of $\mathbb{R}^3 \setminus \widetilde{S}_k(t_1, t_2; t)$. Then by the isoperimetric inequality (see, e.g., [11]), $\mathfrak{H}^2\big(\widetilde{S}_k(t_1, t_2; t)\big) \geq C_*\big(\operatorname{meas} U(t_2)\big)^{\frac{2}{3}}$ for $t \in [t', t'']$. Therefore, (5.77) implies

$$\big(\text{meas}\, U(t_2)\big)^{\frac{4}{3}} (t'' - t')^2 \le C\big((t'')^2 - (t')^2\big)\, \text{meas}(E_k). \tag{5.78}$$

On the other hand, by definition, $S_k(t_1, t_2; t) \subset V(t_2)\setminus V(t_1)$. Consequently, $\widetilde{S}_k(t_1, t_2; t) \subset U(t_1)\setminus U(t_2)$ for all $t \in [t', t'']$, hence from (5.78) we get

$$\begin{aligned}\big(\text{meas}\, U(t_2)\big)^{\frac{4}{3}} &\le C\frac{t''+t'}{t''-t'}\, \text{meas}\big(U(t_1)\setminus U(t_2)\big)\\ &= C\frac{t''+t'}{t''-t'}\big[\text{meas}\, U(t_1) - \text{meas}\, U(t_2)\big].\end{aligned} \tag{5.79}$$

The last estimate is valid for every pair $t'', t' \in (t_1, t_2)$. Taking a limit as $t'' \to t_2$, $t' \to t_1$, we obtain the required estimate (5.74). □

The last estimate leads us to the main result of this subsection.

Lemma 5.10 *Assume that $\Omega \subset \mathbb{R}^3$ is an exterior axially symmetric domain of type* (1.2) *with C^2-smooth boundary $\partial\Omega$, and $\mathbf{f} \in W^{1,2}(\Omega)$, $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ are axially symmetric. Then assumptions (E-NS) and* (5.12) *lead to a contradiction.*

Proof By construction, $U(t_1) \supset U(t_2)$ for $t_1, t_2 \in \mathscr{T}$, $t_1 < t_2$. Thus the just obtained estimate (5.74) contradicts Lemma 5.8. This contradiction finishes the Proof of Lemma 5.10. □

5.2 The case $0 < \widehat{p}_N = \text{ess sup}_{x\in\bar{\Omega}} \Phi(x)$.

Suppose now that (5.13) holds, i.e., the maximum of Φ is attained on the boundary component Γ_N which does not intersect the symmetry axis. Then the proof can be reduced to the case with a bounded domain, which was considered in [20]. Let us describe some details of this reduction.

Repeating the arguments from the first part of the previous Sect. 5.1, we can construct a C^1-smooth cycle $A_N \subset \mathcal{D}$ such that $\psi|_{A_N} = \text{const}$, $0 < \Phi(A_N) < \widehat{p}_N$ and $\breve{\Gamma}_N = P_+ \cap \Gamma_N$ lies in the bounded connected component of the set $P_+\setminus A_N$. Denote this component by $\mathcal{D}_b$. The cycle A_N separates $\breve{\Gamma}_N$ from infinity and from the singularity line O_z. Thus, in order to obtain a contradiction, it is enough to concentrate our attention on the bounded domain $\mathcal{D}_b \cap \mathcal{D}$.

Namely, let

$$\begin{aligned}&\mathcal{D}_b \cap \Gamma_j = \emptyset, \quad j = 1, \dots M_1 - 1,\\ &\mathcal{D}_b \supset \breve{\Gamma}_j, \quad j = M_1, \dots, N\end{aligned}$$

(the case $M_1 = N$ is not excluded). Making a renumeration (if necessary), we may assume without loss of generality that

$$\begin{aligned}&\Phi(\breve{\Gamma}_j) < \widehat{p}_N, \quad j = M_1, \dots, M_2,\\ &\Phi(\breve{\Gamma}_j) = \widehat{p}_N, \quad j = M_2 + 1, \dots, N\end{aligned}$$

