Leray's Problem on Existence of Steady State Solutions for the Navier-Stokes Flow

Mikhail V. Korobkov, Konstantin Pileckas, and Remigio Russo

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M.V. Korobkov (✉)
Sobolev Institute of Mathematics, Novosibirsk, Russia

Novosibirsk State University, Novosibirsk, Russia
e-mail: korob@math.nsc.ru

K. Pileckas
Faculty of Mathematics and Informatics, Vilnius University, Vilnius, Lithuania
e-mail: konstantinas.pileckas@mif.vu.lt

R. Russo
Department of Mathematics and Physics, Second University of Naples, Caserta, Italy
e-mail: remigio.russo@unina2.it

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Abstract

This is a survey of results on the Leray problem (1933) for the nonhomogeneous boundary value problem for the steady Navier–Stokes equations in a bounded domain with multiple boundary components. The boundary conditions are assumed only to satisfy the necessary requirement of zero total flux. The authors have proved that the problem is solvable in arbitrary bounded planar or three-dimensional axially symmetric domains. The proof uses Bernoulli's law for weak solutions of the Euler equations and a generalization of the Morse–Sard theorem for functions in Sobolev spaces. Similar existence results (without any restrictions on fluxes) are proved for steady Navier–Stokes system in two- and three-dimensional exterior domains with multiply connected boundary under assumptions of axial symmetry. In particular, it was shown that in domains with two axes of symmetry and for symmetric boundary datum, the two-dimensional exterior problem has a symmetric solution vanishing at infinity.

1 Introduction

The paper deals with bounded and exterior domains $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with C^2-smooth boundary $\partial\Omega = \cup_{j=0}^{N}\Gamma_j$ consisting of $N + 1$ disjoint components Γ_j, i.e.,

$$\Omega = \Omega_0 \setminus \left(\bigcup_{j=1}^{N} \bar{\Omega}_j\right), \quad \bar{\Omega}_j \subset \Omega_0,\ j = 1, \dots, N, \tag{1}$$

where Ω_j, $j = 1, \dots, N$, are bounded domains, $\Gamma_j = \partial\Omega_j$. In the case of exterior domains, $\Omega_0 = \mathbb{R}^n$. Consider in Ω the stationary Navier–Stokes system with the nonhomogeneous boundary conditions

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega. \end{cases} \tag{2}$$

In (2) as usual, $\nu > 0$ is the kinematical viscosity coefficient, $\mathbf{u}$, p are the (unknown) velocity and pressure fields, and $\mathbf{a}$ and $\mathbf{f}$ are the (assigned) boundary value and the body force density, respectively.

The continuity equation (2_2) implies the necessary compatibility condition for the solvability of problem (2):

$$\int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\, dS = \sum_{j=0}^{N} \int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\, dS = \sum_{j=0}^{N} \mathcal{F}_j = 0, \tag{3}$$

where $\mathbf{n}$ is a unit vector of the outward (with respect to Ω) normal to $\partial\Omega$. The compatibility condition (3) means that the total flux of the fluid over the boundary $\partial\Omega$ is zero.

Starting from the famous paper of J. Leray [49] published in 1933, problem (2) has been studied in many papers. However, only recently [36, 37, 41] the solvability of the problem (2) was proved in bounded $2D$ domains and for the axially symmetric $3D$ case under the sole necessary condition (3). In all previous papers the existence of a weak solution $\mathbf{u} \in W^{1,2}(\Omega)$ to problem (2) was proved only either under the stronger condition that all fluxes $\mathcal{F}_j$ of the boundary value $\mathbf{a}$ are equal to zero separately across each boundary component Γ_j:

$$\mathcal{F}_j = \int_{\Gamma_j} \mathbf{a} \cdot \mathbf{n} \, dS = 0, \qquad j = 0, 2, \dots, N, \tag{4}$$

(see, e.g., [30, 46, 47, 49, 77]), or for sufficiently small fluxes $\mathcal{F}_j$ (see, e.g., [5, 16, 18, 20, 21, 44, 65, 66]), or under certain symmetry conditions on the domain Ω and the boundary value $\mathbf{a}$ (e.g., [2, 14, 19, 52, 63, 64, 70]). (Note that condition (4) does not allow the presence of sinks and sources; further, the condition of smallness of fluxes $\mathcal{F}_j$ does not imply the norm of the boundary value $\mathbf{a}$ to be small.)

Recently the stationary Navier–Stokes problem with nonhomogeneous boundary conditions was also studied in domains with non-compact boundaries. In 1999 [57] this problem was solved in an infinite layer on the bottom of which is a compactly supported sink or source of an arbitrary intensity (without any the smallness assumption on the flux). Later in 2010 [59, 60], the problem was considered in unbounded domains Ω with multiply connected boundaries assuming the "smallness" assumption on fluxes of the boundary value $\mathbf{a}$ over bounded connected components of the boundary and does not impose any restrictions on fluxes over infinite parts of $\partial\Omega$. In [59, 60] only solutions with finite Dirichlet integral were studied, what impose some restrictions on the geometry of the flow domain. In [31, 32] these results were extended to a class of solutions which can have infinite Dirichlet integral in domains $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, having paraboloidal and layer type outlets to infinity. In [53–56] the stationary Navier–Stokes problem was studied in symmetric two-dimensional multiply connected domains Ω with channel-like outlets to infinity containing a finite number of "holes." Under certain symmetry assumptions on the domain and the boundary value $\mathbf{a}$, and assuming that $\mathbf{a}$ is equal to zero on the infinite outer boundary, the authors proved that there exists a solution which tends in every channel to a corresponding Poiseuille flow. Notice that in these results the fluxes of $\mathbf{a}$ over the boundary of each "hole" may be arbitrarily large, but the sum of them has to be equal to the flux of the corresponding Poiseuille flow which needs to be sufficiently small. Finally, in [9] the problem was solved in a symmetric domain $\Omega \subset \mathbb{R}^2$ with either a paraboloidal or a channel-like outlet to infinity assuming that the boundary value $\mathbf{a}$ is a symmetric function. The boundary value $\mathbf{a}$ could be nonzero on the outer boundary, and there are no restrictions on the size of the fluxes over both the inner and the outer boundaries. Furthermore, the solution may have infinite Dirichlet integral and does not oblige to tend (in the case of channel-like outlets) to the Poiseuille flow. Therefore, the restriction that the sum of fluxes has to be small is also relaxed.

Since the study of the existence of a solution to (2) for not small fluxes originates from Leray's paper [49], the question whether (2) is solvable in an arbitrary domain with a multiply connected boundary under the sole condition (3) is today known as **Leray's problem.**

In this paper the results concerning Leray's problem in bounded and exterior two-dimensional and axially symmetric three-dimensional domains are presented. The presentation is based on results obtained by authors in [36]–[43].

The standard notation for Lebesgue and Sobolev spaces, $L^q(\Omega)$ and $W^{k,q}(\Omega)$, $W^{k,q}_{loc}(\Omega)$, $\overset{\circ}{W}{}^{k,q}(\Omega)$, are used in the paper; $W^{k-1/q,q}(\partial\Omega)$ is the trace space on $\partial\Omega$ of functions from $W^{k,q}(\Omega)$. $C_0^\infty(\Omega)$ denotes the set of all infinitely differentiable functions with compact support in Ω. $D(\Omega)$ is the Hilbert space of vector-valued functions formed as the closure of $C_0^\infty(\Omega)$ with respect to the Dirichlet norm $\|\mathbf{u}\|_{D(\Omega)} = \|\nabla\mathbf{u}\|_{L^2(\Omega)}$ induced by the scalar product

$$[\mathbf{u},\mathbf{v}] = \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx, \tag{5}$$

where $\nabla\mathbf{u}\cdot\nabla\mathbf{v} = \sum\limits_{j=1}^n \nabla u_j\cdot\nabla v_j = \sum\limits_{j=1}^n\sum\limits_{k=1}^n \frac{\partial u_j}{\partial x_k}\frac{\partial v_j}{\partial x_k}$.

Denote by $J_0^\infty(\Omega)$ the set of all divergence-free (div $\mathbf{u}=0$) vector fields $\mathbf{u}$ from $C_0^\infty(\Omega)$ and by $H(\Omega)$ the space formed as the closure of $J_0^\infty(\Omega)$ with respect to the Dirichlet norm.

2 On Two Leray's Approaches

Let us describe shortly the history of the topic. Let Ω be a bounded domain. *A weak solution* of the problem (2) is a vector field $\mathbf{u}$ such that $\mathbf{w}=\mathbf{u}-\mathbf{A}\in H(\Omega)$ satisfies the integral identity

$$\begin{aligned}\nu\int_\Omega \nabla\mathbf{w}\cdot\nabla\boldsymbol{\eta}\,dx &= -\nu\int_\Omega \nabla\mathbf{A}\cdot\nabla\boldsymbol{\eta}\,dx + \int_\Omega \big(\mathbf{A}\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{A}\,dx\\ &+\int_\Omega \big(\mathbf{A}\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{w}\,dx + \int_\Omega \big(\mathbf{w}\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{w}\,dx\\ &+\int_\Omega \big(\mathbf{w}\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{A}\,dx + \int_\Omega \mathbf{f}\cdot\boldsymbol{\eta}\,dx\end{aligned} \tag{6}$$

for any $\boldsymbol{\eta}\in H(\Omega)$. Here, $\mathbf{A}\in W^{1,2}(\Omega)$ is a solenoidal extension of the boundary data $\mathbf{a}\in W^{1/2,2}(\partial\Omega)$.

The integral identity (6) can be reduced to an operator equation in the Hilbert space $H(\Omega)$:

$$\mathbf{w} = \mathbf{T}\mathbf{w}, \tag{7}$$

where the compact operator T is defined by the equality

$$[\mathrm{T}\mathbf{w},\ \boldsymbol{\eta}] = \nu^{-1}\int\limits_{\Omega}((\mathbf{w}+\mathbf{A})\cdot\nabla)\,\boldsymbol{\eta}\cdot\mathbf{w}\,dx + \nu^{-1}\int\limits_{\Omega}(\mathbf{w}\cdot\nabla)\,\boldsymbol{\eta}\cdot\mathbf{A}\,dx$$

$$-\int\limits_{\Omega}\nabla\mathbf{A}\cdot\nabla\boldsymbol{\eta}\,dx + \nu^{-1}\int\limits_{\Omega}(\mathbf{A}\cdot\nabla)\,\boldsymbol{\eta}\cdot\mathbf{A}\,dx + \nu^{-1}\int\limits_{\Omega}\mathbf{f}\cdot\boldsymbol{\eta}\,dx \quad \forall\ \mathbf{w},\boldsymbol{\eta}\in H(\Omega)$$

(see (5) for the definition of the inner product $[\cdot,\cdot]$). The existence of a fixed point to (7) follows from the Leray–Schauder Theorem (e.g., [21, 46, 49]). In order to apply the Leray–Schauder theorem, one needs an a priori estimate of solutions to the operator equation with a parameter λ:

$$\mathbf{w}^{(\lambda)} = \lambda\mathrm{T}\mathbf{w}^{(\lambda)}, \qquad \lambda\in[0,\ 1]. \tag{8}$$

In [49] J. Leray introduced two different approaches to get this estimate. The first method uses a special extension of the boundary value $\mathbf{a}$ into Ω as $\mathbf{A}(\varepsilon,x) = \operatorname{curl}\big(\zeta(\varepsilon,x)\mathbf{b}(x)\big)$, where $\zeta(\varepsilon,x)$ is the so-called Hopf's cutoff function [29].

For such extension the following estimate

$$-\int\limits_{\Omega}\big(\mathbf{v}\cdot\nabla\big)\mathbf{A}\cdot\mathbf{v}\,dx \le \varepsilon c\int\limits_{\Omega}|\nabla\mathbf{v}|^2\,dx \quad \forall\ \ \mathbf{v}\in \mathring{W}^{1,2}(\Omega), \tag{9}$$

is valid (e.g., [47]), where $\varepsilon > 0$ can be taken arbitrary small and the constant c is independent of ε. This allows to prove the required a priori estimate for the solutions. Indeed, taking in (6) $\boldsymbol{\eta} = \mathbf{w}$ and using (9), one gets

$$\begin{aligned}\nu\int\limits_{\Omega}|\nabla\mathbf{w}|^2\,dx &= -\lambda\nu\int\limits_{\Omega}\nabla\mathbf{A}\cdot\nabla\mathbf{w}\,dx - \lambda\int\limits_{\Omega}\big(\mathbf{A}\cdot\nabla\big)\mathbf{A}\cdot\mathbf{w}\,dx\\ &-\lambda\int\limits_{\Omega}\big(\mathbf{w}\cdot\nabla\big)\mathbf{A}\cdot\mathbf{w}\,dx + \lambda\int\limits_{\Omega}\mathbf{f}\cdot\mathbf{w}\,dx\\ &\le \varepsilon c\int\limits_{\Omega}|\nabla\mathbf{w}|^2\,dx + c_\varepsilon\Big(\int\limits_{\Omega}|\mathbf{f}|^2dx + \int\limits_{\Omega}|\nabla\mathbf{A}|^2\,dx + \int\limits_{\Omega}|\mathbf{A}|^4\,dx\Big).\end{aligned} \tag{10}$$

If $\varepsilon c < \nu$, from (10), it follows that

$$\int\limits_{\Omega}|\nabla\mathbf{w}|^2\,dx \le c_\varepsilon\left(\int\limits_{\Omega}|\mathbf{f}|^2dx + \int\limits_{\Omega}|\nabla\mathbf{A}|^2\,dx + \int\limits_{\Omega}|\mathbf{A}|^4\,dx\right). \tag{11}$$

Usually (9) is called Leray–Hopf's inequality. It is well known that our boundary value can be extended into the domain as a curl only if the condition (4) is satisfied.

The counterexamples in [27, 76], and [15] (see also [21]) show that if the net flux across some component of the boundary is nonzero, then for arbitrary domain Ω, it is impossible, in general, to extend the boundary value $\mathbf{a}$ in any manner as a solenoidal function $\mathbf{A}$ satisfying Leray–Hopf's inequality (9). Thus, this approach may be applied only when condition (4) is satisfied.

The second approach in Leray's paper [49] is to prove an a priory estimate by a contradiction. Such arguments can be found also in the book of O.A. Ladyzhenskaya [47]. In [2] the solvability of (2) was proved using this method for arbitrary fluxes $\mathcal{F}_j$ assuming only the necessary condition (3). However, the problem was studied for a special class of plane symmetric domains and symmetric boundary values. An effective estimate for the solution of the Navier–Stokes problem with the above symmetry conditions was first obtained by L.I. Sazonov [70], who constructed a symmetric extension of the boundary data satisfying Leray–Hopf's inequality. Analogous results were independently obtained by H. Fujita [19] (see also [52]), who called the proposed method "virtual drains method."

Let us describe the second approach of getting an a priory estimate by a contradiction. Let $\mathbf{A} \in W^{1,2}(\Omega)$ be an arbitrary solenoidal extension of the boundary value $\mathbf{a}$. It is necessary to show that the norms of all possible solutions $\mathbf{w}^{(\lambda)}$ of the operator equation (8) are uniformly bounded by a constant independent of $\lambda \in [0,1]$. Suppose that this is false. Then there exist sequences $\{\lambda_k\}_{k\in\mathbb{N}} \subset [0,1]$ and $\{\widehat{\mathbf{w}}_k = \widehat{\mathbf{w}}^{(\lambda_k)}\}_{k\in\mathbb{N}} \in H(\Omega)$ such that

$$\nu \int\limits_{\Omega} \nabla\widehat{\mathbf{w}}_k \cdot \nabla\boldsymbol{\eta}\, dx - \lambda_k \int\limits_{\Omega} \big((\widehat{\mathbf{w}}_k + \mathbf{A})\cdot\nabla\big)\boldsymbol{\eta}\cdot\widehat{\mathbf{w}}_k\, dx - \lambda_k \int\limits_{\Omega} \big(\widehat{\mathbf{w}}_k\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{A}\, dx$$

$$= -\lambda_k \nu \int\limits_{\Omega} \nabla\mathbf{A}\cdot\nabla\boldsymbol{\eta}\, dx + \lambda_k \int\limits_{\Omega} \big(\mathbf{A}\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{A}\, dx + \lambda_k \int\limits_{\Omega} \mathbf{f}\cdot\boldsymbol{\eta}\, dx \quad \forall\, \boldsymbol{\eta} \in H(\Omega), \tag{12}$$

and

$$\lim_{k\to\infty} \lambda_k = \lambda_0 \in [0,1], \qquad \lim_{k\to\infty} J_k = \lim_{k\to\infty} \|\widehat{\mathbf{w}}_k\|_{H(\Omega)} = \infty. \tag{13}$$

Set $\mathbf{w}_k = J_k^{-1}\widehat{\mathbf{w}}_k$. Since $\|\mathbf{w}_k\|_{H(\Omega)} = 1$, there exists a subsequence $\{\mathbf{w}_{k_l}\}$ which weakly converges in $H(\Omega)$ to some vector field $\mathbf{v} \in H(\Omega)$. By compactness of the imbedding $H(\Omega) \hookrightarrow L^r(\Omega)\ \ \forall\, r \in [1,\infty)$ for $n = 2$, and $\forall\, r \in [1,6)$ for $n = 3$, the subsequence $\{\mathbf{w}_{k_l}\}$ converges strongly in $L^r(\Omega)$. Taking in (12) $\boldsymbol{\eta} = J_{k_l}^{-1}\mathbf{w}_{k_l}$ and passing in the obtained equality to the limit as $k_l \to \infty$ yield

$$\nu = \lambda_0 \int\limits_{\Omega} \big(\mathbf{v}\cdot\nabla\big)\mathbf{v}\cdot\mathbf{A}\, \mathrm{d}x. \tag{14}$$

In particular it follows from (14) that $\lambda_0 > 0$. Hence, λ_k are separated from zero.

