DOI 10.1070/RM2014v069n06ABEH004928

The flux problem for the Navier–Stokes equations

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Abstract. This is a survey of results on the Leray problem (1933) for the Navier–Stokes equations of an incompressible fluid in a domain with multiple boundary components. Imposed on the boundary of the domain are inhomogeneous boundary conditions which satisfy the necessary requirement of zero total flux. The authors have proved that the problem is solvable in arbitrary bounded planar or axially symmetric domains. The proof uses Bernoulli's law for weak solutions of the Euler equations and a generalization of the Morse–Sard theorem for functions in Sobolev spaces. New a priori bounds for the Dirichlet integral of the velocity vector field in symmetric flows, as well as estimates for the regular component of the velocity in flows with singularities of source/sink type are presented.

Bibliography: 60 titles.

Keywords: Navier–Stokes and Euler equations, multiple boundary components, Dirichlet integral, virtual drain, Bernoulli's law, maximum principle.

Contents

The first author was supported by grant no. 14-01-00768-a from the Russian Foundation for Basic Research and grant no. МД-5146.2013.1 for Support of Young Doctors of the Sciences from the President of the Russian Federation, the second author was supported by grant no. CH-ŠMM-01/01 from the Lithuanian–Swiss Cooperation Programme, and the third author was supported by the Siberian Branch of the Russian Academy of Sciences (grant no. 38 of the Programme of Joint Integration Projects of the Siberian, Ural, and Far-Eastern Branches of the Russian Academy of Sciences).

AMS 2010 *Mathematics Subject Classification*. Primary 35Q30, 35Q31, 76D05; Secondary 76D07, 76D10.

1. Introduction

We consider boundary-value problems for the Navier–Stokes equations in domains with multiple boundary components. These equations are the basic mathematical model used in hydrodynamics. For a steady flow of an incompressible fluid they have the form

$$-\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f}, \qquad x\in\Omega, \tag{1.1}$$

$$\operatorname{div}\mathbf{u}=0, \qquad x\in\Omega. \tag{1.2}$$

Here $\mathbf{u}(x)$ is the velocity vector, $p(x)$ is the ratio of the pressure to the constant density of the fluid, $\mathbf{f}$ is the acceleration of the external mass forces, $\nu=\text{const}>0$ is the kinematic viscosity coefficient, and Ω is a bounded domain in $\mathbb{R}^n$ $(n=2,3)$. On its boundary $\partial\Omega$ we impose the boundary condition

$$\mathbf{u}=\mathbf{a}(x), \qquad x\in\partial\Omega. \tag{1.3}$$

Assume that $\partial\Omega$ consists of $N+1$ connected components Γ_i,

$$\partial\Omega=\Gamma_0\cup\Gamma_1\cup\Gamma_2\cup\dots\cup\Gamma_N,$$

and $\Gamma_i\cap\Gamma_j=\varnothing$ for $i\neq j$. Here the surface (curve) Γ_0 bounds Ω from the outside, while the other connected components of the boundary Γ_i, $i=1,\dots,N$, lie inside this surface, so that

$$\Omega=\Omega_0\setminus\biggl(\bigcup_{i=1}^N\overline{\Omega}_i\biggr), \qquad \overline{\Omega}_i\subset\Omega_0, \quad i=1,\dots,N, \tag{1.4}$$

and $\Gamma_i=\partial\Omega_i$ (see Fig. 1). In view of the continuity equation (1.2), the function $\mathbf{a}$ must satisfy the condition

$$\int_{\partial\Omega}\mathbf{a}\cdot\mathbf{n}\,dS=\sum_{i=0}^N\int_{\Gamma_i}\mathbf{a}\cdot\mathbf{n}\,dS=0 \tag{1.5}$$