(the case $M_2 = M_1 - 1$, i.e., when Φ attains maximum value at every boundary component inside the domain $\mathcal{D}_b$, is not excluded). Apply Kronrod results from Sect. 2.4 to the restriction $\psi|_{\mathcal{D}_b}$ of the stream function ψ to the domain $\mathcal{D}_b$ (this is possible because of Remark 2.2). Let T_ψ denote the corresponding Kronrod tree for this restriction. Denote by B_0 the element of T_ψ with $B_0 \supset A_N$. Similarly, denote by B_j, $j = M_1, \dots, N$, the elements of T_ψ with $B_j \supset \breve{\Gamma}_j$. Adding a constant to the pressure $p(x)$, we can assume from this moment that

$$\begin{aligned}
&\Phi(B_0) = \Phi(A_N) < 0,\\
&\Phi(B_j) = \Phi(\breve{\Gamma}_j) < 0 \quad j = M_1, \dots, M_2,\\
&\Phi(B_j) = \Phi(\breve{\Gamma}_j) = 0 \quad j = M_2 + 1, \dots, N.
\end{aligned}$$

Now in order to receive the required contradiction, one need to consider the behavior of Φ on the Kronrod arcs $[B_j, B_N]$ and to repeat word by word the corresponding arguments of Subsection 4.2.2 in [20] starting from Lemma 4.7. The only modifications are as follows: now our sets B_0 and $B_{M_1}, \dots, B_{M_2}$ play the role of the sets $C_{M'}$ and $C_{M'+1}, \dots, C_M$ from [20, Subsection 4.2.2] respectively. Also, the domain $\mathcal{D}_b \cap \mathcal{D}$ from the present case plays the role of the domain D_{r_*} from [20, Subsection 4.2.2].

5.3 The case $\max_{j=1,\dots,N} \widehat{p}_j < \operatorname{ess\,sup}_{x\in\bar{\Omega}} \Phi(x) > 0$.

Suppose that (5.14) holds, i.e., the maximum of Φ is not zero and it is not attained on $\partial\Omega$ (the case $\operatorname{ess\,sup}_{x\in\Omega} \Phi(x) = +\infty$ is not excluded). Again, we reduce the situation to the case of a bounded domain considered in [20].

We start from the following simple fact.

Lemma 5.11 *Under assumptions* (5.14) *there exists a compact connected set* $F \subset \mathcal{D}$ *such that* $\operatorname{diam} F > 0$, $\psi|_F \equiv \text{const}$, *and*

$$0 < \Phi(F) > \max_{j=1,\dots,N} \widehat{p}_j.$$

The proof of this Lemma is quite similar to the proof of [20, Lemma 3.10] which was done for the case of bounded plane domain. But since the present situation has some specific differences, for reader's convenience we reproduce the proof with the corresponding modifications. Denote $\sigma = \max_{j=1,\dots,N} \widehat{p}_j$. By the assumptions, $\Phi(x) \le \sigma$ for every $x \in P_+ \cap \partial\mathcal{D} \setminus A_{\mathbf{v}}$ and there is a set of a positive plane measure $E \subset \mathcal{D} \setminus A_{\mathbf{v}}$ such that $\Phi(x) > \sigma$ at each $x \in E$. In virtue of Theorem 4.3 (iii), there exists a straight-line segment $I = [x_0, y_0] \subset \overline{\mathcal{D}} \cap P_+$ with $I \cap A_{\mathbf{v}} = \emptyset$, $x_0 \in \partial\mathcal{D}$, $y_0 \in E$, such that $\Phi|_I$ is a continuous function. By construction, $\Phi(x_0) \le \sigma$, $\Phi(y_0) \ge \sigma + \delta_0$ with some $\delta_0 > 0$. Take a subinterval $I_1 = [x_1, y_0] \subset \mathcal{D}$ such that $\Phi(x_1) = \sigma + \frac{1}{2}\delta_0$ and $\Phi(x) \ge \sigma + \frac{1}{2}\delta_0$ for each $x \in [x_1, y_0]$. Then by Bernoulli's Law (see Theorem 4.2) $\psi \ne \text{const}$ on I_1. Take a closed rectangle $Q \subset \mathcal{D}$ such that $I_1 \subset Q$. By Theorem 2.1 (iii) applied to $\psi|_Q$ we can take $x \in I_1$ such that the preimage $\{y \in Q : \psi(y) = \psi(x)\}$ consists of a finite union of C^1-curves. Denote by F

the curve containing x. Then by construction $\Phi(F) \geq \sigma + \frac{1}{2}\delta_0$, $F \subset Q \subset \mathcal{D}$ and diam $F > 0$. □