Now take in (12) $\boldsymbol{\eta} = J_{k_l}^{-2}\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is an arbitrary vector field from $H(\Omega)$. Passing again to the limit as $k_l \to \infty$ yields the integral identity

$$\int_\Omega \big(\mathbf{v}\cdot\nabla\big)\mathbf{v}\cdot\boldsymbol{\xi}\,dx = 0 \quad \forall\boldsymbol{\xi}\in H(\Omega). \tag{15}$$

Hence, $\mathbf{v}\in H(\Omega)$ is a weak solution of the Euler equation

$$\begin{cases} \big(\mathbf{v}\cdot\nabla\big)\mathbf{v} + \nabla p = 0, & x\in\Omega,\\ \operatorname{div}\ \mathbf{v} = 0, & x\in\Omega,\\ \mathbf{v} = 0, & x\in\partial\Omega. \end{cases} \tag{16}$$

The function p in (16) belongs to the space $W^{1,s}(\Omega)$, where $s\in[1,2)$ for $n=2$ and $s\in[1,3/2]$ for $n=3$. Since $\mathbf{v}=0$ on $\partial\Omega$, it can be proved, using the equations (16), that the pressure p is equal to some constants $\hat{p}_j$ on the connected components Γ_j of the boundary $\partial\Omega$. More precisely, it was proved in [30, Lemma 4] and independently in [2, Theorem 2.2] that the following equalities

$$p(x)|_{\Gamma_i} = \hat{p}_i, \quad \hat{p}_i\in\mathbb{R}, \quad j = 0,1,\dots,N. \tag{17}$$

hold.

Multiply the Euler system (16) by $\mathbf{A}$ and integrate the obtained equality over Ω. Integrating by parts and using (17), we obtain

$$\int_\Omega \big(\mathbf{v}\cdot\nabla\big)\mathbf{v}\cdot\mathbf{A}\,dx = -\int_{\partial\Omega} p\,\mathbf{A}\cdot\mathbf{n}\,dS = -\sum_{i=0}^{N}\widehat{p}_i\int_{\Gamma_i}\mathbf{A}\cdot\mathbf{n}\,dS = -\sum_{i=0}^{N}\hat{p}_i\mathcal{F}_i. \tag{18}$$

If $N=0$ or $\mathcal{F}_i=0,\ i=0,1,\dots,N$ (the condition (4) is satisfied), then (18) gives

$$\int_\Omega \big(\mathbf{v}\cdot\nabla\big)\mathbf{v}\cdot\mathbf{A}\,dx = 0. \tag{19}$$

The last relation contradicts (14). Therefore, the assumption is wrong and the norms of all possible solutions $\mathbf{w}^{(\lambda)}$ to the operator equation (8) are uniformly bounded with respect to $\lambda\in[0,1]$. Thus, by the Leray–Schauder theorem, equation (7) has at least one solution.

An analogous conclusion is obtained when all constants $\widehat{p}_j$ are equal:

$$\hat{p}_0 = \hat{p}_1 = \ldots = \hat{p}_N. \tag{20}$$

Indeed, in virtue of (3),

$$\sum_{i=0}^{N} \hat{p}_i \mathcal{F}_i = \widehat{p}_0 \sum_{i=0}^{N} \mathcal{F}_i = 0,$$

and from (18) again follows (19).

However, in the general case, one cannot claim that all constants $\hat{p}_i$ are equal. Amick [2] exhibited a solution to problem (16), for which equalities (20) are not valid. Let $\Omega = \{x \in \mathbb{R}^2 : 1 < |x| < 2\}$ be annulus on the plane, $\psi \in C^1([1,2])$, $\psi'(1) = \psi'(2) = 0$, and $\psi'' \in L^2((1,2))$. A solution of the Euler problem (16) is defined by

$$\mathbf{v}(x) = \left(\frac{x_2}{|x|}\psi'(|x|), -\frac{x_1}{|x|}\psi'(|x|)\right) \in H(\Omega), \quad p(x) = \int_1^{|x|} \frac{|\psi'(s)|^2}{s}\, ds. \tag{21}$$

It is easy to see that $p(x)|_{|x|=1} = 0$, and $p(x)|_{|x|=2} = \int_1^2 \frac{|\psi'(s)|^2}{s}\, ds > 0$.

3 An Existence Theorem in the General Planar Case

In this section the problem (2) is studied in the general case. For the two-dimensional domains, the result reads as follows.

Theorem 1. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^2-smooth boundary $\partial\Omega$. If $\mathbf{f} \in W^{1,2}(\Omega)$ and $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ satisfies condition (3), then problem (2) admits at least one weak solution $\mathbf{u}$.*

Remark 1. It is well known (see [47]) that under the hypotheses of Theorem 1, every weak solution $\mathbf{u}$ of problem (2) is more regular, i.e., $\mathbf{u} \in W^{2,2}(\Omega) \cap W^{3,2}_{\mathrm{loc}}(\Omega)$. Generally speaking, the solution is as regular as the data allow; in particular, $\mathbf{u}$ is C^∞-smooth when $\mathbf{f}$, $\mathbf{a}$, and $\partial\Omega$ are C^∞-smooth.

Similar result holds for the $3D$ axially symmetric case (see Theorem 6). Moreover, for the axially symmetric case, also the existence theorem for an exterior domain could be proved (see Theorem 7).

Below (Sect. 3.4) the main ideas of the proof of Theorem 1 are shown. In order to make it easer, consider the case when $\partial\Omega$ has only two connected components of the boundary and assume that $\mathbf{f} = 0$.

Some needed auxiliary results are formulated in the subsections below.

3.1 Properties of Sobolev Functions and an Analog of the Morse–Sard Theorem for Functions from $W^{2,1}(\mathbb{R}^2)$

Recall some classical differentiability properties of Sobolev functions. Working with such functions, we always assume that the "best representatives" are chosen. If $w \in L^1_{\mathrm{loc}}(\Omega)$, then the best representative w^* is defined by

$$w^*(x) = \begin{cases} \lim\limits_{r\to 0} \fint_{B_r(x)} w(z)dz, & \text{if the finite limit exists;} \\ 0 & \text{otherwise,} \end{cases}$$

where $\fint_{B_r(x)} w(z)dz = \frac{1}{\mathrm{meas}(B_r(x))} \int_{B_r(x)} w(z)dz$, $B_r(x) = \{y : |y-x| < r\}$ is a ball of radius r centered at x.

Lemma 1 (see Proposition 1 in [12]). *Let $\psi \in W^{2,1}(\mathbb{R}^2)$. Then the function ψ is continuous, and there exists a set A_ψ such that $\mathfrak{H}^1(A_\psi) = 0$ and the function ψ is differentiable (in the classical sense) at each $x \in \mathbb{R}^2 \setminus A_\psi$. Furthermore, the classical derivative at such points x coincides with $\nabla\psi(x) = \lim\limits_{r\to 0} \fint_{B_r(x)} \nabla\psi(z)dz$, and $\lim\limits_{r\to 0} \fint_{B_r(x)} |\nabla\psi(z) - \nabla\psi(x)|^2 dz = 0$.*

Here and henceforth, denote by $\mathfrak{H}^1$ the one-dimensional Hausdorff measure, i.e., $\mathfrak{H}^1(F) = \lim\limits_{t\to 0+} \mathfrak{H}^1_t(F)$, where $\mathfrak{H}^1_t(F) = \inf\left\{ \sum\limits_{i=1}^{\infty} \mathrm{diam} F_i \,:\, \mathrm{diam} F_i \le t, F \subset \bigcup\limits_{i=1}^{\infty} F_i \right\}$. It is well known that for functions $w \in W^{1,q}_{\mathrm{loc}}(\Omega)$, $\Omega \subset \mathbb{R}^2$, $\mathfrak{H}^1$-almost all points $x \in \Omega$ are the Lebesgue points, i.e., the above limit exists $\mathfrak{H}^1$-almost everywhere in Ω.

The next theorem has been proved recently by J. Bourgain, M. Korobkov, and J. Kristensen [6] (see also [7, 35] for multidimensional case). The statement (i) of this theorem is the analog for Sobolev functions of the classical Morse–Sard Theorem.

Theorem 2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and $\psi \in W^{2,1}(\Omega)$. Then*

(i) $\mathfrak{H}^1(\{\psi(x) : x \in \overline{\Omega} \setminus A_\psi \;\&\; \nabla\psi(x) = 0\}) = 0$;
(ii) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathfrak{H}^1(\psi(U)) < \varepsilon$ for any set $U \subset \overline{\Omega}$ with $\mathfrak{H}^1_\infty(U) < \delta$; in particular, $\mathfrak{H}^1(\psi(A_\psi)) = 0$;*
(iii) *for every $\varepsilon > 0$, there exists an open set $V \subset \mathbb{R}$ with $\mathfrak{H}^1(V) < \varepsilon$ and a function $g \in C^1(\mathbb{R}^2)$ such that for each $x \in \overline{\Omega}$ if $\psi(x) \notin V$, then $x \notin A_\psi$ and $\psi(x) = g(x)$, $\nabla\psi(x) = \nabla g(x) \neq 0$;*
(iv) *for $\mathfrak{H}^1$–almost all $y \in \psi(\overline{\Omega}) \subset \mathbb{R}$, the preimage $\psi^{-1}(y)$ is a finite disjoint family of C^1-curves S_j, $j = 1, 2, \ldots, N(y)$. Each S_j is either a cycle in Ω (i.e., $S_j \subset \Omega$ is homeomorphic to the unit circle $\mathbb{S}^1$) or a simple arc with endpoints on $\partial\Omega$ (in this case S_j is transversal to $\partial\Omega$).*

3.2 Some Facts from Topology

Below some topological definitions and results will be needed. By *continuum* we mean a compact connected set. The connectedness is understood in the sense of general topology. A subset of a topological space is called *an arc* if it is homeomorphic to the unit interval $[0,1]$. A locally connected continuum T is called *a topological tree*, if it does not contain a curve homeomorphic to a circle or, equivalently, if any two different points of T can be joined by a unique arc. This definition implies that T has topological dimension 1. A point $C \in T$ is *an endpoint of* T (resp., *a branching point of* T), if the set $T \setminus \{C\}$ is connected (resp., if $T \setminus \{C\}$ has more than two connected components).

Let us shortly present some results from the classical paper of A.S. Kronrod [45] concerning level sets of continuous functions. Let $Q = [0,1] \times [0,1]$ be a square in $\mathbb{R}^2$, and let f be a continuous function on Q. Denote by E_t a level set of the function f, i.e., $E_t = \{x \in Q : f(x) = t\}$. A connected component K of the level set E_t containing a point x_0 is a maximal connected subset of E_t containing x_0. By T_f denote a family of all connected components of level sets of f. It was established in [45] that T_f equipped by a natural topology is a one-dimensional topological tree. (The convergence in T_f is defined by the following rule: $T_f \ni C_i \to C$ iff $\sup\limits_{x \in C_i} \operatorname{dist}(x, C) \to 0$.) Endpoints of this tree are the components $C \in T_f$ which do not separate Q, i.e., $Q \setminus C$ is a connected set. Branching points of the tree are the components $C \in T_f$ such that $Q \setminus C$ has more than two connected components (see [45, Theorem 5]). By results of [45, Lemma 1], see also [51] and [62], the set of all branching points of T_f is at most countable. The main property of a tree is that any two points could be joined by a unique arc. Therefore, the same is true for T_f.

Lemma 2 (see Lemma 13 in [45]). *If $f \in C(Q)$, then for any two different points $A \in T_f$ and $B \in T_f$, there exists a unique arc $J = J(A,B) \subset T_f$ joining A to B. Moreover, for every inner point C of this arc, the points A, B lie in different connected components of the set $T_f \setminus \{C\}$.*

Remark 2. The assertion of Lemma 2 remains valid for level sets of continuous functions $f : \overline{\Omega} \to \mathbb{R}$, where Ω is a multi-connected bounded domain of type (1), provided $f \equiv \xi_j = \text{const}$ on each inner boundary component Γ_j with $j = 1, \dots, N$. Indeed, f can be extended to the whole $\overline{\Omega}_0$ by putting $f(x) = \xi_j$ for $x \in \overline{\Omega}_j$, $j = 1, \dots, N$. The extended function f will be continuous on the set $\overline{\Omega}_0$ which is homeomorphic to the unit square $Q = [0,1]^2$.

3.3 Euler Equation

Most of the results of this section are obtained under the following assumptions.

(E) *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain of type* (1) *with Lipschitz boundary. Assume that* $\mathbf{v} \in W^{1,2}(\Omega)$ *and* $p \in W^{1,s}(\Omega)$, $s \in [1,2)$, *satisfy the Euler equations*

$$\begin{cases} (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0, \\ \operatorname{div}\mathbf{v} = 0, \end{cases} \tag{22}$$

for almost all $x \in \Omega$, *and let*

$$\int\limits_{\Gamma_i} \mathbf{v}\cdot\mathbf{n}\,ds = 0, \quad i = 0,1,\dots,N, \tag{23}$$

where Γ_i *are connected components of the boundary* $\partial\Omega$.

If instead of (23) the solution $\mathbf{v}$ satisfies the homogeneous boundary conditions

$$\mathbf{v}|_{\Gamma_i} = 0, \quad i = 0,1,\dots,N, \tag{24}$$

it will be said that $\mathbf{v}$ satisfies *the condition* $E_\circ$.

Under the conditions (E), it is easy to see that there exists a stream function $\psi \in W^{2,2}(\Omega)$ such that $\nabla\psi = (-v_2, v_1)$ (note that by the Sobolev embedding theorem, ψ is continuous in $\overline{\Omega}$). Denote by $\Phi = p + \frac{|\mathbf{v}|^2}{2}$ the total head pressure corresponding to the solution $(\mathbf{v}, p)$. Obviously, $\Phi \in W^{1,s}(\Omega)$ for all $s \in [1,2)$. By direct calculations one easily gets the identity

$$\nabla\Phi \equiv \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\left(v_2, -v_1\right) = \omega\nabla\psi \quad \text{ in } \Omega, \tag{25}$$

where ω denotes the corresponding vorticity: $\omega = \partial_2 v^1 - \partial_1 v^2 = \Delta\psi$. Since the stream lines in our case coincide with the level sets of ψ, from (25), in the case of smooth functions ψ, Φ, the classical Bernoulli law follows immediately:

The total head pressure Φ *is constant along any stream line.*

But the Sobolev case is more delicate: now the stream function $\psi \in W^{2,1}(\Omega)$ is not C^1-smooth, and the total head pressure Φ belongs to the spaces $W^{1,q}(\Omega)$ with $q < 2$, but functions of this space need not to be continuous and they are well defined everywhere except for some "bad" set of $\mathfrak{H}^1$-measure zero (see, e.g., Theorem 1 of §4.8 and Theorem 2 of §4.9.2 in [13]). So the formulation of the Bernoulli law for solutions in Sobolev spaces has to be modulo negligible "bad" set $A_{\mathbf{v}}$ of one dimensional Hausdorff measure zero. Such version of Bernoulli's law was obtained in [34, Theorem 1] (see also [36, Theorem 3.2] for a more detailed proof).

Theorem 3 (The Bernoulli law). *Assume the conditions (E). Then there exists a set* $A_{\mathbf{v}}$ *with* $\mathfrak{H}^1(A_{\mathbf{v}}) = 0$ *such that any point* $x \in \overline{\Omega} \setminus A_{\mathbf{v}}$ *is a Lebesgue point for* $\mathbf{v}$, Φ, *and for every compact connected set* $K \subset \overline{\Omega}$, *the following property holds: if*

$$\psi\big|_K = \text{const}, \tag{26}$$

then

$$\Phi(x_1) = \Phi(x_2) \quad \text{for all } x_1, x_2 \in K \setminus A_{\mathbf{v}}. \tag{27}$$

(Here, in order to define a Lebesgue point at $x \in \partial\Omega$, the usual Sobolev extension of $\mathbf{v}$, Φ to the whole $\mathbb{R}^2$ is considered.)