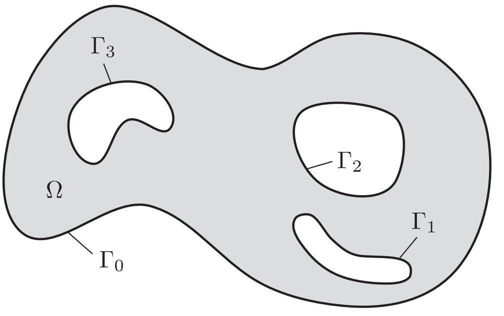

Figure 1. The domain Ω

(here $\mathbf{n}$ is the unit outward normal to $\partial\Omega$), which means that the flux of the incompressible fluid across the boundary of the flow region is equal to zero. In the case when Ω has a connected boundary (that is, $N=0$) and certain smoothness conditions are fulfilled, Leray proved the solvability of (1.1)–(1.3) in his famous 1933 paper [1]. The same result holds when the flux F_i of the velocity vector across each connected component Γ_i of the boundary vanishes:

$$\int_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\,dS \equiv F_i = 0, \qquad i=0,1,\dots,N. \tag{1.6}$$

(This condition means that the flow region contains neither sources nor sinks.)

It remained an open question as to whether the necessary condition $\sum_{i=1}^N F_i = 0$ (see (1.5)) is sufficient for (1.1)–(1.3) to be solvable. We are interested in the situation when (1.6) fails to hold. In that case we shall call the problem (1.1)–(1.3) the steady flux problem for the Navier–Stokes equations. It is also called the **Leray problem** because it actually goes back to his paper [1] cited above.

The Navier–Stokes equations have long attracted the interest of mathematicians: suffice it to mention that Leray's result on the solvability of (1.1)–(1.3), (1.6) was the first example when the topological Leray–Schauder fixed point principle was applied to a concrete problem in mechanics. The method of matching asymptotic expansions was developed by Prandtl in his analysis of the flow problem for (1.1) as $\nu\to 0$. The extension of the notion of an elliptic system of differential equations was a by-product of Douglis and Nirenberg's investigation of a linearized version of (1.1).

Let us now consider a non-stationary analogue of (1.1)–(1.3). In it one must find a solution $\mathbf{u}(x,t)$, $p(x,t)$ of the system

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \qquad x\in\Omega, \quad t\in(0,T), \tag{1.7}$$

$$\operatorname{div}\mathbf{u} = 0, \qquad x\in\Omega, \quad t\in(0,T), \tag{1.8}$$

which satisfies the boundary condition

$$\mathbf{u} = \mathbf{a}, \qquad x\in\partial\Omega, \quad t\in(0,T), \tag{1.9}$$

and the initial condition

$$\mathbf{u} = \mathbf{u}_0(x), \qquad x\in\Omega, \quad t=0. \tag{1.10}$$

A generalized solution of (1.7)–(1.10) satisfies the energy identity, which implies that the $L^2(\Omega)$-norm of $\mathbf{u}(x,t)$ is bounded for almost all $t \in (0,T)$. No restrictions on the values of the partial fluxes across the connected boundary components Γ_i of $\partial\Omega$ are imposed. This estimate can be improved using methods presented in Ladyzhenskaya's book [2], resulting eventually in a proof of the unique solvability of the two-dimensional problem for any $T > 0$. As for the three-dimensional problem, the unique solvability of the problem (1.7)–(1.10) in a suitable class of generalized solutions could be proved only on a finite time interval whose length tends to zero with increasing norm of $\mathbf{u}_0$. Hopf [3] proved the existence of a weak generalized solution of the three-dimensional problem on each finite time interval. To prove that the Hopf solution is smooth (and therefore unique) is one of the seven Millenium Prize Problems. It is indisputably the central problem of mathematical hydrodynamics, but its inclusion in the Millenium list had the consequence of overshadowing other problems connected with the Navier–Stokes equations. On the other hand, there exist many other interesting current problems in this list; several such problems were formulated by Yudovich [4]. The steady flux problem (1.1)–(1.3) was among these.