Fix a compact set F from Lemma 5.11. Using the arguments from the first part of Sect. 5.1, we can construct a C^1-smooth cycle $A_F \subset \mathcal{D}$ such that $\psi|_{A_F} = \text{const}$, $0 < \Phi(A_F) < \Phi(F)$ and F lies in the bounded connected component of the set $P_+ \setminus A_F$, denote this component by $\mathcal{D}_b$ [in this procedure the set F plays the role of the set $\breve{\Gamma}_j$ from the beginning of Sect. 5.1, where the cycles $A_j(t)$ were constructed with properties (i)–(iii)]. As before, A_F separates F from infinity and from the singularity line O_z, so, to extract a contradiction, it is enough to consider only the bounded domain $\mathcal{D}_b \cap \mathcal{D}$.

Namely, let

$$\mathcal{D}_b \cap \Gamma_j = \emptyset, \quad j = 1, \dots M_1 - 1,$$
$$\mathcal{D}_b \supset \breve{\Gamma}_j, \quad j = M_1, \dots, N$$

(the case $M_1 = N + 1$, i.e., when $\mathcal{D}_b \cap \partial\mathcal{D} = \emptyset$, is not excluded). Apply the Kronrod results from Sect. 2.4 to the restriction $\psi|_{\mathcal{D}_b}$ of the stream function ψ to the domain $\mathcal{D}_b$. Let T_ψ denote the corresponding Kronrod tree for this restriction. Denote by B_0, B_F the element of T_ψ with $B_0 \supset A_F$, $B_F \supset F$. Similarly, denote by B_j, $j = M_1, \dots, N$, the elements of T_ψ with $B_j \supset \breve{\Gamma}_j$. Adding a constant to the pressure $p(x)$, we can assume from this moment that

$$\Phi(B_0) = \Phi(A_F) < 0,$$
$$\Phi(B_j) = \Phi(\breve{\Gamma}_j) < 0 \quad j = M_1, \dots, N,$$
$$\Phi(B_F) = \Phi(F) = 0.$$

Now in order to receive the required contradiction, one need to consider the behavior of Φ on the Kronrod arcs $[B_j, B_F]$ and to repeat almost word by word the corresponding arguments of Subsection 3.3.2 in [20] after Lemma 3.10. The only modifications are as follows: now our sets B_0, $B_{M_1}, \dots, B_N$, and B_F play the role of the sets $B_0, \dots, B_N$ and F from [20, Subsection 3.3.2] respectively. Also, the domain $\mathcal{D}_b \cap \mathcal{D}$ from the present case plays the role of the domain Ω from [20, Subsection 3.3.2], and on the final stage we have to integrate identity (5.40) of the present paper over the three-dimensional domains $\Omega_{ik}(t)$ with $\partial\Omega_{ik}(t) = \widetilde{S}_{ik}(t)$. □

We can summarize the results of Sects. 5.2–5.3 in the following statement.

Lemma 5.12 *Assume that $\Omega \subset \mathbb{R}^3$ is an exterior axially symmetric domain of type* (1.2) *with C^2-smooth boundary $\partial\Omega$ and $\mathbf{f} \in W^{1,2}(\Omega)$, $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ are axially symmetric. Let (E-NS) be fulfilled. Then each assumptions* (5.13)*,* (5.14) *lead to a contradiction.*

Proof of Theorem 1.1 Let the hypotheses of Theorem 1.1 be satisfied. Suppose that its assertion fails. Then, by Lemma 3.1, there exist $\mathbf{v}$, p and a sequence $(\mathbf{u}_k, p_k)$ satisfying (E-NS), and by Lemmas 5.10, 5.12 these assumptions lead to a contradiction. □

Remark 5.1 Let in Lemma 3.1 the data **f** and **a** be axially symmetric with no swirl. If the corresponding assertion of Theorem 1.1 fails, then that conditions (E-NS) are satisfied with $\mathbf{u}_k$ axially symmetric with no swirl as well (see Theorem 3.1). But since we have proved that assumptions (E-NS) lead to a contradiction in the more general case (with possible swirl), we get also the validity of second assertions of Theorem 1.1.