Remark 3. In particular, if $\mathbf{v} = 0$ on $\partial\Omega$ (in the sense of trace), then by the Morse–Sard Theorem 2, there exist constants $\xi_0, \dots, \xi_N \in \mathbb{R}$ such that $\psi(x) \equiv \xi_j$ on each component Γ_j, $j = 0, \dots, N$ (Indeed, if $\mathbf{v} = 0$ on $\partial\Omega$, then $\nabla\psi = 0$ on $\partial\Omega$, and by Theorem 2 (i)–(ii), the image $\psi(\partial\Omega)$ has zero $\mathfrak{H}^1$-measure. This implies, by continuity of ψ, that $\psi \equiv$ const on each connected subset of $\partial\Omega$.). Therefore, by the above Bernoulli law, the pressure $p(x)$ is constant on $\partial\Omega$. Note that $p(x)$ could take different constant values $\widehat{p}_j = p(x)\big|_{\Gamma_j}$ on different connected components Γ_j of the boundary $\partial\Omega$. This fact was already mentioned in Sect. 2 (see example (21)).

Using the assertion of Remark 3, one could prove the following regularity result for the pressure.

Theorem 4. *Let the conditions* $(E_\circ)$ *be satisfied. Then*

$$p \in C(\overline{\Omega}) \cap W^{2,1}(\Omega). \tag{28}$$

The proof of this theorem is based on the div–curl lemma with two cancelations (e.g., [10, Theorem II.1]) and classical results concerning the Poisson equation (see, e.g., [48, Chapter II]).

Under $(E_\circ)$-conditions by Remarks 2 and 3, one can apply Kronrod's results to the stream function ψ. Define the total head pressure on the Kronrod tree T_ψ (see Sect. 3.2) as follows. Let $K \in T_\psi$ with diam $K > 0$. Take any $x \in K \setminus A_{\mathbf{v}}$ and put $\Phi(K) = \Phi(x)$. This definition is valid by Bernoulli's law (see Theorem 3).

Lemma 3. *Assume that the conditions* $(E_\circ)$ *are satisfied. Let* $A, B \in T_\psi$, diam $A >$ 0, diam $B > 0$. *Consider the corresponding arc* $[A, B] \subset T_\psi$ *joining* A *to* B *(see Lemma 2). Then the restriction* $\Phi|_{[A,B]}$ *is a continuous function.*

Remark 4. The continuity of $\Phi|_{[A,B]}$ was proved in [41, Lemma 3.5]. The proof relies on the fact that each Sobolev function is continuous (in classical sense) on almost all straight line. Note that the total head pressure $\Phi(x)$ itself is not necessary continuous function in the whole Ω since about the velocity field $\mathbf{v}$, it is only known that $\mathbf{v} \in W^{1,2}(\Omega)$.

For $x \in \overline{\Omega}$ denote by K_x the connected component of the level set $\{z \in \overline{\Omega} : \psi(z) = \psi(x)\}$ containing the point x. Under $(E_\circ)$-conditions by Remark 3, $K_x \cap$

$\partial\Omega = \emptyset$ for every $y \in \psi(\overline{\Omega}) \setminus \{\xi_0, \ldots, \xi_N\}$ and for every $x \in \psi^{-1}(y)$. Thus, Theorem 2 (ii), (iv) implies that for almost all $y \in \psi(\overline{\Omega})$ and for every $x \in \psi^{-1}(y)$, the equality $K_x \cap A_{\mathbf{v}} = \emptyset$ holds, and the component $K_x \subset \Omega$ is a C^1– curve homeomorphic to the circle. Such K_x is called *an admissible cycle*.

The next lemma was obtained in [36, Lemma 3.3].

Lemma 4. *Let the conditions $(E_\circ)$ be satisfied. Assume that there exists a sequence of functions $\{\Phi_\mu\}$ such that $\Phi_\mu \in W^{1,q}_{\rm loc}(\Omega)$ and $\Phi_\mu \rightharpoonup \Phi$ in the space $W^{1,q}_{\rm loc}(\Omega)$ for all $q \in [1, 2)$. Then there exists a subsequence Φ_{k_l} such that $\Phi_{k_l}|_S$ converges to $\Phi|_S$ uniformly $\Phi_{k_l}|_S \rightrightarrows \Phi|_S$ on almost all admissible cycles S (here, "almost all cycles" means cycles in preimages $\psi^{-1}(y)$ for almost all values $y \in \psi(\overline{\Omega})$).*

In connection with Lemma 4, note that in [2] Amick proved the uniform convergence $\Phi_k \rightrightarrows \Phi$ on almost all circles. However, his method can be easily modified to prove the uniform convergence on almost all level lines of every C^1-smooth function with nonzero gradient. Such modification was done in the proof of Lemma 3.3 of [36].

Below assume (without loss of generality) that the subsequence Φ_{k_l} of Lemma 4 coincides with Φ_k. Admissible cycles S satisfying the statement of Lemma 4 will be called *regular cycles*.

Let Ω be a bounded domain with Lipschitz boundary. The function $f \in W^{1,s}(\Omega)$ is said to satisfy a *one-side maximum principle locally* in Ω if

$$\operatorname*{ess\,sup}_{x\in\Omega'} f(x) \le \operatorname*{ess\,sup}_{x\in\partial\Omega'} f(x) \tag{29}$$

holds for any strictly interior subdomain Ω' ($\overline{\Omega}' \subset \Omega$) with the boundary $\partial\Omega'$ not containing singleton connected components. (In (29) negligible sets are the sets of two-dimensional Lebesgue measure zero in the left *esssup* and the sets of one-dimensional Hausdorff measure zero in the right *esssup*.)

If (29) holds for any $\Omega' \subset \Omega$ (not necessary strictly interior) with the boundary $\partial\Omega'$ not containing singleton connected components, then $f \in W^{1,s}(\Omega)$ satisfies a *one-side maximum principle* in Ω (in particular, we can take $\Omega' = \Omega$ in (29)).

Using Lemma 4, it could be proved that the one-side maximum principle is inherited by the limiting solutions.

Theorem 5. *Let the conditions (E) be satisfied. Assume that there exists a sequence of functions $\{\Phi_\mu\}$ such that $\Phi_\mu \in W^{1,q}_{\rm loc}(\Omega)$ and $\Phi_\mu \rightharpoonup \Phi$ in the space $W^{1,q}_{\rm loc}(\Omega)$ for all $q \in [1, 2)$. If all Φ_μ satisfy the one-side maximum principle locally in Ω, then Φ satisfies the one-side maximum principle in Ω.*

Theorem 5 was obtained in [34, Theorem 2] (see also [36, Theorem 3.4] for the detailed proof).

Note that some version of a local weak one-side maximum principle was proved by Ch. Amick [2] (see Theorem 3.2 and Remark thereafter in [2]).

3.4 Arriving at a Contradiction

To prove the solvability of problem (2), we follow the arguments described in Sect. 2. First repeating Leray's argument of getting an a priori estimate by a contradiction, we arrive to the following assertion:

Lemma 5. *Assume that* $\Omega \subset \mathbb{R}^2$ *is a bounded domain with* C^2*-smooth boundary* $\partial\Omega$*,* $\mathbf{f} \in W^{1,2}(\Omega)$*, and the boundary value* $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ *satisfies the necessary condition* (3)*. Then, if problem* (2) *admits no weak solutions, then there exists a sequence of functions* $\mathbf{u}_k \in W^{1,2}(\Omega)$*,* $p_k \in W^{1,q}(\Omega)$ *and numbers* $\nu_k \to 0+$*,* $\lambda_k \to \lambda_0 > 0$ *with the following properties:*

(E-NS) the norms $\|\mathbf{u}_k\|_{W^{1,2}(\Omega)}$*,* $\|p_k\|_{W^{1,q}(\Omega)}$ *are uniformly bounded for every* $q \in [1,2)$*, and the pairs* $(\mathbf{u}_k, p_k)$ *satisfy the system of equations*

$$\begin{cases} -\nu_k \Delta \mathbf{u}_k + \big(\mathbf{u}_k \cdot \nabla\big)\mathbf{u}_k + \nabla p_k = \mathbf{f}_k, \quad x \in \Omega, \\ \qquad\qquad\qquad\qquad \operatorname{div} \mathbf{u}_k = 0, \quad x \in \Omega, \\ \qquad\qquad\qquad\qquad\quad \mathbf{u}_k = \mathbf{a}_k, \quad x \in \partial\Omega, \end{cases} \tag{30}$$

with $\mathbf{f}_k = \dfrac{\lambda_k \nu_k^2}{\nu^2}\mathbf{f}$*,* $\mathbf{a}_k = \dfrac{\lambda_k \nu_k}{\nu}\mathbf{a}$*, and*

$$\|\nabla \mathbf{u}_k\|_{L^2(\Omega)} \to 1, \quad \mathbf{u}_k \rightharpoonup \mathbf{v} \ \text{in}\ W^{1,2}(\Omega), \quad p_k \rightharpoonup p \ \text{in}\ W^{1,q}(\Omega) \quad \forall\, q \in [1,2),$$

where the pair of functions $\mathbf{v} \in W^{1,2}(\Omega)$*,* $p \in W^{1,q}(\Omega)$ *is a solution to the Euler system* (16)*.*

(In this lemma $\mathbf{u}_k = \mathbf{w}_k + J_k^{-1}\mathbf{A}$, $\nu_k = (\lambda_k J_k)^{-1}\nu$, $\mathbf{f}_k = \lambda_k \nu_k^2 \nu^{-2}\mathbf{f}$, where the objects $J_k, \mathbf{w}_k$ were defined in Sect. 2.)

Assume, in what follows, that the conditions (E-NS) are satisfied. As it is shown in Sect. 2, if all the fluxes $\mathcal{F}_i$ are zero (see (4)), then the conditions (E-NS) lead to a contradiction, thereby proving that (2) is solvable. In this section the goal is to demonstrate that these conditions also lead to a contradiction in the general case when the boundary data satisfy only the necessary condition (3). This will justify the existence of Theorem 1.

Assume for simplicity that $\partial\Omega$ consists of two connected components Γ_0 and Γ_1. Moreover, suppose that $\mathbf{f} = 0$. The pressure p is equal to constants on Γ_0 and Γ_1:

$$p|_{\Gamma_0} = \hat{p}_0, \qquad p|_{\Gamma_1} = \hat{p}_1$$

(see (17)). If $\hat{p}_0 = \hat{p}_1$, then, as it is shown in Sect. 2, a contradiction arises with the equality (14), and the required a priori estimate follows. Assume that $\widehat{p}_0 \neq \hat{p}_1$. Normalizing the pressure (and changing the numeration of the components Γ_i, if necessary), one can assume without loss of generality that

$$\hat{p}_0 = 0, \quad \hat{p}_1 < 0. \tag{31}$$

Introduce the main idea of the proof in a heuristic way. It is well known that total head pressures $\Phi_k = p_k + \frac{1}{2}\mathbf{u}_k^2$ (under above assumptions $\mathbf{f} = 0$) satisfy the linear elliptic equation

$$\Delta\Phi_k = \omega_k^2 + \frac{1}{\nu_k}\text{div}\,(\Phi_k \mathbf{u}_k), \tag{32}$$

where $\omega_k = \partial_2 u_k^1 - \partial_1 u_k^2$ is the corresponding vorticity. By Hopf's maximum principle, in a subdomain $\Omega' \Subset \Omega$ with C^2– smooth boundary $\partial\Omega'$, the maximum of Φ_k is attained at the boundary $\partial\Omega'$, and if $x_* \in \partial\Omega'$ is a maximum point, then the normal derivative of Φ_k at x_* is strictly positive. It is not sufficient to apply this property directly. Instead we will use some "integral analogs" that lead to a contradiction by using the Coarea formula. Namely, we construct a set $E_i \subset \Omega$ consisting of level lines of Φ_k such that $\Phi_k|_{E_i} \to 0$ as $i \to \infty$ and E_i separates the boundary component Γ_0 (where $\Phi = 0$) from the boundary component Γ_1 (where $\Phi < 0$). On the one hand, the length of each of these level lines is bounded from below by a positive constant (since they separate the boundary components), and by the Coarea formula, this implies the estimate from below for $\int_{E_i} |\nabla\Phi_k|$. On the other hand, elliptic equation (32) for Φ_k and boundary conditions allow us to estimate $\int_{E_i} |\nabla\Phi_k|^2$ from above, and this asymptotically contradicts the previous one.

Describe this heuristic idea in more details. From (32) and the mentioned Hopf theorem, one concludes that all Φ_k satisfy the strong maximum principle globally in Ω. Then by conditions (E-NS) and Theorem 5, the limiting total head pressure Φ satisfies the weak maximum principle globally in Ω, i.e.,

$$\max_{j=1,2} \widehat{p}_j = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) = 0. \tag{33}$$

Using the results of Kronrod (see Sect. 3.2), one can construct a decreasing sequence of domains with the following properties. Let T_ψ be a family of all connected components of level sets of ψ. Take $B_0, B_1 \in T_\psi$, $B_0 \supset \Gamma_0$, and $B_1 \supset \Gamma_1$, and set

$$\alpha = \min_{C\in[B_1,B_0]} \Phi(C) < 0.$$

(this minimum exists by Lemma 3). Let $t_i \in (0, -\alpha)$, $t_{i+1} = \frac{1}{2}t_i$ and t_i is such that

$$\Phi(C) = -t_i \quad \Rightarrow \quad C \in (B_1, B_0) \text{ is a regular cycle.}$$

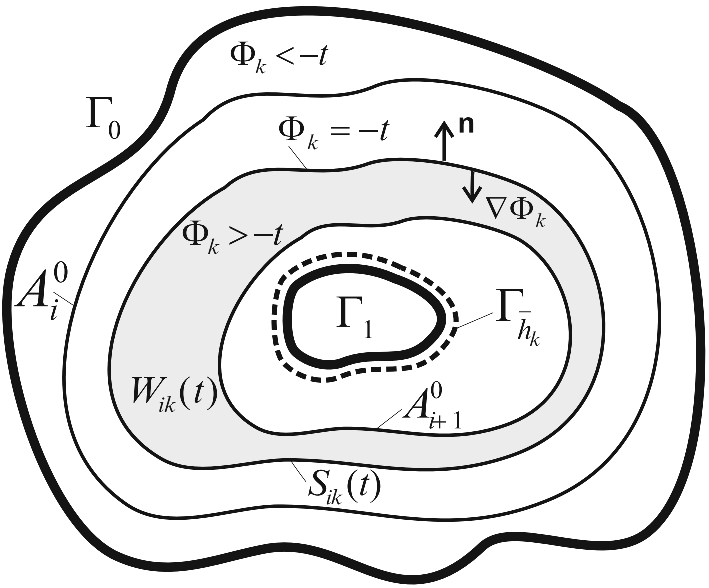

Fig. 1 The case of annulus-type domain (here, A_i is denoted as A_i^0)

(See the definition of the regular cycles in the commentary to Lemma 4.) Note that the existence of such a sequence t_i follows from the fact that

$$\mathfrak{H}^1\big(\{\Phi(C) : C \in [B_1, B_0] \text{ and } C \text{ is not a regular cycle}\}\big) = 0;$$

see [41, Corollary 3.2]. The proof of this equality is based on the Coarea formula (see [41]).

Denote by A_i an element from the set $\{C \in [B_1, B_0] : \Phi(C) = -t_i\}$ which is closest to Γ_0. Let V_i be a connected component of the set $\Omega \setminus A_i$ such that $\Gamma_0 \subset \partial V_i$, i.e., $\partial V_i = A_i \cup \Gamma_0$. Obviously, $V_i \supset V_{i+1}$ (since $t_{i+1} = \frac{1}{2}t_i$). Note that A_i are regular cycles and, therefore, $\Phi_k|_{A_i} \rightrightarrows \Phi|_{A_i} = -t_i$.

Take $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$. Let $W_{ik}(t)$ be the connected component of the set $\{x \in V_i \setminus \overline{V}_{i+1} : \Phi_k(x) > -t\}$ such that $\partial W_{ik}(t) \supset A_{i+1}$ (see Fig. 1). Put $S_{ik}(t) = (\partial W_{ik}(t)) \cap V_i \setminus \overline{V}_{i+1}$. Then $\Phi_k|_{S_{ik}(t)} = -t$, $\partial W_{ik}(t) = S_{ik}(t) \cup A_{i+1}$. Since $\Phi_k \in W^2_{2,\mathrm{loc}}(\Omega)$, by the Morse–Sard theorem for almost all $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$, the level set $S_{ik}(t)$ consists of a finite number of C^1-cycles; moreover, Φ_k is differentiable at every point $x \in S_{ik}(t)$ and $\nabla\Phi_k(x) \neq 0$. Such values t are called (k, i)-*regular*.