Assume that the domain Ω has a Lipschitz boundary $\partial\Omega$ and that the function $\mathbf{a}$ in (1.3) belongs to the class $W^{1/2,2}(\partial\Omega)$. If the boundary vector field $\mathbf{a}$ satisfies (1.5), then it has a divergence-free extension $\mathbf{A}$ to Ω such that $\mathbf{A} \in W^{1,2}(\Omega)$ and

$$\|\mathbf{A}\|_{W^{1,2}(\Omega)} \leqslant c\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)} \tag{1.11}$$

(see [5]; here and below, c denotes various positive constants). Let $H(\Omega)$ be the Hilbert space equal to the completion of the set of divergence-free vector-valued functions $\boldsymbol{\eta} \in C_0^\infty(\Omega)$ in the metric corresponding to the inner product

$$[\mathbf{v}, \boldsymbol{\eta}] = \int_\Omega \nabla\mathbf{v} \cdot \nabla\boldsymbol{\eta}\, dx, \qquad \text{where} \quad \nabla\mathbf{v} \cdot \nabla\boldsymbol{\eta} = \sum_{i,j=1}^n \frac{\partial v_i}{\partial x_j}\,\frac{\partial \eta_i}{\partial x_j}\,.$$

We call a vector field $\mathbf{u} \in W^{1,2}(\Omega)$ a generalized solution of (1.1)–(1.3) if the following conditions hold:

(a) there exists a divergence-free vector-valued function $\mathbf{A} \in W^{1,2}(\Omega)$ such that $\mathbf{A}\big|_{\partial\Omega} = \mathbf{a}$ and $\mathbf{u} - \mathbf{A} = \mathbf{w} \in H(\Omega)$;

(b) $\mathbf{w}$ satisfies the integral identity

$$\begin{aligned} &\nu \int_\Omega \nabla\mathbf{w} \cdot \nabla\boldsymbol{\eta}\, dx - \int_\Omega ((\mathbf{w} + \mathbf{A}) \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{w}\, dx - \int_\Omega (\mathbf{w} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A}\, dx \\ &\qquad = -\nu \int_\Omega \nabla\mathbf{A} \cdot \nabla\boldsymbol{\eta}\, dx + \int_\Omega (\mathbf{A} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A}\, dx + \int_\Omega \mathbf{f} \cdot \boldsymbol{\eta}\, dx \qquad \forall\, \boldsymbol{\eta} \in H(\Omega). \end{aligned} \tag{1.12}$$

This identity is equivalent to the operator equation

$$\mathbf{w} = \mathrm{T}\mathbf{w} \tag{1.13}$$

in the Hilbert space $H(\Omega)$, where T is the operator defined by

$$[\mathrm{T}\mathbf{w}, \boldsymbol{\eta}] = \nu^{-1}\int_\Omega ((\mathbf{w}+\mathbf{A})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{w}\,dx + \nu^{-1}\int_\Omega (\mathbf{w}\cdot\nabla)\,\boldsymbol{\eta}\cdot\mathbf{A}\,dx$$
$$-\int_\Omega \nabla\mathbf{A}\cdot\nabla\boldsymbol{\eta}\,dx + \nu^{-1}\int_\Omega (\mathbf{A}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{A}\,dx + \nu^{-1}\int_\Omega \mathbf{f}\cdot\boldsymbol{\eta}\,dx \quad \forall\,\mathbf{w},\boldsymbol{\eta}\in H(\Omega).$$

This operator is completely continuous, so we can use the Leray–Schauder fixed point theorem [2] to prove that (1.13) is solvable. To do this it is sufficient to show that all solutions of the equation

$$\mathbf{w}^{(\lambda)} = \lambda\mathrm{T}\mathbf{w}^{(\lambda)}, \qquad \lambda\in[0,1], \tag{1.14}$$

are bounded in the norm of $H(\Omega)$, or in other words, they have a bounded Dirichlet integral

$$I = [\mathbf{w},\mathbf{w}] \equiv \int_\Omega \nabla\mathbf{w}\cdot\nabla\mathbf{w}\,dx. \tag{1.15}$$

In his pathbreaking paper [1] Leray proved that all solutions of (1.1)–(1.3) satisfying the additional condition (1.6) have a bounded Dirichlet integral. His proof was by contradiction and did not give any a priori estimate for the Dirichlet integral (1.15) in terms of the problem data. Under the additional assumption (1.6) such a bound was first obtained by Hopf [6] in 1941. Both these papers prompted further investigations of the flux problem.