6 The case $\mathbf{u}_0 \neq 0$

In the Proof of Theorem 1.1 we have assumed that the assigned value of the velocity at infinity is zero: $\mathbf{u}_0 = 0$. If $\mathbf{u}_0 \neq 0$, we can use the same arguments with some modifications. First, we need some additional facts on Euler equations.

6.1 Some identities for solutions to the Euler system

Let the conditions (E) be fulfilled, i.e., axially symmetric functions $(\mathbf{v}, p)$ satisfy the Euler equations (3.8) and

$$\begin{aligned} &\mathbf{v} \in L^6(\mathbb{R}^3), \quad p \in L^3(\mathbb{R}^3),\\ &\nabla \mathbf{v} \in L^2(\mathbb{R}^3), \quad \nabla p \in L^{3/2}(\mathbb{R}^3), \quad \nabla^2 p \in L^1(\mathbb{R}^3) \end{aligned}$$

(these properties were discussed in Sect. 4).

For a C^1-cycle $\mathcal{S} \subset P_+$ (i.e., $\mathcal{S}$ is a curve homeomorphic to the circle) denote by $\Omega_{\mathcal{S}}$ the bounded domain in $\mathbb{R}^3$ such that $\partial\Omega_{\mathcal{S}} = \widetilde{\mathcal{S}}$, where, recall, $\widetilde{\mathcal{S}}$ means the surface obtained by rotation of the curve $\mathcal{S}$ around the symmetry axis.

Lemma 6.1 *If conditions (E) are satisfied, then for any C^1-cycle $\mathcal{S} \subset P_+$ with $\psi|_{\mathcal{S}} \equiv$ const we have*

$$\int\limits_{\Omega_{\mathcal{S}}} \mathbf{v} \cdot \partial_z \mathbf{v}\, dx = 0.$$

Proof By Bernoulli Law (see Theorem 4.2) we have $\Phi \equiv$ const on $\mathcal{S}$, therefore,

$$\int\limits_{\Omega_{\mathcal{S}}} \partial_z \Phi\, dx = \int\limits_{\Omega_{\mathcal{S}}} \big[\partial_z p + \mathbf{v} \cdot \partial_z \mathbf{v}\big]\, dx = 0.$$

Thus, to finish the proof of the Lemma, we need to check the equality

$$\int\limits_{\Omega_{\mathcal{S}}} \partial_z p \;dx = 0. \tag{6.1}$$

Denote by $\mathcal{D}_{\mathcal{S}}$ the open bounded domain in the half-plane P_+ such that $\partial\mathcal{D}_{\mathcal{S}} = \mathcal{S}$. Of course, $\Omega_{\mathcal{S}} = \widetilde{\mathcal{D}}_{\mathcal{S}}$. Then the required assertion (6.1) could be rewritten in the following form

$$\int_{\mathcal{D}_{\mathcal{S}}} r\partial_z p \, drdz = -\int_{\mathcal{D}_{\mathcal{S}}} r\big[v_r\partial_r v_z + v_z\partial_z v_z\big]\, drdz = 0, \tag{6.2}$$

where we have used the Euler equation (4.1_1) for $\partial_z p$. Since the gradient of the stream function ψ satisfies $\nabla\psi \equiv (-rv_z, rv_r)$, we could rewrite (6.2), using the Coarea formula, in the following equivalent form

$$\int_{\mathbb{R}} dt \int_{\psi^{-1}(t)\cap\mathcal{D}_{\mathcal{S}}} \nabla v_z \cdot \mathbf{l}\, ds = \int_{\mathbb{R}} dt \int_{\psi^{-1}(t)\cap\mathcal{D}_{\mathcal{S}}} \frac{\partial v_z}{\partial s}\, ds = 0, \tag{6.3}$$

where $\mathbf{l} = \frac{1}{|\nabla\psi|}(rv_r, rv_z)$ is the tangent vector to the streamline $\psi^{-1}(t)$. The last equality in (6.3) is evident because almost all level lines of ψ in $\mathcal{D}_{\mathcal{S}}$ are C^1-curves homeomorphic to the circle [see the Morse–Sard Theorem 2.1 (iii)]. The Lemma is proved. □

We need also the following simple technical fact.