By construction

$$\int\limits_{S_{ik}(t)} \nabla\Phi_k \cdot \mathbf{n}\, dS = -\int\limits_{S_{ik}(t)} |\nabla\Phi_k|\, dS < 0,$$

where $\mathbf{n}$ outward with respect to $W_{ik}(t)$ normal to $\partial W_{ik}(t)$ (see Fig. 1). Indeed, $S_{ik}(t)$ is a subset of the level set $\{x \in \Omega : \Phi(x) = -t\}$, and by construction the nonzero gradient $\nabla\Phi(x)$ is directed *inside* the domain $W_{ik}(t)$ for $x \in S_{ik}(t)$, i.e., $\frac{\nabla\Phi_k(x)}{|\nabla\Phi_k(x)|} = -\mathbf{n}$.

The key step in the proof is the following estimate

Lemma 6. *For any $i \in \mathbb{N}$, there exists $k(i) \in \mathbb{N}$ such that the inequality*

$$\int\limits_{S_{ik}(t)} |\nabla\Phi_k(x)|dS \le C_* t$$

holds for every $k \ge k(i)$ and for almost all $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$. The constant C_* is independent of t, k, and i.

The proof of Lemma 6 is based on the integration of the equality (32) over the suitable subdomain $\Omega_k(t)$ with $\partial\Omega_k(t) = S_{ik}(t) \cup \Gamma_{h_k}$, where the cycle $\Gamma_{h_k} = \{x \in \Omega : \operatorname{dist}(x, \Gamma_0) = h_k\}$ lies near the boundary component Γ_0 and the parameter h_k is taken in such a way that

$$\int\limits_{\Gamma_{h_k}} \Phi_k^2\, ds < \sigma^2, \qquad \left|\int\limits_{\Gamma_{h_k}} \nabla\Phi_k \cdot \mathbf{n}\, ds\right| = \left|\int\limits_{\Gamma_{h_k}} \omega_k \mathbf{u}_k^{\perp} \cdot \mathbf{n}\, ds\right| < \varepsilon, \tag{34}$$

$$\int\limits_{\Gamma_{h_k}} |\mathbf{u}_k|^2\, ds < C_\varepsilon \nu_k^2, \tag{35}$$

where σ and ε are some fixed sufficiently small numbers and C_ε **does not depend on** k **and** σ. For sufficiently large $k \ge k(i)$ such h_k can be found, using the weak convergences $\Phi_k \rightharpoonup \Phi$, $\mathbf{u}_k \to \mathbf{v}$ from the assumptions (E-NS) and the boundary conditions $\|\mathbf{u}_k\|_{L^2(\partial\Omega)} \sim \nu_k$, $\mathbf{v} \equiv \mathbf{0}$, $\Phi \equiv 0$ on $\partial\Omega$ (see (30_3) and (16_3))

When Lemma 6 is proved, the required contradiction can be obtained using the Coarea formula. For $i \in \mathbb{N}$ and $k \ge k(i)$, put

$$E_i = \bigcup_{t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]} S_{ik}(t).$$

By the Coarea formula (see, e.g., [50]), for any integrable function $g : E_i \to \mathbb{R}$, the equality

$$\int\limits_{E_i} g|\nabla\Phi_k|\, dx = \int\limits_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \int\limits_{S_{ik}(t)} g(x)\, d\mathfrak{H}^1(x)\, dt$$

holds. In particular, taking $g = |\nabla\Phi_k|$ and using Lemma 6 yield

$$\int\limits_{E_i} |\nabla\Phi_k|^2\,dx = \int\limits_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \int\limits_{S_{ik}(t)} |\nabla\Phi_k|\,d\mathfrak{H}^1(x)dt \le \int\limits_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} C_* t\,dt = C_{**}t_i^2.$$

Now, taking $g = 1$ in the Coarea formula and using the Hölder Inequality, we get

$$\int\limits_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \mathfrak{H}^1\big(S_{ik}(t)\big)dt = \int\limits_{E_i} |\nabla\Phi_k|\,dx$$

$$\le \left(\int\limits_{E_i} |\nabla\Phi_k|^2\,dx\right)^{\frac{1}{2}} \big(\,\mathrm{meas}(E_i)\big)^{\frac{1}{2}} \le \sqrt{C_{**}}t_i\big(\,\mathrm{meas}(E_i)\big)^{\frac{1}{2}}.$$

By construction, for almost all $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$, the set $S_{ik}(t)$ is a smooth cycle and $S_{ik}(t)$ separates A_i from A_{i+1}. Thus, each set $S_{ik}(t)$ separates Γ_0 from Γ_1. In particular, $\mathfrak{H}^1(S_{ik}(t)) \ge C = \min\big(\,\mathrm{diam}\,\Gamma_0, \mathrm{diam}\,\Gamma_1\big)$. Hence,

$$\int\limits_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \mathfrak{H}^1\big(S_{ik}(t)\big)dt \ge \frac{1}{4}Ct_i.$$

So, it holds

$$\frac{1}{4}Ct_i \le \sqrt{C_{**}}t_i\big(\,\mathrm{meas}(E_i)\big)^{\frac{1}{2}},$$

or

$$\frac{1}{4}C \le \sqrt{C_{**}}\big(\,\mathrm{meas}(E_i)\big)^{\frac{1}{2}}. \tag{36}$$

By construction $\mathrm{meas}(E_i) \le \mathrm{meas}\,\big(V_i \setminus V_{i+1}\big)$. But since $V_i \supset V_{i+1}$ is a decreasing sequence of bounded sets, we have $\mathrm{meas}\,\big(V_i \setminus V_{i+1}\big) \to 0$ as $i \to \infty$; therefore, inequality (36) gives the contradiction. Thus, our assumption is wrong; the norms of all possible solutions $\mathbf{w}^{(\lambda)}$ to the operator equation (8) are uniformly bounded with respect to $\lambda \in [0, 1]$, and by the Leray–Schauder theorem, the equation (7) and equivalently the problem (2) has at least one solution.

Hence, in the case when $\mathbf{f} = 0$, Theorem 1 is proved. If $\mathbf{f} \ne 0$, then the maximum principle is not valid, and one has to consider two cases

(a) Maximum of Φ is attained on the boundary $\partial\Omega$:

$$\max\{\hat{p}_0, \widehat{p}_1\} = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x).$$

(b) Maximum of Φ is not attained on $\partial\Omega$:

$$\max\{\hat{p}_0, \widehat{p}_1\} < \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x)$$

(the case $\operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) = +\infty$ is also possible).

In the case (a) the proof is literally the same as above, while in the case (b) it can be proved that there exists a regular cycle $F \in T_\psi$ such that $\operatorname{diam} F > 0$, $F \cap \partial\Omega = \emptyset$, and $\Phi(F) > \beta$, where $\beta = \max\{\hat{p}_0, \hat{p}_1\}$. For such F we consider the behavior of Φ on the Kronrod arcs $[B_j, F]$, $j = 0, 1$. The remaining part of the proof is similar to that of the proof for the case (a) with the following difference: F plays now the role which was played before by B_0, and the calculations become easier since F lies strictly inside Ω.

The main idea of the proof for a general multiply connected domain is the same as in the case of annulus-like domains (when $\partial\Omega = \Gamma_1 \cup \Gamma_0$). The proof has an analytical nature and unessential differences concern only well-known geometrical properties of level sets of continuous functions of two variables.

4 3*D* Axially Symmetric Case

First, specify some notations. Let $O_{x_1}, O_{x_2}, O_{x_3}$ be the coordinate axis in $\mathbb{R}^3$ and $\theta = \operatorname{arctg}(x_2/x_1)$, $r = (x_1^2 + x_2^2)^{1/2}$, $z = x_3$ be the cylindrical coordinates. Denote by v_θ, v_r, v_z the projections of the vector $\mathbf{v}$ on the axes θ, r, z.

A function f is said to be *axially symmetric* if it does not depend on θ. A vector-valued function $\mathbf{h} = (h_r, h_\theta, h_z)$ is called *axially symmetric* if h_r, h_θ, and h_z do not depend on θ. A vector-valued function $\mathbf{h}' = (h_r, h_\theta, h_z)$ is called *axially symmetric with no swirl* if $h_\theta = 0$, while h_r and h_z do not depend on θ.

4.1 Bounded 3*D* Axially Symmetric Domains

The main result of this section is as follows.

Theorem 6 ([37,41]). *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded axially symmetric domain of type (1) with C^2-smooth boundary $\partial\Omega$ (Fig. 2). If $\mathbf{f} \in W^{1,2}(\Omega)$, $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ are axially symmetric and $\mathbf{a}$ satisfies condition (3), then (2) admits at least one weak axially symmetric solution. Moreover, if $\mathbf{f}$ and $\mathbf{a}$ are axially symmetric with no swirl, then (2) admits at least one weak axially symmetric solution with no swirl.*

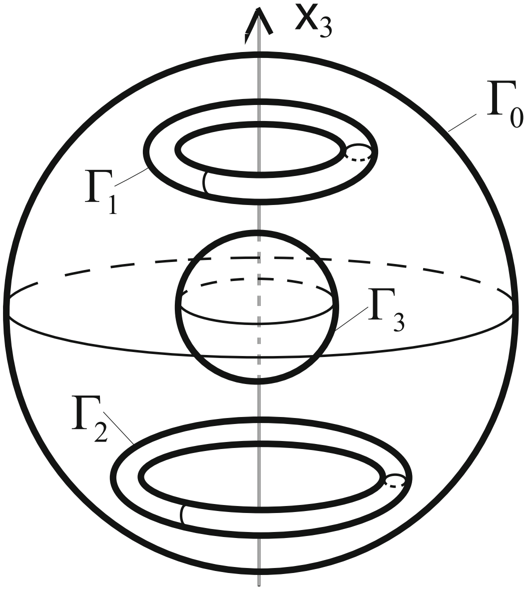

Fig. 2 Axially symmetric domain ($N = 3$)

The proof of Theorem 6 follows the same ideas as for the two-dimensional case, so it is not discussed here. (Some specific details for axially symmetric case could be found in the next section where the more complicated case of exterior domains is discussed.)

4.2 Exterior 3*D* Axially Symmetric Domains

This section is based on results of the paper [39]. Consider the Navier–Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div}\,\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to+\infty} \mathbf{u}(x) = \mathbf{u}_0 \end{cases} \tag{37}$$

in the exterior domain of $\mathbb{R}^3$

$$\Omega = \mathbb{R}^3 \setminus \Big(\bigcup_{j=1}^{N} \bar{\Omega}_j\Big), \tag{38}$$

where Ω_i are bounded domains with connected C^2-smooth boundaries Γ_i, $\bar{\Omega}_j \cap \bar{\Omega}_i = \emptyset$ for $i \neq j$, and $\mathbf{u}_0$ is a constant assigned vector.

Let

$$\mathcal{F}_i = \int_{\Gamma_i} \mathbf{a} \cdot \mathbf{n}\, dS, \quad i = 1, \dots N, \tag{39}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. Under suitable regularity hypotheses on Ω and $\mathbf{a}$ and assuming that

$$\mathcal{F}_i = 0, \quad i = 1, \dots N, \tag{40}$$

in the celebrated paper [49] of 1933, J. Leray showed that (37) has a solution $\mathbf{u}$ with finite Dirichlet integral:

$$\int_{\Omega} |\nabla \mathbf{u}|^2\, dx < +\infty, \tag{41}$$

and $\mathbf{u}$ satisfies (37_4) in a suitable sense for general $\mathbf{u}_0$ and uniformly for $\mathbf{u}_0 = \mathbf{0}$. In the 1950s, the problem was reconsidered by R. Finn [16] and O.A. Ladyzhenskaya [46, 47]. They showed that the solution satisfies the condition at infinity uniformly. Moreover, condition (40) and the regularity of $\mathbf{a}$ have been relaxed by requiring $\sum_{i=1}^{N} |\mathcal{F}_i|$ to be sufficiently small [16] and $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ [47].

In 1973 K.I. Babenko [4] proved that if $(\mathbf{u}, p)$ is a solution to (37), (41) with $\mathbf{u}_0 \neq \mathbf{0}$, then $(\mathbf{u} - \mathbf{u}_0, p)$ behaves at infinity as the solutions to the linear Oseen system. In particular,

$$\mathbf{u}(x) - \mathbf{u}_0 = O(r^{-1}), \quad p(x) = O(r^{-2}). \tag{42}$$

(See also [21]. Here the symbol $f(x) = O(g(r))$ means that there is a positive constant c such that $|f(x)| \leq cg(r)$ for large r.) However, nothing is known, in general, on the rate of convergence at infinity for $\mathbf{u}_0 = \mathbf{0}$. (For small $\|\mathbf{a}\|_{L^\infty(\partial\Omega)}$ existence of a solution $(\mathbf{u}, p)$ to (37) such that $\mathbf{u} = O(r^{-1})$ is a simple consequence of Banach contractions theorem [69]. Moreover, one can show that $p = O(r^{-2})$, and the derivatives of order k of $\mathbf{u}$ and p behave at infinity as r^{-k-1}, r^{-k-2}, respectively [75]; see also [21, 58].)

One of the most important problems in the theory of the steady-state Navier–Stokes equations concerns the possibility to prove the existence of a solution to (37) without any assumptions on the fluxes $\mathcal{F}_i$ (see, e.g., [21]). To the best of our knowledge, the most general assumptions assuring the existence is expressed by

$$\sum_{i=1}^{N} \max_{\Gamma_i} \frac{|\mathcal{F}_i|}{|x - x_i|} < 8\pi\nu \tag{43}$$

(see [67]), where $\mathcal{F}_i$ is defined by (39) and x_i is a fixed point of Ω_i (see also [5] for analogous conditions in bounded domains).

In the recent paper [39], the above question was solved for the axially symmetric case. Note that for axially symmetric solutions $\mathbf{u}$ of (37), the vector $\mathbf{u}_0$ has to be parallel to the symmetry axis. The main result is as follows.

Theorem 7 ([39]). *Assume that $\Omega \subset \mathbb{R}^3$ is an exterior axially symmetric domain (38) with C^2-smooth boundary $\partial\Omega$, $\mathbf{u}_0 \in \mathbb{R}^3$ is a constant vector parallel to the symmetry axis, and $\mathbf{f} \in W^{1,2}(\Omega) \cap L^{6/5}(\Omega)$, $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ are axially symmetric. Then (37) admits at least one weak axially symmetric solution $\mathbf{u}$ satisfying (41). Moreover, if $\mathbf{a}$ and $\mathbf{f}$ are axially symmetric with no swirl, then (37) admits at least one weak axially symmetric solution with no swirl satisfying (41).*

Remark 5. It is well known (see, e.g., [47]) that under hypothesis of Theorem 7, every weak solution $\mathbf{u}$ of the problem (37) is more regular, i.e., $\mathbf{u} \in W^{2,2}_{\mathrm{loc}}(\overline{\Omega}) \cap W^{3,2}_{\mathrm{loc}}(\Omega)$.

Emphasize that Theorem 7 furnishes the first existence result without any assumption on the fluxes for the stationary Navier–Stokes problem in exterior three-dimensional domains.

4.2.1 Extension of the Boundary Values

The next lemma concerns the existence of a solenoidal extensions of boundary values.

Lemma 7 (see, e.g., [39]). *Let $\Omega \subset \mathbb{R}^3$ be an exterior axially symmetric domain (38). If $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$, then there exists a solenoidal extension $\mathbf{A} \in W^{2,2}(\Omega)$ of $\mathbf{a}$ such that $\mathbf{A}(x) = \boldsymbol{\sigma}(x)$ for sufficiently large $|x|$, where*

$$\boldsymbol{\sigma}(x) = -\frac{x}{4\pi|x|^3}\sum_{i=1}^{N}\mathcal{F}_i \tag{44}$$

and $\mathcal{F}_i$ are defined by (39). Moreover, the following estimate

$$\|\mathbf{A}\|_{W^{2,2}(\Omega)} \le c\|\mathbf{a}\|_{W^{3/2,2}(\partial\Omega)} \tag{45}$$

holds. Furthermore, if $\mathbf{a}$ is axially symmetric (axially symmetric with no swirl), then $\mathbf{A}$ is axially symmetric (axially symmetric with no swirl) too.

4.2.2 Leray's Argument "Invading Domains"

Consider the Navier–Stokes problem (37) with $\mathbf{f} \in W^{1,2}(\Omega) \cap L^{6/5}(\Omega)$ in the C^2-smooth axially symmetric exterior domain $\Omega \subset \mathbb{R}^3$ defined by (38). Without loss of generality, assume that $\mathbf{f} = \operatorname{curl}\mathbf{b} \in W^{1,2}(\Omega) \cap L^{6/5}(\Omega)$. (By the Helmholtz–Weyl decomposition, $\mathbf{f}$ can be represented as the sum $\mathbf{f} = \operatorname{curl}\mathbf{b} + \nabla\varphi$ with $\operatorname{curl}\mathbf{b} \in$

$W^{1,2}(\Omega) \cap L^{6/5}(\Omega)$, and the gradient part is included then into the pressure term; see, e.g., [21, 47].)