Hopf's construction was also used to replace the restrictive condition (1.6) by the assumption that the fluxes F_i are small [7]–[13]. Parts of this construction, in combination with the notion of a virtual drain introduced in [10] and [14], underlie the solvability proof for two-dimensional flux problems with the additional assumptions of symmetry of the flow region and the boundary vector field $\mathbf{a}$ (see [14]–[16]). In the earlier paper [17] Amick proved the solvability of the planar symmetric flux problem by contradiction. In [14]–[16] the proof was based on a priori bounds for the Dirichlet integral of the velocity vector field. Important in themselves, these bounds are also needed in justifying methods used for the numerical solution of the flux problem, such as finite difference methods or the Galerkin method.

Leray's argument, enriched with new methods from the theory of functions and the theory of elliptic equations, resulted in the solution of the Leray problem in the planar and axially symmetric cases: the planar and the axially symmetric flux problems were proved to be solvable for any values of the fluxes of the velocity vector across the connected boundary components of the flow region [18]–[24]. It is important that no constraints were imposed on the topology of the flow region. Results from [18]–[24] are central in this survey. The main result was proved by three of the authors in the planar case [22], and it can be stated as follows.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with C^2-boundary $\partial\Omega$. If $\mathbf{f}\in W^{1,2}(\Omega)$ and the boundary data $\mathbf{a}\in W^{3/2,2}(\partial\Omega)$ satisfy the condition* (1.5), *then the problem* (1.1)–(1.3) *has at least one generalized solution $\mathbf{u}\in W^{1,2}(\Omega)$.*

Remark 1.1. It is well known (see, for instance, [2]) that under the assumptions of Theorem 1.1 each weak solution $\mathbf{u}$ of (1.1)–(1.3) has enhanced smoothness: $\mathbf{u}\in W^{2,2}(\Omega)\cap W^{3,2}_{\mathrm{loc}}(\Omega)$. That is, the solution is as regular as the problem data allow. In particular, $\mathbf{u}$ is C^∞-smooth if $\mathbf{f}$, $\mathbf{a}$, and $\partial\Omega$ are in the smoothness class C^∞.

The proof of the existence theorem is based on a priori bounds derived by means of a *reductio ad absurdum* argument put forward already by Leray [1]. A novel feature in the application of this argument is the use of Bernoulli's law for Sobolev solutions of the Euler equations, which was obtained in [18] (a proof with all details was presented in [19]). The results on Bernoulli's law are based on the recent version of the Morse–Sard theorem established by Bourgain, Korobkov, and Kristensen in the joint paper [25]. In particular, it follows from this theorem that for a function $\psi \in W^{2,1}(\Omega)$ almost all its level sets are unions of finitely many C^1-curves. This enables one to construct an appropriate subregion (bounded by smooth streamlines) and to find an estimate for the L^2-norm of the total head pressure gradient. Some ideas which are close (heuristically) to Hopf's maximum principle for solutions of elliptic systems of partial differential equations are used in the proof (see § 3.3.1 for more details). Eventually, the desired contradiction is obtained with the use of the co-area formula.

Apart from the main problem (1.1)–(1.3) we shall investigate here its singular analogues: the axially symmetric problem with sources or sinks on the axis of symmetry and the planar problem with a source or a sink in the flow region. In addition, we look at the axially symmetric flux problem in the 'stream function–vorticity' variables in a domain like a spherical layer. We conclude the paper by listing possible generalizations of our results and by stating unsolved problems.

2. Historical survey

2.1. Hopf's lemma. The flux problem for the Navier–Stokes equations has been the subject of more than 100 papers by authors from 11 countries. We start by presenting Hopf's results from [6], where he gave an estimate of the Dirichlet integral for generalized solutions of the problem (1.1)–(1.3), (1.6). Setting $\boldsymbol{\eta} = \mathbf{w}$ in the identity (1.12) and using the equalities

$$\int_\Omega (\mathbf{A}\cdot\nabla)\mathbf{w}\cdot\mathbf{w}\,dx = 0, \qquad \int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{w}\,dx = 0 \qquad \forall\,\mathbf{w}\in H(\Omega),$$

we arrive at the equality

$$\begin{aligned}\nu\int_\Omega \nabla\mathbf{w}\cdot\nabla\mathbf{w}\,dx = \int_\Omega \mathbf{A}\cdot(\mathbf{w}\cdot\nabla)\mathbf{w}\,dx - \nu\int_\Omega \nabla\mathbf{A}\cdot\nabla\mathbf{w}\,dx \\ + \int_\Omega \mathbf{A}\cdot(\mathbf{A}\cdot\nabla)\mathbf{w}\,dx.\end{aligned} \tag{2.1}$$

The main difficulty in finding a priori estimates for solutions is to estimate the first integral on the right-hand side of (2.1). If (1.6) holds, then we can use Hopf's lemma to this end.