Lemma 6.2 *If* $\mathbf{u} = (u_\theta, u_r, u_z)$ *is* C^1*-smooth axially-symmetric vector field in* Ω *with* $\operatorname{div}\mathbf{u} \equiv 0$*, then for any Lipschitz curve* $\mathcal{S} \subset P_+ \cap \Omega$ *such that* $\widetilde{\mathcal{S}}$ *is a compact closed Lipschitz surface the identity*

$$\int_{\widetilde{\mathcal{S}}} \mathbf{n}\cdot\partial_z\mathbf{u}\, dS = 0$$

holds.

Proof There are two possibilities: $\mathcal{S}$ is homeomorphic to the circle, or $\mathcal{S}$ is homeomorphic to the straight segments with endpoints on symmetry axis. Consider the last case (the first case could be done analogously). By identity $\operatorname{div}\mathbf{u} = 0$, we have

$$r\partial_z u_z \equiv -\partial_r(ru_r).$$

Then by direct calculation we have

$$\begin{aligned}
\int_{\widetilde{\mathcal{S}}} \mathbf{n}\cdot\partial_z\mathbf{u}\, dS &= \int_{\mathcal{S}} r\mathbf{n}\cdot\partial_z\mathbf{u}\, ds = \int_{\mathcal{S}} r(-dz, dr)\cdot\partial_z\mathbf{u} \\
&= \int_{\mathcal{S}} \left[-\frac{\partial(ru_r)}{\partial z}dz + \frac{\partial(ru_z)}{\partial z}dr\right] = \int_{\mathcal{S}} \left[-\frac{\partial(ru_r)}{\partial z}dz - \frac{\partial(ru_r)}{\partial r}dr\right] \\
&= \int_{\mathcal{S}} d(ru_r) = 0.
\end{aligned}$$

The Lemma is proved. □

6.2 The existence theorem

Let the hypotheses of Theorem 1.1 be satisfied. Suppose that its assertion fails. Then, as in the first part of Sect. 3, we obtain the sequence of solutions $(\tilde{\mathbf{u}}_k, p_k)$ to problems (3.10) with $\tilde{\mathbf{u}}_k = \mathbf{w}_k + \frac{\nu_k}{\nu}(\mathbf{A} + \mathbf{u}_0)$, where $\mathbf{A} = \mathbf{a} - \mathbf{u}_0$ on $\partial\Omega$, $\mathbf{A} = \boldsymbol{\sigma}$ for sufficiently large $|x|$, $\|\mathbf{w}_k\|_{H(\Omega_{bk})} \equiv 1$, and $\nu_k \to 0$ as $k \to \infty$. Take $\mathbf{u}_k = \mathbf{w}_k + \frac{\nu_k}{\nu}\mathbf{A}$ and note that $\mathbf{u}_k$ is a solution to the Navier–Stokes system

$$\begin{cases} -\nu_k \Delta \mathbf{u}_k + \big(\mathbf{u}_k \cdot \nabla\big)\mathbf{u}_k + \nabla p_k = \mathbf{f}_k + \tilde{\mathbf{f}}_k \quad \text{in} \quad \Omega_{bk}, \\ \qquad\qquad \operatorname{div} \mathbf{u}_k = 0 \quad \text{in} \quad \Omega_{bk}, \\ \qquad\qquad \mathbf{u}_k = \mathbf{a}_k \quad \text{on} \quad \partial\Omega_{bk}, \end{cases} \tag{6.4}$$

with $\Omega_{bk} = \Omega \cap B_k = \{x \in \Omega : |x| < R_k\}$, $\mathbf{f}_k = \frac{\nu_k^2}{\nu^2}\mathbf{f}$, $\mathbf{a}_k = \frac{\nu_k}{\nu}\mathbf{A}$, and

$$\tilde{\mathbf{f}}_k = -\frac{\nu_k}{\nu}(\mathbf{u}_0 \cdot \nabla)\mathbf{u}_k = \nu_k \alpha \partial_z \mathbf{u}_k,$$