Below the proof of Theorem 7 will be discussed in the case

$$\mathbf{u}_0 = \mathbf{0}.$$

The proof for $\mathbf{u}_0 \neq \mathbf{0}$ follows the same steps with minor standard modification.

A function $\mathbf{u}$ is called *a weak solution* of problem (37) if $\mathbf{w} - \mathbf{A} \in H(\Omega)$ and the integral identity

$$\begin{aligned} \nu \int_\Omega \nabla \mathbf{w} \cdot \nabla \boldsymbol{\eta}\, dx &= -\nu \int_\Omega \nabla \mathbf{A} \cdot \nabla \boldsymbol{\eta}\, dx - \int_\Omega \big(\mathbf{A} \cdot \nabla\big)\mathbf{A} \cdot \boldsymbol{\eta}\, dx \\ &- \int_\Omega \big(\mathbf{A} \cdot \nabla\big)\mathbf{w} \cdot \boldsymbol{\eta}\, dx - \int_\Omega \big(\mathbf{w} \cdot \nabla\big)\mathbf{w} \cdot \boldsymbol{\eta}\, dx \\ &- \int_\Omega \big(\mathbf{w} \cdot \nabla\big)\mathbf{A} \cdot \boldsymbol{\eta}\, dx + \int_\Omega \mathbf{f} \cdot \boldsymbol{\eta}\, dx \end{aligned} \tag{46}$$

holds for any $\boldsymbol{\eta} \in J_0^\infty(\Omega)$. Here $\mathbf{A}$ is the extension of the boundary data constructed in Lemma 7. We shall look for the axially symmetric (axially symmetric with no swirl) weak solution of problem (37) and find this solution as a limit of weak solution to the Navier–Stokes problem in a sequence of bounded domain Ω_k that in the limit exhausts the unbounded domains Ω (this is the main idea of the "invading domain" method). Namely, consider the sequence of the boundary value problems

$$\begin{cases} -\nu \Delta \widehat{\mathbf{u}}_k + (\widehat{\mathbf{u}}_k \cdot \nabla)\widehat{\mathbf{u}}_k + \nabla \hat{p}_k = \mathbf{f} & \text{in } \Omega_k, \\ \operatorname{div} \widehat{\mathbf{u}}_k = 0 & \text{in } \Omega_k, \\ \widehat{\mathbf{u}}_k = \mathbf{A} & \text{on } \partial\Omega_k, \end{cases} \tag{47}$$

where $\Omega_k = B_k \cap \Omega$ for $k \geq k_0$, $B_k = \{x : \ |x| < k\}$, $\frac{1}{2}B_{k_0} \supset \bigcup_{i=1}^{N} \bar{\Omega}_i$. By Theorem 6, each problem (47) has an axially symmetric solution $\widehat{\mathbf{u}}_k = \mathbf{A} + \widehat{\mathbf{w}}_k$ with $\widehat{\mathbf{w}}_k \in H(\Omega_k)$.

To prove the assertion of Theorem 7, it is sufficient to establish the uniform estimate

$$\int_\Omega |\nabla \widehat{\mathbf{w}}_k|^2 \leq c. \tag{48}$$

Estimate (48) will be proved following a classical *reductio ad absurdum* argument of J. Leray and O.A. Ladyzhenskaia (see [47, 49]). Indeed, if (48) is not true, then there exists a sequence $\{\widehat{\mathbf{w}}_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty} J_k^2 = +\infty, \quad J_k^2 = \int_{\Omega_k} |\nabla \widehat{\mathbf{w}}_k|^2.$$

The sequence $\mathbf{w}_k = \widehat{\mathbf{w}}_k/J_k$ is bounded in $H(\Omega)$. Extracting a subsequence (if necessary), one can assume that $\mathbf{w}_k$ converges weakly in $H(\Omega)$ and strongly in $L^q_{\mathrm{loc}}(\overline{\Omega})$ $(q < 6)$ to a vector field $\mathbf{v} \in H(\Omega)$ with

$$\int_\Omega |\nabla \mathbf{v}|^2 \le 1. \tag{49}$$

It is easy to check that $\mathbf{v} \in H(\Omega)$ is a weak solution to the Euler equations, and for some $p \in D^{1,3/2}(\Omega)$ the pair $(\mathbf{v}, p)$ satisfies the Euler equations almost everywhere:

$$\begin{cases} \big(\mathbf{v}\cdot\nabla\big)\mathbf{v} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{50}$$

Adding some constants to p (if necessary) by virtue of the Sobolev inequality (see, e.g., [21] II.6), it may be assumed without loss of generality that

$$\|p\|_{L^3(\Omega)} \le \text{const}. \tag{51}$$

Put $\nu_k = (J_k)^{-1}\nu$. Multiplying equations (47) by $\frac{1}{J_k^2} = \frac{\nu_k^2}{\nu^2}$, one sees that the pair $\big(\mathbf{u}_k = \frac{1}{J_k}\widehat{\mathbf{w}}_k + \frac{1}{J_k}\mathbf{A},\ p_k = \frac{1}{J_k^2}\hat{p}_k\big)$ satisfies the following system:

$$\begin{cases} -\nu_k \Delta \mathbf{u}_k + \big(\mathbf{u}_k\cdot\nabla\big)\mathbf{u}_k + \nabla p_k = \mathbf{f}_k & \text{in } \Omega_k, \\ \operatorname{div} \mathbf{u}_k = 0 & \text{in } \Omega_k, \\ \mathbf{u}_k = \mathbf{a}_k & \text{on } \partial\Omega_k, \end{cases} \tag{52}$$

where $\mathbf{f}_k = \frac{\nu_k^2}{\nu^2}\mathbf{f}$, $\mathbf{a}_k = \frac{\nu_k}{\nu}\mathbf{A}$, $\mathbf{u}_k \in W^{3,2}_{\mathrm{loc}}(\Omega)$, $p_k \in W^{2,2}_{\mathrm{loc}}(\Omega)$ (the interior regularity of the solution depends on the regularity of $\mathbf{f} \in W^{1,2}(\Omega)$, but not on the regularity of the boundary value $\mathbf{a}$; see [47]). Using the classical local estimates for ADN-elliptic problems (see [1, 73]), one could prove the following uniform estimate:

$$\|\mathbf{u}_k\|_{L^6(\Omega_k)} + \|\nabla p_k\|_{L^{3/2}(\Omega_k)} \le C, \tag{53}$$

where C does not depend on k. By construction, there holds the weak convergences $\mathbf{u}_k \rightharpoonup \mathbf{v}$ in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$, $p_k \rightharpoonup p$ in $W^{1,3/2}_{\mathrm{loc}}(\overline{\Omega})$ (the weak convergence in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$ means the weak convergence in $W^{1,2}(\Omega')$ for every bounded subdomain $\Omega' \subset \Omega$).

As in the two-dimensional case, in conclusion, one can prove the following lemma.

Lemma 8. *Assume that $\Omega \subset \mathbb{R}^3$ is an exterior axially symmetric domain of type (38) with C^2-smooth boundary $\partial\Omega$, and $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$, $\mathbf{f} = \operatorname{curl} \mathbf{b} \in W^{1,2}(\Omega) \cap L^{6/5}(\Omega)$ are axially symmetric. If the assertion of Theorem 7 is false, then there exist $\mathbf{v}$, p with the following properties.*

(E-AX) The axially symmetric functions $\mathbf{v} \in H(\Omega)$, $p \in D^{1,3/2}(\Omega)$ satisfy the Euler system (50) and $\|p\|_{L^3(\Omega)} < \infty$.
(E-NS-AX) Condition (E-AX) is satisfied, and there exist sequences of axially symmetric functions $\mathbf{u}_k \in W^{1,2}(\Omega_k)$, $p_k \in W^{1,3/2}(\Omega_k)$, $\Omega_k = \Omega \cap B_{R_k}$, $R_k \to \infty$ as $k \to \infty$, and numbers $\nu_k \to 0+$, such that estimate (53) holds, the pair $(\mathbf{u}_k, p_k)$ satisfies (52) with $\mathbf{f}_k = \frac{\nu_k^2}{\nu^2}\mathbf{f}$, $\mathbf{a}_k = \frac{\nu_k}{\nu}\mathbf{A}$ (here $\mathbf{A}$ is solenoidal extension of $\mathbf{a}$ from Lemma 7), and

$$\|\nabla \mathbf{u}_k\|_{L^2(\Omega_k)} \to 1, \quad \mathbf{u}_k \rightharpoonup \mathbf{v} \;\; in \;\; W^{1,2}_{\mathrm{loc}}(\overline{\Omega}), \quad p_k \rightharpoonup p \;\; in \;\; W^{1,3/2}_{\mathrm{loc}}(\overline{\Omega}), \tag{54}$$

$$\nu = \int\limits_{\Omega} (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \mathbf{A}\, dx \tag{55}$$

Moreover, $\mathbf{u}_k \in W^{3,2}_{\mathrm{loc}}(\Omega)$ and $p_k \in W^{2,2}_{\mathrm{loc}}(\Omega)$.

4.2.3 Euler Equation in 3*D* Axially Symmetric Case (Exterior Domains)

Suppose that the assumptions (E-AX) (from Lemma 8) are satisfied, and, for definiteness, assume that

(SO) Ω is the domain (38) symmetric with respect to the axis O_{x_3} and

$$\Gamma_j \cap O_{x_3} \neq \emptyset, \quad j = 1, \ldots, M',$$
$$\Gamma_j \cap O_{x_3} = \emptyset, \quad j = M' + 1, \ldots, N.$$

(The cases $M' = N$ or $M' = 0$, i.e., when all components (resp., no components) of the boundary intersect the axis of symmetry, are also allowed.)

Denote $P_+ = \{(0, x_2, x_3) : x_2 > 0, \; x_3 \in \mathbb{R}\}$, $\mathcal{D} = \Omega \cap P_+$, $\mathcal{D}_j = \Omega_j \cap P_+$. Of course, on P_+ the coordinates x_2, x_3 coincide with coordinates r, z.

Then $\mathbf{v}$ and p satisfy the following system in the plane domain $\mathcal{D}$:

$$\begin{cases} \dfrac{\partial p}{\partial z} + v_r \dfrac{\partial v_z}{\partial r} + v_z \dfrac{\partial v_z}{\partial z} = 0, \\ \dfrac{\partial p}{\partial r} - \dfrac{(v_\theta)^2}{r} + v_r \dfrac{\partial v_r}{\partial r} + v_z \dfrac{\partial v_r}{\partial z} = 0, \\ \dfrac{v_\theta v_r}{r} + v_r \dfrac{\partial v_\theta}{\partial r} + v_z \dfrac{\partial v_\theta}{\partial z} = 0, \\ \dfrac{\partial (r v_r)}{\partial r} + \dfrac{\partial (r v_z)}{\partial z} = 0 \end{cases} \tag{56}$$

(these equations are fulfilled for almost all $x \in \mathcal{D}$).

There hold the following integral estimates: $\mathbf{v} \in W^{1,2}_{\mathrm{loc}}(\mathcal{D})$,

$$\int\limits_{\mathcal{D}} r|\nabla \mathbf{v}(r,z)|^2\,drdz + \int\limits_{\mathcal{D}} r|\mathbf{v}(r,z)|^6\,drdz < \infty. \tag{57}$$

Also, the inclusions $\nabla p \in L^{3/2}(\Omega)$, $p \in L^3(\Omega)$ can be rewritten in the following two-dimensional form:

$$\int\limits_{\mathcal{D}} r|\nabla p(r,z)|^{3/2}\,drdz + \int\limits_{\mathcal{D}} r|p(r,z)|^3\,drdz < \infty. \tag{58}$$

The next statement was proved in [30, Lemma 4] and in [2, Theorem 2.2].

Theorem 8. *Let the conditions (E-AX) be fulfilled. Then*

$$\forall j \in \{1,\dots,N\}\ \exists\, \hat{p}_j \in \mathbb{R}: \quad p(x) \equiv \hat{p}_j \quad \textit{for } \mathfrak{H}^2 - \textit{almost all } x \in \Gamma_j. \tag{59}$$

In particular, by axial symmetry,

$$p(x) \equiv \hat{p}_j \quad \textit{for } \mathfrak{H}^1 - \textit{almost all } x \in \breve{\Gamma}_j. \tag{60}$$

Here and below the following convenient agreement is used: for a set $A \subset \mathbb{R}^3$ put $\breve{A} := A \cap P_+$, and for $B \subset P_+$ denote by $\tilde{B}$ the set in $\mathbb{R}^3$ obtained by rotation of B around O_z-axis.

The following result gives more precise information about the constants from the previous theorem.

Corollary 1 ([39]). *Assume that the conditions (E-AX) are satisfied. Then* $\Phi|_{\Gamma_j} \equiv 0$ *whenever* $\Gamma_j \cap O_z \neq \emptyset$, *i.e.,*

$$\hat{p}_1 = \cdots = \hat{p}_{M'} = 0,$$

where $\hat{p}_j$ *are the constants from Theorem 8.*

This phenomenon is connected with the fact that the symmetry axis can be approximated by stream lines (see Theorem 10 below), where the total head pressure is constant according to the Bernoulli law (see Theorem 9 below). To formulate the last result, some preparation is needed.

Below without loss of generality, assume that the functions $\mathbf{v}$, p are extended to the whole half-plane P_+ as follows:

$$\mathbf{v}(x) := 0, \quad x \in P_+ \setminus \mathcal{D}, \tag{61}$$

$$p(x) := \hat{p}_j, \ \ x \in P_+ \cap \bar{\mathcal{D}}_j, \ \ j = 1,\dots,N. \tag{62}$$

Obviously, the extended functions inherit the properties of the previous ones. Namely, $\mathbf{v} \in W^{1,2}_{\rm loc}(P_+)$, $p \in W^{1,3/2}_{\rm loc}(P_+)$, and the Euler equations (56) are fulfilled almost everywhere in P_+.

The last equality in (56) (which is fulfilled, after the above extension agreement, in the whole half-plane P_+) implies the existence of a continuous stream function $\psi \in W^{2,2}_{\rm loc}(P_+)$ such that

$$\frac{\partial \psi}{\partial r} = -r v_z, \quad \frac{\partial \psi}{\partial z} = r v_r. \tag{63}$$

Denote by $\Phi = p + \dfrac{|\mathbf{v}|^2}{2}$ the total head pressure corresponding to the solution $(\mathbf{v}, p)$. From (57) we get

$$\int\limits_{P_+} r|\Phi(r,z)|^3 \, drdz + \int\limits_{P_+} r|\nabla \Phi(r,z)|^{3/2} \, drdz < \infty. \tag{64}$$

By direct calculations one easily gets the identity

$$v_r \Phi_r + v_z \Phi_z = 0 \tag{65}$$

for almost all $x \in P_+$. The identities (61)–(62) mean that

$$\Phi(x) \equiv \hat{p}_j \qquad \forall x \in P_+ \cap \bar{\mathcal{D}}_j, \;\; j = 1, \dots, N. \tag{66}$$

Theorem 9 (Bernoulli law for Sobolev solutions [39]). *Let the conditions (E-AX) be valid. Then there exists a set $A_{\mathbf{v}} \subset P_+$ with $\mathfrak{H}^1(A_{\mathbf{v}}) = 0$, such that every point $x \in P_+ \setminus A_{\mathbf{v}}$ is a Lebesgue point for $\mathbf{v}, \Phi$, and for any compact connected set $K \subset P_+$, the following property holds : if*

$$\psi\big|_K = \text{const}, \tag{67}$$

then

$$\Phi(x_1) = \Phi(x_2) \quad \textit{for all } \; x_1, x_2 \in K \setminus A_{\mathbf{v}}. \tag{68}$$

Theorem 9 is a space version of the above Theorem 3 (for the plane case). For the axially symmetric bounded domains, the result was proved in [37, Theorem 3.3]. The proof for exterior axially symmetric domains is similar: one has to overcome two obstacles. First difficulty is the lack of the classical regularity, and here the results of [6] have a decisive role (according to these results, almost all level sets of plane $W^{2,1}$-functions are C^1-curves; see Sect. 3.1). The second obstacle is the set where $\nabla \psi(x) = 0 \neq \nabla \Phi(x)$, i.e., where $v_r(x) = v_z(x) = 0$, but $v_\theta(x) \neq 0$. Namely, without assuming the boundary conditions (50_3), in general, the equality (67) even for smooth functions does not imply (68). For example, if $v_r = v_z = 0$ in the whole domain, $v_\theta = r$, then $\psi \equiv$ const on the whole domain, while $\Phi = r^2 \neq$ const. Without the boundary assumptions, one can prove only that $\Phi(r,z) = f(r)$ along every level set K of the stream function ψ for some absolutely continuous function $f(r)$ (see [39, Lemma 4.5]). But the last equality together with the boundary conditions (50_3), (66) easily implies Theorem 9.