Lemma 2.1 (Hopf). *Let Ω be a domain with Lipschitz boundary $\partial\Omega$ and let $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$. If (1.6) is satisfied, then for any $\varepsilon > 0$ the field $\mathbf{a}$ has a divergence-free extension $\mathbf{A}(\,\cdot\,,\varepsilon) \in W^{1,2}(\Omega)$ to Ω such that*

$$\biggl|\int_\Omega \mathbf{A}\cdot(\mathbf{w}\cdot\nabla)\mathbf{w}\,dx\biggr| \leqslant \varepsilon\|\nabla\mathbf{w}\|^2_{L^2(\Omega)} \qquad \forall\,\mathbf{w}\in H(\Omega). \tag{2.2}$$

The proof of this lemma is carried out in two steps. First we extend the boundary field $\mathbf{a}$ as $\mathbf{B} = \operatorname{curl} \mathbf{D}$, where $\mathbf{D} \in W^{2,2}(\Omega)$, and then we modify this extension using the Hopf cutoff function $\zeta(x,\varepsilon)$, which depends on the parameter $\varepsilon > 0$, in such a way that $\mathbf{A} = \operatorname{curl}(\zeta(x,\varepsilon)\mathbf{D}(x))$ [6]. The function ζ has the following properties: $\zeta = 1$ for $x \in \Gamma_i$ with $i = 1,\dots,N$; the support of ζ lies in a small neighbourhood of the surface $\partial\Omega$; ζ decays rapidly away from this surface; finally,

$$|\nabla\zeta(x,\varepsilon)| \leqslant \frac{c\varepsilon}{d(x)},$$

where $d(x) = \operatorname{dist}(x,\partial\Omega)$, and the constant c is independent of ε. A thorough proof of Hopf's lemma for $\mathbf{a}$ satisfying (1.6), provided that there exists a $\mathbf{B} = \operatorname{curl} \mathbf{D}$ such that $\mathbf{B}\big|_{\partial\Omega} = \mathbf{a}$, was presented by Ladyzhenskaya in [2]. An effective construction of a vector field $\mathbf{B} = \operatorname{curl} \mathbf{D}$ for smooth surfaces Γ_i was given by Fujita [7] and Finn [8]. We can extend the boundary field $\mathbf{a}$ to Ω as a curl only when all the fluxes of $\mathbf{a}$ across the boundary components Γ_i are zero, that is, when (1.6) holds.

For planar or axially symmetric flows the required representation for the field $\mathbf{B}$ can easily be obtained in terms of the stream function. For a planar flow the stream function $\psi(x_1,x_2)$ of a divergence-free vector field $\mathbf{B}$ is determined by the equations

$$\frac{\partial\psi}{\partial x_2} = B_1, \qquad \frac{\partial\psi}{\partial x_1} = -B_2.$$

If (1.6) holds, then we can construct ψ as a solution of the problem

$$\begin{gathered} \Delta\Delta\psi = 0, \qquad x \in \Omega, \\ \psi = \int_0^s \mathbf{a}\cdot\mathbf{n}\,dS, \quad \frac{\partial\psi}{\partial n} = \mathbf{a}\cdot\mathbf{s}, \qquad x \in \Gamma_i, \quad i = 1,\dots,N. \end{gathered}$$

Here s and 0 are the current point and a fixed point on Γ_i and $\mathbf{s}$ is the unit tangent vector to this curve. The solution of this problem is uniquely defined because of (1.6).