where $\alpha \in \mathbb{R}$ is a constant. Since by Hölder inequality $\|\mathbf{f}\|_{L^{3/2}(\Omega_R)} \le \|\mathbf{f}\|_{L^2(\Omega_R)}\sqrt{R}$, and, consequently, $\|\mathbf{f}_k + \tilde{\mathbf{f}}_k\|_{L^{3/2}(\Omega_R)} \le C\sqrt{R}$, we conclude that Corollary 2.1 implies the uniform estimate

$$\|\nabla p_k\|_{L^{3/2}(\Omega_R)} \le C\sqrt{R} \quad \text{for all } R \in [R_0, R_k], \tag{6.5}$$

i.e., the estimate (3.12) holds. The another needed estimate $\|\mathbf{u}_k\|_{L^6(\Omega_{bk})} \le C$ follows from the Sobolev Imbedding Theorem. So, for the total head pressure $\Phi_k = p_k + \frac{1}{2}|\mathbf{u}_k|^2$ we have

$$\|\nabla \Phi_k\|_{L^{3/2}(\Omega_R)} \le C\sqrt{R} \quad \text{for all } R \in [R_0, R_k]. \tag{6.6}$$

By the same reasons as before,

$$\mathbf{u}_k \rightharpoonup \mathbf{v} \text{ in } W^{1,2}_{\text{loc}}(\overline{\Omega}), \quad p_k \rightharpoonup p \text{ in } W^{1,3/2}_{\text{loc}}(\overline{\Omega}), \tag{6.7}$$

$$\nu = \int\limits_{\Omega} (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \mathbf{A}\, dx, \tag{6.8}$$

where the limit functions $(\mathbf{v}, p)$ satisfy the Euler equation with zero boundary conditions, i.e., condition (E) is fulfilled. Our goal is to receive a contradiction.

We need to discuss only the case (5.12), since other cases (5.13)–(5.14) are reduced to the consideration of bounded domains (see Sects. 5.2–5.3) and zero or nonzero condition at infinity has no influence on the proof.

The arguments in Sect. 5 up to the 4th Step of the Proof of Lemma 5.7 could be repeated almost word by word. But the 4th Step of the Proof of Lemma 5.7 needs some modifications. Recall that the main idea there was to use the identity

$$\begin{aligned}\int\limits_{\partial\Omega_k(t)} \nabla\Phi_k \cdot \mathbf{n}\, dS &= \int\limits_{\widetilde{S}_k(t_1,t_2;t)} \nabla\Phi_k \cdot \mathbf{n}\, dS + \int\limits_{S_{R_k}\cup\Gamma_0} \nabla\Phi_k \cdot \mathbf{n}\, dS \\ &= \int\limits_{\Omega_k(t)} \Delta\Phi_k\, dx.\end{aligned} \tag{6.9}$$

Since now

$$\nabla\Phi_k = -\nu_k \,\mathrm{curl}\, \boldsymbol{\omega}_k + \mathbf{u}_k \times \boldsymbol{\omega}_k + \mathbf{f}_k + \tilde{\mathbf{f}}_k, \tag{6.10}$$

$$\Delta\Phi_k = \omega_k^2 + \frac{1}{\nu_k}\mathrm{div}(\Phi_k \mathbf{u}_k) - \frac{1}{\nu_k}\mathbf{f}_k \cdot \mathbf{u}_k - \frac{1}{\nu_k}\tilde{\mathbf{f}}_k \cdot \mathbf{u}_k, \tag{6.11}$$

we need to prove the smallness of the following integrals generated by the additional term $\tilde{\mathbf{f}}_k$:

$$\int\limits_{S_{R_k}\cup\Gamma_0} \tilde{\mathbf{f}}_k \cdot \mathbf{n}\, dS, \tag{6.12}$$

$$\frac{1}{\nu_k}\int\limits_{\Omega_k(t)} \tilde{\mathbf{f}}_k \cdot \mathbf{u}_k\, dx = \alpha \int\limits_{\Omega_k(t)} \mathbf{u}_k \cdot \partial_z \mathbf{u}_k\, dx. \tag{6.13}$$

The difficulty is that the term $\tilde{\mathbf{f}}_k = \nu_k \alpha \partial_z \mathbf{u}_k$ in the right-hand side of (6.4) is not "small enough" (it is of order $O(\nu_k)$ only, not of order $O(\nu_k^2)$ as $\mathbf{f}_k$). However, this term has very good symmetry properties, and by Lemma 6.2 we immediately obtain that the integral (6.12) is negligible.