For $\varepsilon > 0$ and $R > 0$ denote by $S_{\varepsilon,R}$ the set $S_{\varepsilon,R} = \{(r,z) \in P_+ : r \geq \varepsilon,\ r^2 + z^2 = R^2\}$. From the assumptions (64) one gets

Lemma 9. *For any $\varepsilon > 0$, there exists a sequence $\rho_j > 0$, $\rho_j \to +\infty$, such that $S_{\varepsilon,\rho_j} \cap A_{\mathbf{v}} = \emptyset$ and*

$$\sup_{x \in S_{\varepsilon,\rho_j}} |\Phi(x)| \to 0 \quad \text{as } j \to \infty. \tag{69}$$

One of the main results of this section is the following.

Theorem 10. *Assume that the conditions (E-AX) are satisfied. Let K_j be a sequence of continuums with the following properties: $K_j \subset \bar{\mathcal{D}} \setminus O_z$, $\psi|_{K_j} = \text{const}$, and $\lim\limits_{j\to\infty} \inf\limits_{(r,z)\in K_j} r = 0$, $\varliminf\limits_{j\to\infty} \sup\limits_{(r,z)\in K_j} r > 0$. Then $\Phi(K_j) \to 0$ as $j \to \infty$. Here we denote by $\Phi(K_j)$ the corresponding constant $c_j \in \mathbb{R}$ such that $\Phi(x) = c_j$ for all $x \in K_j \setminus A_{\mathbf{v}}$ (see Theorem 9).*

4.2.4 Obtaining a Contradiction

From now on assume that the assumptions (E-NS-AX) (see Lemma 8) are satisfied. The goal is to prove that they lead to a contradiction. This implies the validity of Theorem 7.

For simplicity assume that $\mathbf{f} = 0$, $N = 2$ and $M' = 1$, i.e., the boundary $\partial\Omega$ splits into the two components $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 \cap O_z \neq \emptyset, \qquad \Gamma_2 \cap O_z = \emptyset. \tag{70}$$

The main idea of the proof is similar to that for the two-dimensional case. Since every $\Phi_k = p_k + \frac{1}{2}|\mathbf{u}_k|^2$ satisfies the linear elliptic equation

$$\Delta\Phi_k = \omega_k^2 + \frac{1}{\nu_k}\operatorname{div}(\Phi_k \mathbf{u}_k), \tag{71}$$

where $\boldsymbol{\omega}_k = \operatorname{curl} \mathbf{u}_k$ and $\omega_k(x) = |\boldsymbol{\omega}_k(x)|$, a contradiction is obtained by using some "integral analog" of Hopf's maximum principle and the Coarea formula.

Consider the constants $\hat{p}_j$ from Theorem 8 (see also Theorem 1). From the equality (55), the Euler equations (50_1), and the regularity assumptions (64), the identity follows

$$-\nu = \sum_{j=M'+1}^{N} \hat{p}_j \mathcal{F}_j = \widehat{p}_2 \mathcal{F}_2 \tag{72}$$

(since $\hat{p}_1 = 0$ by Theorem 1). Therefore, $\hat{p}_2 \neq 0$.

Further consider separately three possible cases.

(a) The maximum of Φ is attained at infinity, i.e., it is zero:

$$0 = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x). \tag{73}$$

(b) The maximum of Φ is attained on a boundary component which does not intersect the symmetry axis:

$$0 < \hat{p}_2 = \max_{j=M'+1,\dots,N} \widehat{p}_j = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x), \tag{74}$$

(c) The maximum of Φ is not zero and it is not attained on $\partial\Omega$:

$$\max_{j=M'+1,\dots,N} \widehat{p}_j < \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) > 0 \tag{75}$$

(the case $\operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) = +\infty$ is not excluded).

4.2.5 The Case $\operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) = 0$.

Let us consider the case (73). By Theorem 1,

$$\hat{p}_1 = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) = 0. \tag{76}$$

Then $\hat{p}_2 < 0$.

The arguments below are similar to the plane situation (see Sect. 3.4). Take the positive constant $\delta_p = -\hat{p}_2$. The first goal is to separate the boundary components where $\Phi < 0$ from infinity and from the singularity axis O_z by level sets of Φ compactly supported in $\mathcal{D}$. More precisely, for any $t \in (0, \delta_p)$ we construct a continuum $A(t) \Subset P_+$ with the following properties:

(i) The set $\breve{\Gamma}_j$ (recall that for a set $A \subset \mathbb{R}^3$ by definition $\breve{A} := A \cap P_+$) lies in a bounded connected component of the open set $P_+ \setminus A(t)$;
(ii) $\psi|_{A(t)} \equiv \text{const}, \ \Phi(A(t)) = -t$;
(iii) (monotonicity) If $0 < t_1 < t_2 < \delta_p$, then the set $A(t_1) \cup \breve{\Gamma}_1$ lies in the unbounded connected component of the set $P_+ \setminus A(t_2)$ (in other words, $A(t_2) \cup \breve{\Gamma}_2$ lies in the bounded connected component of the set $P_+ \setminus A(t_1)$; see Fig. 3).

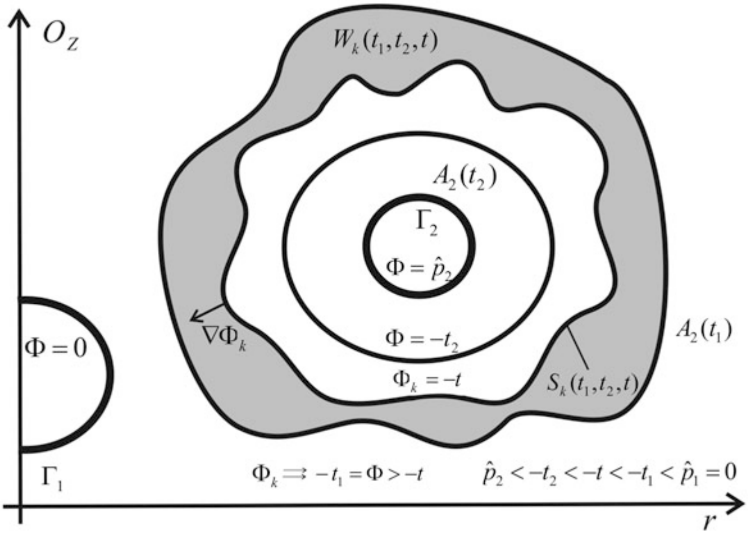

Fig. 3 The surface $S_k(t_1, t_2, t)$ (here $A(t)$ is denoted as $A_2(t)$)

For this construction, the results of Sect. 3.2 are used for the restrictions of the stream function ψ on suitable compact subdomains of P_+ (see details in [39]).

We have also one additional property (cf. with Lemma 4 for the plane case):

(iv) There exists a set $\mathscr{T} \subset (0, \delta_p)$ of full measure (i.e., meas $\big((0, \delta_p) \setminus \mathscr{T}\big) = 0$) such that for all $t \in \mathscr{T}$ the set $A(t)$ is *a regular cycle*, i.e., it is a C^1-curve homeomorphic to the circle and

$$\Phi_k(x) \rightrightarrows \Phi(x) \equiv -t \qquad \text{on } A(t). \tag{77}$$

Let $t_1, t_2 \in \mathscr{T}$ and $t_1 < t' < t'' < t_2$. The very important issue is to construct for sufficiently large $k \geq k_0$ and for almost all $t \in \big[t', t''\big]$ a C^1-circle $S_k(t)$ which separates $A(t_1)$ from $A(t_2)$ and satisfies $\Phi_k|_{S_k(t)} \equiv -t$. Moreover, the gradient of Φ_k is directed toward Γ_1.

For this purpose, for $t \in [t', t'']$ denote by $W_k(t_1, t_2; t)$ the bounded connected component of the open set $\{x \in P_+ \setminus A(t_1) : \Phi_k(x) > -t\}$ such that $\partial W_k(t_1, t_2; t) \supset A(t_1)$ (see Fig. 3). This definition is valid since for sufficiently large k by the convergence (77), the estimates

$$\Phi_k|_{A(t_1)} > -t, \qquad \Phi_k|_{A(t_2)} < -t \qquad \forall t \in [t', t''] \tag{78}$$

hold. Now put

$$S_k(t_1, t_2; t) = (\partial W_k(t_1, t_2; t)) \setminus A(t_1).$$

Clearly, $\Phi_k \equiv -t$ on $S_k(t_1, t_2; t)$. Moreover, $S_k(t_1, t_2; t)$ separates $A(t_1)$ from $A(t_2)$ because of the monotonicity property (iii) and (78) (see Fig. 3).

Recall, that for a set $A \subset P_+$, $\tilde{A}$ denotes the set in $\mathbb{R}^3$ obtained by rotation of A around O_z-axis. By construction, for every regular value $t \in [t', t''] \Subset (t_1, t_2)$, the set $S_k(t_1, t_2; t)$ is a C^1-circle; consequently, $\tilde{S}_k(t_1, t_2; t)$ is a torus, and

$$\int\limits_{\tilde{S}_k(t_1,t_2;t)} \nabla\Phi_k \cdot \mathbf{n}\, dS = -\int\limits_{\tilde{S}_k(t_1,t_2;t)} |\nabla\Phi_k|\, dS < 0, \tag{79}$$

where $\mathbf{n}$ is the unit outward normal vector to $\partial\tilde{W}_k(t_1, t_2; t)$.

Now formulate the key estimate.

Lemma 10. *For any $t_1, t_2, t', t'' \in \mathscr{T}$ with $t_1 < t' < t'' < t_2$, there exists $k_* = k_*(t_1, t_2, t', t'')$ such that for every $k \geq k_*$ and for almost all $t \in [t', t'']$, the inequality*

$$\int\limits_{\tilde{S}_k(t_1,t_2;t)} |\nabla\Phi_k|\, dS < \mathcal{F}t, \tag{80}$$

holds with the constant $\mathcal{F}$ independent of t, t_1, t_2, t', t'' and k.

Proof. Fix $t_1, t_2, t', t'' \in \mathscr{T}$ with $t_1 < t' < t'' < t_2$. Below always assume that k is sufficiently large; in particular, the set $\tilde{S}_k(t_1, t_2; t)$ is well defined for all $t \in [t', t'']$.

The main idea of the proof of (80) is quite simple: to integrate the equation

$$\Delta\Phi_k = \omega_k^2 + \frac{1}{\nu_k}\text{div}\,(\Phi_k \mathbf{u}_k) \tag{81}$$

over the suitable domain $\Omega_k(t)$ with $\partial\Omega_k(t) \supset \tilde{S}_k(t_1, t_2; t)$ such that the corresponding boundary integrals

$$\Bigg|\int\limits_{\big(\partial\Omega_k(t)\big)\setminus\tilde{S}_k(t_1,t_2;t)} \nabla\Phi_k \cdot \mathbf{n}\, dS\Bigg| \tag{82}$$

$$\frac{1}{\nu_k}\Bigg|\int\limits_{\big(\partial\Omega_k(t)\big)\setminus\tilde{S}_k(t_1,t_2;t)} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, dS\Bigg| \tag{83}$$

are negligible. Such domain $\Omega_k(t)$ can be constructed because of the weak convergences $\Phi_k \rightharpoonup \Phi$, $\mathbf{u}_k \to \mathbf{v}$ from the assumption (E-NS-AX) and the boundary conditions $\|\mathbf{u}_k\|_{L^2(\partial\Omega)} \sim \nu_k$, $\mathbf{v} \equiv \mathbf{0}$, $\Phi \equiv 0$ on $\partial\Omega$ (see (52_3) and (50_3)).

The following technical fact from the one-dimensional real analysis is needed.

Lemma 11. *Let $f : \mathscr{S} \to \mathbb{R}$ be a positive decreasing function defined on a measurable set $\mathscr{S} \subset (0,\delta)$ with* $\operatorname{meas}[(0,\delta)\setminus\mathscr{S}] = 0$. *Then*

$$\sup_{t_1,t_2\in\mathscr{S}} \frac{[f(t_2)]^{\frac{4}{3}}(t_2-t_1)}{(t_2+t_1)(f(t_1)-f(t_2))} = \infty. \tag{84}$$

The proof of this fact is elementary (see, e.g., [39, Appendix]).

For $t \in \mathscr{T}$ denote by $U(t)$ the bounded connected component (the torus) of the set $\mathbb{R}^3 \setminus \tilde{A}(t)$. By construction, $U(t_2) \Subset U(t_1)$ for $t_1 < t_2$.

From estimate (80), the isoperimetric inequality and from the Coarea formula, one can easily deduce

Lemma 12. *For any $t_1, t_2 \in \mathscr{T}$ with $t_1 < t_2$, the estimate*

$$\operatorname{meas} U(t_2)^{\frac{4}{3}} \le C\,\frac{t_2+t_1}{t_2-t_1}\big[\operatorname{meas} U(t_1) - \operatorname{meas} U(t_2)\big] \tag{85}$$

holds with the constant C independent of t_1, t_2.

The proof of this lemma is based on the same idea $\big($Coarea formula for $\int|\nabla\Phi_k|$ and $\int|\nabla\Phi_k|^2\big)$ as in Lemma 6 discussed for the plane case.

The last estimate leads us to the main result of this subsection.

Lemma 13. *Assume that $\Omega \subset \mathbb{R}^3$ is an exterior axially symmetric domain of type (38) with C^2-smooth boundary $\partial\Omega$ and $\mathbf{f} \in W^{1,2}(\Omega) \cap L^{6/5}(\Omega)$, $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ are axially symmetric. Then the assumptions (E-NS-AX) and (73) lead to a contradiction.*

Proof. By construction, $U(t_1) \supset U(t_2)$ for $t_1, t_2 \in \mathscr{T}$, $t_1 < t_2$. Thus the just obtained estimate (85) contradicts Lemma 11. This contradiction finishes the proof of Lemma 13. □

4.2.6 The Case $\operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) > 0$.

The cases (b) and (c), where $\operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) > 0$ (see (74) and (75)), are easily reduced to the plane case, because now one can separate, by the level sets of Φ, the region where Φ is close to maximum both from infinity and from the singularity axis O_z and carry out all arguments in the constructed bounded plane domain.

4.3 A Simple Proof of the Existence Theorem in the Case Without Swirl

Here we discuss an alternative (and much more simple) approach to the existence result for the $3D$ exterior domain in the axially symmetric case without swirl. The proof is based on the idea of [36] of using some divergence identities for solutions to the Euler equations (see also [42]).

4.3.1 Some Identities for Solutions to the Euler System

Let the conditions (E-AX) from Lemma 8 be fulfilled, i.e., the axially symmetric functions $(\mathbf{v}, p)$ satisfy the Euler equations (50) and

$$\mathbf{v} \in L^6(\mathbb{R}^3), \quad p \in L^3(\mathbb{R}^3),$$
$$\nabla \mathbf{v} \in L^2(\mathbb{R}^3), \quad \nabla p \in L^{3/2}(\mathbb{R}^3), \quad \nabla^2 p \in L^1(\mathbb{R}^3)$$

(these properties were discussed in Sect. 4.2.3; without loss of generality, assume that $\mathbf{v}$ is extended by zero into $\mathbb{R}^3 \setminus \Omega$ and put $p(x) = \hat{p}_j$ for $x \in \Omega_j$, $j = 1, \ldots, N$). Assume also that (73) is valid, i.e.,

$$\Phi(x) \leq 0. \tag{86}$$

First of all, discuss the integrability properties of these functions on half-plane P_+. For any axially symmetric vector function $\mathbf{g} = (g_\theta, g_r, g_z)$, the following equality

$$\begin{aligned} |\nabla \mathbf{g}|^2 = \frac{|g_\theta|^2}{r^2} + \frac{|g_r|^2}{r^2} + |\partial_r g_r|^2 + |\partial_z g_r|^2 + |\partial_r g_\theta|^2 \\ + |\partial_z g_\theta|^2 + |\partial_r g_z|^2 + |\partial_z g_z|^2 \end{aligned} \tag{87}$$

holds. Thus, $\frac{|\mathbf{g}_r|}{r} \leq |\nabla \mathbf{g}|$. Applying this formula to $\mathbf{g} = \nabla p = (\partial_r p, 0, \partial_z p)$ we get, by virtue of $\nabla^2 p \in L^1(\mathbb{R}^3)$,

$$\frac{\partial_r p}{r} \in L^1(\mathbb{R}^3). \tag{88}$$

Hence $\partial_r p \in L^1(P_+)$. Since $p(r, z) \to 0$ for each $z \in \mathbb{R}$ as $r \to \infty$, the inclusion

$$p(r, \cdot) \in L^1(\mathbb{R}) \tag{89}$$

is valid for each $r > 0$; moreover,

$$\int_{\mathbb{R}} p(t, z)\, dz = -\int_t^\infty \int_{\mathbb{R}} \partial_r p(r, z)\, dz\, dr \to 0 \qquad \text{as } t \to \infty. \tag{90}$$

From the last formula and the inequality $\Phi \le 0$, it follows that

$$|\mathbf{v}|^2(r,\cdot) \in L^1(\mathbb{R}) \tag{91}$$

for each $r > 0$. Further, (87) and $\nabla \mathbf{v} \in L^2(\mathbb{R}^3)$ imply

$$\frac{|v_\theta|^2}{r} + \frac{|v_r|^2}{r} \in L^1(P_+). \tag{92}$$

From the Euler system (50), it follows by direct calculation that for any smooth function $\mathbf{g}$, the following identity

$$\operatorname{div}\,[p\,\mathbf{g} + (\mathbf{v}\cdot\mathbf{g})\mathbf{v}] = p\operatorname{div}\mathbf{g} + \big[(\mathbf{v}\cdot\nabla)\mathbf{g}\big]\cdot\mathbf{v} \tag{93}$$

holds. Apply this formula two times for $\mathbf{g} = r\mathbf{e}_r$ and $\mathbf{g} = \frac{1}{r}\mathbf{e}_r$, where $\mathbf{e}_r$ is the unit vector parallel to the r-axis.