Corollary 6.1 *The identity*

$$\int\limits_{S_{R_k}\cup\Gamma_0} \tilde{\mathbf{f}}_k \cdot \mathbf{n}\, dS = 0$$

holds.

Let us estimate the integral which is in formula (6.13). We need to use the limit solution of the Euler equations.

For $t \in \mathscr{T}$ denote $S(t) = \bigcup_{j=M+1}^{N} A_j(t)$ (recall that the set $\mathscr{T}$ and the C^1-cycles $A_j(t)$ were defined in the beginning of Sect. 5). Denote by $\Omega_{S(t)}$ the bounded open set in $\mathbb{R}^3$ such that $\partial\Omega_{S(t)} = \widetilde{S}(t)$. Further, put $\Omega'_{S(t)} = \Omega \cap \Omega_{S(t)}$. Convergence (6.7) implies, in particular, that

$$\mathbf{u}_k \to \mathbf{v} \text{ in } L^q_{\mathrm{loc}}(\overline{\Omega}) \quad \text{for any } q \in [1,6). \tag{6.14}$$

Therefore, by Lemma 6.1 we obtain immediately

Lemma 6.3 *For any $t \in \mathscr{T}$ the convergence*

$$\int\limits_{\Omega'_{S(t)}} \mathbf{u}_k \cdot \partial_z \mathbf{u}_k \, dx \to 0$$

holds.

For $t \in \mathscr{T}$ denote $\mu(t) = \operatorname{meas} \Omega'_{S(t)}$. By construction, the function $\mu(t)$ is strictly decreasing, therefore it is continuous on $\mathscr{T}$ except for at most countable set. Removing the discontinuity points from $\mathscr{T}$, we could assume without loss of generality that the function $\mu : \mathscr{T} \to (0, +\infty)$ is continuous. By constructions of Sect. 5, it is easy to see, that if $t_1, \tau_1, t, \tau_2, t_2 \in \mathscr{T}$ and $t_1 < \tau_1 < t < \tau_2 < t_2$, then

$$S_k(t_1, t_2; t) = S_k(\tau_1, \tau_2; t) \quad \text{and} \quad \Omega'_{S(\tau_2)} \subset \Omega'_{S_k(t_1,t_2;t)} \subset \Omega'_{S(\tau_1)} \tag{6.15}$$

for sufficiently large k. Using these facts, the continuity of $\mu(t)$, the convergence (6.14) and Lemma 6.3, we obtain

Lemma 6.4 *For any $t_1, t, t_2 \in \mathscr{T}$ with $t_1 < t < t_2$ the convergence*

$$\int\limits_{\Omega'_{S_k(t_1,t_2,t)}} \mathbf{u}_k \cdot \partial_z \mathbf{u}_k \, dx \to 0$$

holds as $k \to \infty$.

Moreover, since the function $\mu(t)$ is *uniformly* continuous on each compact subset of $\mathscr{T}$, we have by the same reasons that

Lemma 6.5 *For any $\epsilon > 0$ and $t_1, t_2 \in \mathscr{T}$ with $t_1 < t_2$ the convergence*

$$\operatorname{meas}\Big\{t \in (t_1, t_2) : \Big| \int\limits_{\Omega'_{S_k(t_1,t_2,t)}} \mathbf{u}_k \cdot \partial_z \mathbf{u}_k \, dx \Big| > \epsilon \Big\} \to 0$$

holds as $k \to \infty$.

On the other hand, by constructions of Sect. 5 [see, in particular, (5.68)],

$$\Omega_k(t) = \Omega_{bk} \setminus \Omega'_{S_k(t_1,t_2,t)}, \quad \text{where} \quad \Omega_{bk} = \Omega \cap B_k \tag{6.16}$$

and

$$\|\mathbf{u}_k\|_{L^2(\partial \Omega_{bk})} \to 0$$

as $k \to \infty$ [see (6.4$_3$)]. Therefore, for any $t_1, t, t_2 \in \mathscr{T}$ with $t_1 < t < t_2$ we have

$$\int\limits_{\Omega_{bk}} \mathbf{u}_k \cdot \partial_z \mathbf{u}_k \, dx \to 0$$

as $k \to \infty$. This fact together with (6.16) and Lemma 6.5 implies the required assertion:

Lemma 6.6 *For any $\epsilon > 0$ and $t_1, t_2 \in \mathscr{T}$ with $t_1 < t_2$ the convergence*

$$\operatorname{meas}\Big\{t \in (t_1, t_2) : \Big|\int\limits_{\Omega_k(t)} \mathbf{u}_k \cdot \partial_z \mathbf{u}_k \, dx\Big| > \epsilon\Big\} \to 0$$

holds as $k \to \infty$.

Thus, the smallness of the second additional term (6.13) is established.

Now repeating the arguments of Step 4 of the Proof of Lemma 5.7, we obtain its assertion with the following modification

Lemma 6.7 *There exists a constant $\mathcal{F} > 0$ such that for any $t_1, t_2 \in \mathscr{T}$ with $t_1 < t_2$ the convergence*

$$\operatorname{meas}\Big\{t \in (t_1, t_2) : \int\limits_{\widetilde{S}_k(t_1,t_2;t)} |\nabla \Phi_k| \, dS \geq \mathcal{F} t\Big\} \to 0$$

holds as $k \to \infty$.

It is easy to see that Lemma 6.7 allow us to obtain the assertion of Lemma 5.9 (without any changes) with almost the same proof. We need only to define the corresponding set E_k as

$$E_k = \bigcup_{t \in \mathscr{T}_k} \widetilde{S}_k(t_1, t_2; t),$$

where

$$\mathscr{T}_k = \Big\{t \in [t', t''] : \int\limits_{\widetilde{S}_k(t_1,t_2;t)} |\nabla \Phi_k| \, dS \leq \mathcal{F} t\Big\},$$

and to use the fact that, by Lemma 6.7, $\operatorname{meas}\big([t', t''] \setminus \mathscr{T}_k\big) \to 0$ as $k \to \infty$. Therefore, Lemma 6.7 together with Lemma 5.8 give us the required contradiction. This contradiction finishes the proof of the Existence Theorem for the case $\mathbf{u}_0 \neq \mathbf{0}$.

7 Appendix: Lemma from real analysis

We need the following elementary fact.

Lemma 7.1 *Let $f : (0, \delta] \to \mathbb{R}$ be a positive decreasing function. Then*

$$\operatorname*{ess\,sup}_{t \in (0,\delta]} \frac{[f(t)]^{\frac{4}{3}}}{t|f'(t)|} = \infty. \tag{7.1}$$

Proof Recall that by the Lebesgue theorem, the derivative $f'(t)$ exists almost everywhere. Suppose that the assertion (7.1) fails. Then, taking into account that $f'(t) \le 0$, we have

$$-f'(t)[f(t)]^{-\frac{4}{3}} \ge \frac{C}{t} \quad \text{for almost all } t \in (0, \delta], \tag{7.2}$$

with some positive constant C independent of t. Put $g(t) = [f(t)]^{-\frac{1}{3}}$. Then $g(t)$ is positive increasing function on $(0, \delta]$. By the Lebesgue theorem,

$$g(t_2) - g(t_1) \ge \int_{t_1}^{t_2} g'(t)\,dt \tag{7.3}$$

for any pair $t_1, t_2 \in (0, \delta]$ with $t_1 < t_2$. On the other hand, (7.2) implies

$$g'(t) \ge \frac{C}{3t} \quad \text{for almost all } t \in (0, \delta]. \tag{7.4}$$

The estimates (7.3)–(7.4) contradict the boundedness of g: $0 < g(t) \le g(\delta)$ for all $t \in (0, \delta]$. This contradiction finishes the proof. □

Proof of Lemma 5.8 We can extend the function f by one-sided continuity rule to the whole interval $(0, \delta]$ and to apply Lemma 7.1. □

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