(I) $\mathbf{g} = r\mathbf{e}_r$, $\operatorname{div}\,[p\,\mathbf{g} + (\mathbf{v}\cdot\mathbf{g})\mathbf{v}] = 2p + v_\theta^2 + v_r^2$. Integrating this identity over the 3D infinite cylinder $C_t = \Big\{(x_1,x_2,x_3) \in \mathbb{R}^3 : r = \sqrt{x_1^2 + x_2^2} < t\Big\}$ yields

$$t^2 \int\limits_{\mathbb{R}} \big[p(t,z) + v_r^2(t,z)\big]\,dz = \iint\limits_{P_t} r\big[2p + v_\theta^2 + v_r^2\big]\,dz\,dr, \tag{94}$$

where $P_t = \{(r,z) \in P_+ : r < t\}$.

(II) $\mathbf{g} = \frac{1}{r}\mathbf{e}_r$, $\operatorname{div}\,[p\,\mathbf{g} + (\mathbf{v}\cdot\mathbf{g})\mathbf{v}] = \frac{1}{r^2}(v_\theta^2 - v_r^2)$. Since there is an essential singularity at $r = 0$, one needs to integrate this identity over $C_{t_0t} = \Big\{(x_1,x_2,x_3) \in \mathbb{R}^3 : r = \sqrt{x_1^2 + x_2^2} \in (t_0,t)\Big\}$ to obtain

$$\begin{aligned} &\textstyle\int_{\mathbb{R}} \big[p(t,z) + v_r^2(t,z)\big]\,dz - \int_{\mathbb{R}} \big[p(t_0,z) + v_r^2(t_0,z)\big]\,dz \\ &= \iint\limits_{P_{t_0t}} \big[\tfrac{1}{r}(v_\theta^2 - v_r^2)\big]\,dz\,dr, \end{aligned} \tag{95}$$

where $P_{t_0t} = \{(r,z) \in P_+ : r \in (t_0,t)\}$.

Since $\int_{\mathbb{R}} \big[p(t,z) + v_r^2(t,z)\big]\,dz \to 0$ as $t \to +\infty$ and

$$\iint\limits_{P_+} \big|\frac{1}{r}\left(v_\theta^2 - v_r^2\right)\big|\,dz\,dr < \infty,$$

it follows from the above formulas that

$$\int\limits_{\mathbb{R}} \left[p(t,z) + v_r^2(t,z)\right] dz = \iint\limits_{P_{t\infty}} \left[\frac{1}{r}\left(v_r^2 - v_\theta^2\right)\right] dz\, dr, \tag{96}$$

where $P_{t\infty} = \{(r,z) \in P_+ : r \in (t, +\infty)\}$.

4.3.2 Proof of the Existence Theorem

As in the beginning of Sect. 4.2.4, assume that the assumptions (E-NS-AX) (see Lemma 8) are satisfied, but now suppose, additionally, that all functions $\mathbf{a}, \mathbf{f}, \mathbf{u}_k, \mathbf{v}$ have no swirl. Our goal is to receive a contradiction. This implies the validity of Theorem 7 for the case with no swirl.

It turns out that a contradiction for this case could be obtained extremely easy. Consider the limit solution $(\mathbf{v}, p)$ to the Euler equations (50) from Lemma 8 (E-AX). It is necessary to discuss only the case (73), since for other two cases (74)–(75), the arguments are carried out for bounded plane domains (see Sect. 4.2.6). From (73), (94), it follows that $\int_{\mathbb{R}} \left[p(t,z) + v_r^2(t,z)\right] dz \le 0$ for all $t > 0$. But in the case $v_\theta \equiv 0$, the equality (96) implies $\int_{\mathbb{R}} \left[p(t,z) + v_r^2(t,z)\right] dz > 0$ a contradiction.

Remark 6. The similar idea was used in [36] to obtain the existence theorem for annulus-type plane domain under inflow conditions (the flux through the external boundary component is nonpositive) and in [42] to prove the Liouville theorem in $\mathbb{R}^3$ for the D-solution without swirl of the stationary Navier–Stokes system. Note that this result of [42] could be easily derived from the Liouville-type theorem for *ancient solutions* of nonsteady Navier–Stokes system in [33].

5 2*D* Axially Symmetric Case: Exterior Domain

5.1 Formulation of the Problem and Historical Review

Let Ω be an exterior domain of $\mathbb{R}^2$:

$$\Omega = \mathbb{R}^2 \setminus \Big(\bigcup_{j=1}^{N} \bar{\Omega}_j \Big), \tag{97}$$

where $\Omega_j \subset \mathbb{R}^2$, $j = 1, \dots, N$, are bounded, simply connected domains with Lipschitz boundaries and $\bar{\Omega}_j \cap \bar{\Omega}_i = \emptyset$ for $i \ne j$. Look for a solution of the steady-state Navier–Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{h} & \text{on } \partial\Omega, \end{cases} \tag{98}$$

satisfying the additional condition at infinity

$$\lim_{|x|\to+\infty} \mathbf{u}(x) = \xi\mathbf{e}_1, \tag{99}$$

where for simplicity it is assumed that $\mathbf{f}$ vanishes outside a disk.

The two-dimensional problem in an exterior domain is much harder than the above three-dimensional case (see Sect. 4.2). The main difficulty is to find a solution satisfying the condition at infinity (99). In 1933 J. Leray [49] proved that if the boundary data are sufficiently regular, $\mathbf{f} = 0$, and the fluxes through every $\partial\Omega_i$ vanish

$$\mathcal{F}_i = \int\limits_{\partial\Omega_i} \mathbf{h}\cdot\mathbf{n}\, dS = 0, \tag{100}$$

then problem (98) has a weak solution $(\mathbf{u}, p)$ with finite Dirichlet integral

$$\int\limits_{\Omega} |\nabla\mathbf{u}|^2 dx < +\infty. \tag{101}$$

To show this, Leray introduced an elegant argument, known as *invading domains method*, which consists in proving first that the Navier–Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{u}_k + \big(\mathbf{u}_k\cdot\nabla\big)\mathbf{u}_k + \nabla p_k = 0 & \text{in } \Omega_k, \\ \operatorname{div}\, \mathbf{u}_k = 0 & \text{in } \Omega_k, \\ \mathbf{u}_k = \mathbf{h} & \text{on } \partial\Omega, \\ \mathbf{u}_k = \xi\mathbf{e}_1 & \text{on } \partial B_k \end{cases} \tag{102}$$

has a weak solution $\mathbf{u}_k$ for every bounded domain $\Omega_k = \Omega \cap B_k$, $B_k = \{x \in \mathbb{R}^2 : |x| < k\}$, $B_k \Supset \complement\Omega$, and shows that the following estimate holds

$$\int\limits_{\Omega_k} |\nabla\mathbf{u}_k|^2 dx \le c, \tag{103}$$

for some positive constant c independent of k. While (103) is sufficient to assure existence of a subsequence $\mathbf{u}_{k_l}$ which converges weakly to a solution $\mathbf{u}$ of (98) satisfying (101), it does not give any information about the behavior at infinity of the velocity $\mathbf{u}$ (Indeed, the unbounded function $\log^\alpha |x|$ ($\alpha \in (0, 1/2)$) satisfies (101).), i.e., we do not know whether this limiting solution $\mathbf{u}$ satisfies the condition at infinity (99). That means that the limiting solution $\mathbf{u}$ does not "remember" about the boundary value $\xi\mathbf{e}_1$ despite the fact that this boundary value was used in the construction of $\mathbf{u}_{k_l} \rightharpoonup \mathbf{u}$ (cf. with Sect. 4.2.2 for 3D case).

In 1961 H. Fujita recovered, by means of a different method, Leray's result. Nevertheless, due to the lack of a uniqueness theorem, the solutions constructed by Leray and Fujita are not comparable, even for very small ν. The solution to (98) constructed by the invading domains method is called *Leray's solution*, while any solution satisfying (101) is called *D-solution*.

Only 40 years after Leray's paper, D. Gilbarg and H.F. Weinberger [25] were able to show that the velocity $\mathbf{u}$ in Leray's solution is bounded, p converges uniformly to a constant at infinity, and there is a constant vector $\bar{\mathbf{u}}$ such that

$$\lim_{r\to+\infty}\int_0^{2\pi}|\mathbf{u}(r,\theta)-\bar{\mathbf{u}}|^2 d\theta = 0 \tag{104}$$

(here (r,θ) denote polar coordinates with center at O). Moreover, they proved that if $\bar{\mathbf{u}} = 0$, then the convergence is uniform and $\nabla\mathbf{u}$ decays at infinity as $r^{-3/4}\log r$. In the subsequent paper [26], the same authors proved that a bounded D-solution meets the same asymptotic properties as the Leray solution (see also [2]). One of the most difficult and unanswered questions is the relation between $\xi\mathbf{e}_1$ and $\bar{\mathbf{u}}$. To point out the difficulties of the problem, recall that even the linear Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{u}+\nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{h} & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to+\infty}\mathbf{u}(x) = \xi\mathbf{e}_1, \end{cases} \tag{105}$$

does not have, in general, a solution. Indeed, the solutions of the homogeneous problem

$$\begin{cases} -\nu\Delta\mathbf{v}+\nabla Q = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = 0 & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to+\infty}\dfrac{\mathbf{v}(x)}{|x|} = 0, \end{cases}$$

spans a two-dimensional linear space $\mathfrak{C}$, and (105) is solvable *if and only if* the data satisfy the following compatibility condition (Stokes' paradox)

$$\int_{\partial\Omega}(\mathbf{h}-\xi\mathbf{e}_1)\cdot[\nu(\nabla\mathbf{v}+\nabla\mathbf{v}^\top)\cdot\mathbf{n}-Q\mathbf{n}]dS = 0, \quad \forall\,\mathbf{v}\in\mathfrak{C} \tag{106}$$

(see [8, 23]). Let us observe, by the way, that this is not surprising. Indeed, the natural solution to $(105)_{1,2,3}$ should behave at infinity as the fundamental solution to (105) ($\mathbf{u} = O(\log r)$), and the addition of $(105)_4$ makes (105) overdetermined. Therefore, (106) appears to be quite natural.

Unexpectedly, in 1967 R. Finn and D.R. Smith [17] discovered the existence of a solution to (98), (99) without any compatibility relation between $\mathbf{h}$ and $\xi \neq 0$, for ν sufficiently large. They also showed that $(\mathbf{u} - \xi\mathbf{e}_1, p)$ behaves at infinity as the fundamental solution of the linear Oseen system (see also [22]). In particular, taking also into account the results in [11, 72], one obtains the following behavior:

$$\begin{aligned} u_1 - \xi &= O(r^{-1/2}), \quad u_2 = O(r^{-1}\log r), \\ \nabla\mathbf{u} &= O(r^{-1}\log^2 r), \quad p = O(r^{-1}\log r), \end{aligned} \tag{107}$$

and outside a parabolic "wake region" around axis $\mathbf{e}_1$, the decay is more rapid; in particular, $\omega = \partial_1 u_2 - \partial_2 u_1$ behaves according to

$$\omega(x) = O(e^{c(x_1 - |x|)}) \tag{108}$$

for some absolute constant c. R. Finn and D.R. Smith called a solution $(\mathbf{u}, p)$ to (98), (99) *physically reasonable* provided $\mathbf{u} - \xi\mathbf{e}_1 = O(r^{-1/4-\varepsilon})$ for some positive ε. D.R. Smith [72] proved that a physically reasonable solution satisfies (107) and D.C. Clark [11] that (107) implies (108). More recently, V. I. Sazonov [71] showed that a D-solution such that $\mathbf{u} - \xi\mathbf{e}_1 = o(1)$, with $\xi \neq 0$, is physically reasonable (see also [21, 24]). Notice that nothing is currently known about the asymptotic behavior, in general, for $\xi = 0$ or for arbitrary ν.

Later, in 1988, problem (98), (99) was taken up by C.J. Amick [3] under the assumption $\mathbf{f} = 0$. He proved that if $\mathbf{h} = 0$, then any D-solution is bounded and converges to $\bar{\mathbf{u}}$ in the sense of (104). Moreover, he considered a particular but physically interesting class of solutions $\mathbf{u} = (u_1, u_2)$ such that u_1 is an even function of x_2 and u_2 is an odd function of x_2:

$$u_1(x_1, x_2) = u_1(x_1, -x_2), \quad u_2(x_1, x_2) = -u_2(x_1, -x_2) \tag{109}$$

in the symmetric domain

$$(x_1, x_2) \in \Omega \Leftrightarrow (x_1, -x_2) \in \Omega. \tag{110}$$

Using Leray's argument C.J. Amick showed that for symmetric solutions the convergence of $\mathbf{u}$ at infinity is uniform; moreover, if $\partial\Omega$ is regular enough and $\mathbf{h} = 0$, then $\mathbf{u}$ is nontrivial, i.e., $\mathbf{u} \neq 0$ whenever $\xi \neq 0$. (Amick assumes Ω to be of class C^3. Recently, this result has been extended to Lipschitz domains [68].) These last results rule out the Stokes paradox for the nonlinear case for symmetric domains and homogeneous boundary data. A clear exposition of Amick's results, as well as the results outlined above, can be found in [22]. For an exterior domain condition (100) has been replaced in [66] by the weaker assumption that the sum $\sum_i |\mathcal{F}_i|$ is sufficiently small. An interesting approach to the existence of solutions to (98)–(99) with $\xi = 0$ and small data has been recently proposed by M. Hillairet and P. Wittwer [28].

Finally, in the recent paper [61] mentioned by the authors, the problem (98), (99) with $\xi = 0$ was considered in exterior plane domains symmetric with respect to both coordinate axes and a solution was found in the class of vector fields $\mathbf{v} \in \mathfrak{C}_0$ satisfying the following symmetry conditions:

$$\begin{aligned} v_1(x_1, x_2) &= v_1(x_1, -x_2) = -v_1(-x_1, x_2), \\ v_2(x_1, x_2) &= -v_2(x_1, -x_2) = v_2(-x_1, x_2). \end{aligned} \tag{111}$$

It is proved in [61] that if data $\mathbf{h}, \mathbf{f} \in \mathfrak{C}_0$ satisfy only natural regularity assumptions, then (98) has a D-solution in $\mathfrak{C}_0$ which converges uniformly to zero at infinity. The flux of the boundary value $\mathbf{h}$ over $\partial\Omega$ in this case is arbitrary.

All abovementioned results (except [61]) were proved either under the condition that all fluxes F_i are equal to zero (see (100)) or assuming that fluxes F_i are "small." Besides the relation between $\xi \mathbf{e}_1$ and $\bar{\mathbf{u}}$, another relevant problem in the theory of the stationary Navier–Stokes equations is to ascertain whether a solution to problem (98) exists without any restriction on the fluxes $\mathcal{F}_i$. For exterior plane domains, this problem, in general, is unsolved until now (solutions of the problem for bounded plane and $3D$ axially symmetric domains as well as for $3D$ axially symmetric exterior domains were discussed above in Sects. 3 and 4).

In the last paper [40] it is proved for arbitrary fluxes $\mathcal{F}_i$ the existence of a D-solution to problem (98) for exterior plane domains in the case when only Amick's symmetric conditions (109)–(110) are satisfied and every Ω_i intersects the x_1–axis, i.e.,

$$\Omega_i \bigcap O_{x_1} \neq \emptyset \quad \text{for all} \quad i = 1, \dots, N. \tag{112}$$

5.2 Formulation of the Main Result

Theorem 11 ([40]). *Let $\Omega \subset \mathbb{R}^2$ be a symmetric exterior domain* (97), (110), (110), (112) *with multiply connected Lipschitz boundary $\partial\Omega$ consisting of N disjoint components Γ_j, $j = 1, \dots, N$. Assume that $\boldsymbol{f}$ is a symmetric* (*in the sense of* (109)) *distribution such that the corresponding linear functional $H(\Omega) \ni \boldsymbol{\eta} \mapsto \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\eta}$ is continuous* (*with respect to the norm $\|\cdot\|_{H(\Omega)}$*) *and $\mathbf{h}$ is a symmetric field in $W^{1/2,2}(\partial\Omega)$. Then problem* (98) *admits at least one symmetric weak solution $\mathbf{u}$. The following estimate*

$$\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} \leq c\Big(\|\mathbf{h}\|^2_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{h}\|^4_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{f}\|^2_*\Big) \tag{113}$$

is valid.

Here the total flux

$$\mathcal{F} = \int_{\partial\Omega} \mathbf{h}(x) \cdot \mathbf{n}(x)\, dS = \sum_{i=1}^{N} \mathcal{F}_i \tag{114}$$

is not required to be zero or small. By what was said before, if $\mathbf{f}$ has a compact support, then the solution converges uniformly at infinity to a constant vector $\alpha\mathbf{e}_1$; moreover, for $\alpha \neq 0$, it behaves at large distance according to (107), (108). However, it is not known whether this solution satisfies the condition at infinity (99).

The proof of Theorem 11 is based on Leray's method of invading domains. The needed a priori estimate is obtained using the special extension of the boundary value satisfying the Leray–Hopf inequality (cf. with (9)) which is obtained by applying a new inequality of Poincaré type (see Lemma 16) that could be useful also in other contexts.

5.3 Some Estimates for Plane Functions with Finite Dirichlet Integral

Lemma 14. *Let Ω be the exterior domain (97), $v \in D(\Omega)$. Then the following inequality*

$$\int_{\Omega} \frac{|v(x)|^2}{|x|^2 \log^2 |x|} dx \leq c \int_{\Omega} |\nabla v(x)|^2 dx \tag{115}$$

holds.

Inequality (115) is well known (e.g., [47]).

As it follows from (115), functions $v \in D(\Omega)$ do not have to vanish at infinity. The next assertion gives some information about the behavior of a function of $D(\Omega)$ as $|x| \to \infty$.

Lemma 15. *Let Ω be the exterior domain (97), $v \in D(\Omega)$. Then*

$$\limsup_{r\to\infty} \frac{1}{\log r} \int_0^{2\pi} |v(r,\theta)|^2 d\theta \leq 2 \int_{\Omega} |\nabla v(x)|^2 dx. \tag{116}$$

Inequality (116) is proved in [26] (see Lemma 2.1).

Lemma 16 ([40]). *Let Ω be the exterior domain (97), $v \in D(\Omega)$, $\kappa > 0$, $\alpha \in (0,1)$, $R_* \geq R_0 > 1$. Then the following inequality*

$$\int\limits_{\mathbb{R}\setminus(-R_*,R_*)} \int\limits_0^{\kappa|x_1|^\alpha} \frac{|v(x_1,x_2)|^2}{|x|^2} dx_1 dx_2 \le c \int\limits_\Omega |\nabla v(x)|^2 dx \tag{117}$$

holds. The constant c in (117) *depends only on* R_0, κ, *and* α.

5.4 Construction of the Extension of the Boundary Value

Below only the construction of the extension of the boundary value is given. Other details of the proof can be found in [40].

Let $\psi \in C^\infty(\mathbb{R})$ be a nonnegative function such that $0 \le \psi(t) \le 1$,

$$\psi(t) = \begin{cases} 1, \ t \ge 1, \\ 0, \ t \le 0, \end{cases}$$

and $\gamma \in C^\infty(\mathbb{R})$ be a monotone function on $\mathbb{R}_+$ with $\gamma(t) \ge \gamma_0 > 0$,

$$\gamma(t) = \begin{cases} |t|^\alpha, \ |t| \ge 3R_0, \\ 1, \quad |t| \le 2R_0, \end{cases}$$

where $\alpha \in (0, 1)$.

Let $\Omega_+ = \{x \in \Omega : \ x_2 > 0\}$ and $\Omega_- = \{x \in \Omega : \ x_2 < 0\}$. Set

$$\Delta_+(x) = x_2\big(\chi(x_1) + (1 - \chi(x_1))\delta(x)\big), \quad x \in \Omega_+,$$

where $\chi \in C^\infty(\mathbb{R})$ is a monotone function with

$$\chi(t) = \begin{cases} 1, \ |t| \ge 2R_0, \\ 0, \ |t| \le \dfrac{3}{2}R_0, \end{cases}$$

and $\delta(x)$ is the regularized distance from the point $x \in \Omega$ to $\partial\Omega = \bigcup\limits_{j=1}^N \partial\Omega_j$. Notice that $\delta(x)$ is infinitely differentiable function in $\mathbb{R}^2\setminus\partial\Omega$ and the following inequalities

$$a_1 d(x) \le \delta(x) \le a_2 d(x), \ \ |D^\alpha \delta(x)| \le a_3 d^{1-|\alpha|}(x)$$

hold. Here $d(x) = \operatorname{dist}(x, \partial\Omega)$ is the Euclidean distance from x to $\partial\Omega$ (see [74]).

Let $\varepsilon \in (0, 1)$ be an arbitrary number. In the domain Ω_+, define the cutoff function

$$\zeta_+(x,\varepsilon) = \psi\left(\varepsilon \ln\left(\frac{\varepsilon\gamma(x_1)}{\Delta_+(x)}\right)\right).$$

Obviously,

$$\zeta_+(x,\varepsilon) = \begin{cases} 0, \ \varepsilon\gamma(x_1) < \Delta_+(x), \\ 1, \ \Delta_+(x) < \varepsilon e^{-\frac{1}{\varepsilon}}\gamma(x_1). \end{cases}$$

Define

$$\mathbf{b}(x) = \frac{1}{2\pi}\nabla \ln|x| = \frac{1}{2\pi}\left(\frac{x_1}{|x|^2}, \frac{x_2}{|x|^2}\right).$$

The vector field $\mathbf{b}(x)$ satisfies the symmetry conditions (109). Moreover, it is well known that the flux of $\mathbf{b}(x)$ over a closed curve γ is equal to 1,

$$\int_\gamma \mathbf{b}(x)\cdot\mathbf{n}(x)\,d\gamma = 1,$$

if and only if the domain bounded by γ contains the point $x = 0$. Here $\mathbf{n}$ is unit vector of outward (with respect to the domain bounded by γ) normal to γ. Otherwise the flux is equal to zero.

Let $x^{(j)} = \left(x_1^{(j)}, 0\right) \in \Omega_j$, $j = 1, \ldots, N$. Put

$$\mathbf{b}^{(j)}(x) = -\mathcal{F}_j\,\mathbf{b}\left(x - x^{(j)}\right).$$

Then

$$\int_{\Gamma_j} \mathbf{b}^{(j)}(x)\cdot\mathbf{n}(x)\,dS = \mathcal{F}_j, \quad \int_{\Gamma_i} \mathbf{b}^{(j)}(x)\cdot\mathbf{n}(x)\,dS = 0, \ \ i \neq j.$$

In the domain Ω_+, the functions $\mathbf{b}^{(j)}(x)$ could be represented in the form

$$\mathbf{b}^{(j)}(x) = \frac{\mathcal{F}_j}{2\pi}\nabla^\perp\varphi_+^{(j)}(x), \quad \varphi_+^{(j)}(x) = \mathrm{arctg}\frac{x_1 - x_1^{(j)}}{x_2}, \ x \in \Omega_+, \ \ j = 1, \ldots, N,$$

where $\nabla^\perp = \left(\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}\right)$. Notice that $|\varphi_+^{(j)}(x)| \leq \pi/2$ for $x \in \overline{\Omega}_+$ and $j = 1, \ldots, N$. Define

$$\begin{aligned}\mathbf{B}_+^{(j)}(x,\varepsilon) &= \frac{\mathcal{F}_j}{2\pi}\nabla^\perp\Big(\zeta_+(x,\varepsilon)\varphi_+^{(j)}(x)\Big) \\ &= \frac{\mathcal{F}_j}{2\pi}\Big(\varphi_+^{(j)}(x)\nabla^\perp\zeta_+(x,\varepsilon) + \zeta_+(x,\varepsilon)\nabla^\perp\varphi_+^{(j)}(x)\Big).\end{aligned}$$

Then div $\mathbf{B}_+^{(j)}(x,\varepsilon) = 0$, and, since $\zeta_+(x,\varepsilon) = 1$ in the neighborhood of $\partial\Omega_+$, it follows that

$$\mathbf{B}_+^{(j)}(x,\varepsilon)\Big|_{\partial\Omega_+} = \frac{\mathcal{F}_j}{2\pi}\nabla^\perp\varphi_+^{(j)}(x)\Big|_{\partial\Omega_+}.$$

Lemma 17. *Let* $j = 1,\dots,N$. *Then for every* $\delta > 0$, *there exists* $\varepsilon = \varepsilon(\delta)$ *such that the following inequality*

$$\Big|\int\limits_{\Omega_+}\big(\mathbf{u}(x)\cdot\nabla\big)\mathbf{u}(x)\cdot\mathbf{B}_+^{(j)}(x,\varepsilon)\,dx\Big| \le \delta\int\limits_{\Omega_+}|\nabla\mathbf{u}(x)|^2\,dx \quad \forall\,\mathbf{u}\in H_S(\Omega) \tag{118}$$

holds. $H_S(\Omega)$ *is the subspace of functions from* $H(\Omega)$ *satisfying the symmetry conditions* (109).

The proof of the Leray–Hopf inequality (118) is based on Lemmas 14–16 and is true only for functions $\mathbf{u}$ in $H_S(\Omega)$, i.e., satisfying the symmetry conditions (109). For an arbitrary function $\mathbf{u}\in H(\Omega)$, this inequality can be wrong.

Define

$$\mathbf{B}^{(j)}(x,\varepsilon)=\begin{cases}\big(B_{+,1}^{(j)}(x_1,x_2,\varepsilon),\,B_{+,2}^{(j)}(x_1,x_2,\varepsilon)\big),\ x\in\Omega_{+,\varepsilon},\\ \big(B_{+,1}^{(j)}(x_1,-x_2,\varepsilon),-B_{+,2}^{(j)}(x_1,-x_2,\varepsilon)\big),\ x\in\Omega_{-,\varepsilon},\end{cases}$$

and

$$\mathbf{B}(x,\varepsilon) = \sum_{j=1}^{N}\mathbf{B}^{(j)}(x,\varepsilon).$$

The vector field $\mathbf{B}$ is symmetric,

$$\mathrm{div}\mathbf{B} = 0, \qquad \int\limits_{\Gamma_j}\mathbf{B}\cdot\mathbf{n}dS = \mathcal{F}_j,\ \ j = 1,\dots,N.$$

Let $\mathbf{h}_1(x) = \mathbf{h}(x) - \mathbf{B}(x,\varepsilon)\big|_{\partial\Omega}$. Then

$$\int\limits_{\Gamma_j}\mathbf{h}_1(x)\cdot\mathbf{n}(x)dS = 0, \quad j = 1,\dots,N. \tag{119}$$

If $\mathbf{h} \in W^{1/2,2}(\partial\Omega)$, then obviously $\mathbf{h}_1 \in W^{1/2,2}(\partial\Omega)$ and

$$\|\mathbf{h}_1\|_{W^{1/2,2}(\partial\Omega)} \le c\Big(\|\mathbf{h}\|_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{B}\big|_{\partial\Omega}\|_{W^{1/2,2}(\partial\Omega)}\Big)$$
$$\le c\Big[\|\mathbf{h}\|_{W^{1/2,2}(\partial\Omega)} + \Big(\sum_{j=1}^{N} \mathcal{F}_j^2\Big)^{1/2}\Big] \le c\|\mathbf{h}\|_{W^{1/2,2}(\partial\Omega)}.$$

Because of condition (119), there exists a function $\mathbf{H} \in H(\Omega)$ such that $\operatorname{supp}\mathbf{H}(x,\varepsilon)$ is contained in a small neighborhood of the boundary $\partial\Omega$,

$$\operatorname{div}\mathbf{H} = 0, \quad \mathbf{H}(x,\varepsilon)|_{\partial\Omega} = \mathbf{h}_1(x), \quad \mathbf{H} \in L^4(\Omega), \ \ \nabla\mathbf{H} \in L^2(\Omega),$$
$$\|\mathbf{H}\|_{L^4(\Omega)} + \|\nabla\mathbf{H}\|_{L^2(\Omega)} \le c\|\mathbf{h}_1\|_{W^{1/2,2}(\partial\Omega)} \le c\|\mathbf{h}\|_{W^{1/2,2}(\partial\Omega)}.$$

Moreover, $\mathbf{H}(x,\varepsilon)$ satisfies Leray–Hopf's inequality, i.e., for every $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta)$ such that

$$\Big|\int_\Omega \big(\mathbf{u}(x)\cdot\nabla\big)\mathbf{u}(x)\cdot\mathbf{H}(x,\varepsilon)\,dx\Big| \le \delta\int_\Omega |\mathbf{u}(x)|^2\,dx \quad \forall\,\mathbf{u} \in H(\Omega)$$

holds (see [47]).

The function $\mathbf{H}(x,\varepsilon)$ is not necessarily symmetric. However, its boundary value is symmetric and, therefore, $\mathbf{H}(x,\varepsilon)$ can be symmetrized defining the function $\widetilde{\mathbf{H}}(x,\varepsilon)$ as follows:

$$\tilde{H}_1(x,\varepsilon) = \frac{1}{2}\big[H_1(x_1,x_2,\varepsilon) + H_1(x_1,-x_2,\varepsilon)\big],$$
$$\tilde{H}_2(x,\varepsilon) = \frac{1}{2}\big[H_2(x_1,x_2,\varepsilon) - H_2(x_1,-x_2,\varepsilon)\big].$$

Put

$$\mathbf{A}(x,\varepsilon) = \mathbf{B}(x,\varepsilon) + \widetilde{\mathbf{H}}(x,\varepsilon).$$

Lemma 18. *(i) The vector field $\mathbf{A}(x,\varepsilon)$ is symmetric,*

$$div\ \mathbf{A}(x,\varepsilon) = 0, \quad \mathbf{A}(x,\varepsilon)\big|_{\partial\Omega} = \mathbf{h}(x),$$

(ii) $\mathbf{A} \in L^4(\Omega)$, $\nabla\mathbf{A} \in L^2(\Omega)$,

$$\|\mathbf{A}\|_{L^4(\Omega)} + \|\nabla\mathbf{A}\|_{L^2(\Omega)} \le c\|\mathbf{h}\|_{W^{1/2,2}(\partial\Omega)}. \tag{120}$$

(iii) For every $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta)$ such that the inequality

$$\Big| \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{A} \, dx \Big| \leq \delta \int_\Omega |\nabla \mathbf{u}|^2 \, dx \quad \forall \, \mathbf{u} \in H_S(\Omega)$$

holds.

The constant c in (120) depends on ε and tends to infinity as $\varepsilon \to 0$. This inequality is used with sufficiently small but fixed ε.

Remark 7. If the domain Ω and the data are symmetric with respect to both coordinate axes, the existence of a weak solution which also satisfies symmetry conditions (111) can be proved. In this case the solution satisfies the condition at infinity (99) with $\xi = 0$:

$$\lim_{|x| \to \infty} \mathbf{v}(|x|, \theta) = 0$$

uniformly in θ, i.e.,

$$\lim_{(x_1, x_2) \to \infty} \mathbf{v}(x_1, x_2) = 0$$

(see [61]).

6 Conclusion

The first paper devoted to the existence of solutions to the stationary Navier–Stokes problem without smallness assumptions on data was that of J. Leray [49], under the sole hypothesis that the fluxes through any connected component of the boundary vanish. The question whether this condition could be removed was by then a fundamental open problem in the mathematical theory of incompressible fluid dynamics and was the object of researches of several outstanding mathematicians. A comprehensive account of attempts devoted to give an answer to this question is contained in the book of G.P. Galdi [21]. Recently, the problem has been solved for (i) two-dimensional bounded domains [36, 41]; (ii) two-dimensional exterior axially symmetric domains and symmetric data [40, 61]; and (iii) three-dimensional bounded and exterior axially symmetric domains and symmetric data [37, 39, 41]. However, it remains much to do in order to get a complete picture of the flow of an incompressible fluid under adherence boundary conditions. Among the still open problems of particular interest are the following: (i′) to remove the symmetry assumptions required in [37, 39, 41]; (ii′) to determine the behavior at infinity of the solutions found in [39, 40], also under symmetry assumptions; and (iii′) to prove or disprove the Liouville theorem in the class of D-solutions vanishing at infinity in the three-dimensional case [21, 42] (see also [33]).

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Cross-References

- ▶ Existence of Strong Stationary Solutions in Bounded and Exterior Domains Including Spatial Asymptotics
- ▶ Stationary Navier–Stokes Flow Around Rotating Bodies and Moving Bodies
- ▶ Stationary Navier–Stokes Flow in Exterior Domains and Landau Solutions

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