Now we take a smooth function $\gamma \in C^\infty(\mathbb{R})$ such that $0 \leqslant \gamma(t) \leqslant 1$ and

$$\gamma(t) = \begin{cases} 1, & t \geqslant 1, \\ 0, & t \leqslant 0, \end{cases} \tag{2.3}$$

and we let

$$\zeta(x,\delta) = \gamma\biggl(\delta\log\frac{\rho}{\Delta(x)}\biggr), \tag{2.4}$$

where ρ is a sufficiently small positive number and $\Delta(x)$ is the generalized distance from the point $x \in \Omega$ to the boundary $\partial\Omega$. We recall that $\Delta(x)$ is an infinitely differentiable function of $x \in \mathbb{R}^n \setminus \partial\Omega$ with the following properties:

$$a_1 d(x) \leqslant \Delta(x) \leqslant a_2 d(x), \qquad |D^\alpha\Delta(x)| \leqslant a_3(\alpha) d^{1-|\alpha|}(x) \tag{2.5}$$

(see [26]). It is easy to see that ζ is infinitely differentiable, $\zeta(x,\delta)\big|_{\partial\Omega} = 1$, ζ vanishes identically outside a small neighbourhood of the boundary,

$$\zeta(x,\delta) = \begin{cases} 1, & \Delta(x) \leqslant e^{-1/\delta}\rho, \\ 0, & \Delta(x) \geqslant \rho, \end{cases}$$

and we have

$$|\nabla\zeta(x,\delta)| \leqslant \frac{c\delta}{d(x)} \tag{2.6}$$

with a constant c independent of δ.

We set $\mathbf{A}(x,\delta) = \operatorname{curl}\big(\zeta(x,\delta)\mathbf{D}(x)\big)$. Clearly, $\mathbf{A} \in W^{1,2}(\Omega)$,

$$\operatorname{div}\mathbf{A} = 0, \quad \mathbf{A}\big|_{\partial\Omega} = \mathbf{a}, \quad \operatorname{supp}\mathbf{A} \subset \{x \in \Omega\colon \Delta(x) \leqslant \rho\},$$

$$\mathbf{A}(x,\delta) = \nabla\zeta(x,\delta) \times \mathbf{D}(x) + \zeta(x,\delta)\operatorname{curl}\mathbf{D}(x) =: \mathbf{A}_1(x,\delta) + \mathbf{A}_2(x,\delta),$$

and

$$|\mathbf{A}_1(x,\delta)| \leqslant \frac{c\delta}{d(x)}, \qquad |\mathbf{A}_2(x,\delta)| \leqslant |\operatorname{curl}\mathbf{D}(x)|. \tag{2.7}$$

We can now estimate the integral $\int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{A}\,dx$ as follows:

$$\begin{aligned}
\int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{A}\,dx &\leqslant \biggl(\int_\Omega |\mathbf{w}|^2|\mathbf{A}|^2\,dx\biggr)^{1/2}\biggl(\int_\Omega |\nabla\mathbf{w}|^2\,dx\biggr)^{1/2} \\
&\leqslant 2\biggl(\int_\Omega |\mathbf{w}|^2|\mathbf{A}_1|^2\,dx + \int_\Omega |\mathbf{w}|^2|\mathbf{A}_2|^2\,dx\biggr)^{1/2}\biggl(\int_\Omega |\nabla\mathbf{w}|^2\,dx\biggr)^{1/2} \\
&\leqslant c\biggl(\delta^2\int_\Omega \frac{|\mathbf{w}(x)|^2}{d^2(x)}\,dx + \int_{\{x\in\Omega\colon \Delta(x)\leqslant\rho\}} |\mathbf{w}|^2|\operatorname{curl}\mathbf{D}|^2\,dx\biggr)^{1/2}\biggl(\int_\Omega |\nabla\mathbf{w}|^2\,dx\biggr)^{1/2}.
\end{aligned} \tag{2.8}$$

If the domain has a Lipschitz boundary, then we can use Hardy's inequality to prove the estimate

$$\int_\Omega \frac{|\mathbf{w}(x)|^2}{d^2(x)}\,dx \leqslant c\int_\Omega |\nabla\mathbf{w}(x)|^2\,dx$$

(see [2]). Furthermore, by embedding theorems and Friedrichs' inequality,

$$\begin{aligned}
&\int_{\{x\in\Omega\colon \Delta(x)\leqslant\rho\}} |\mathbf{w}|^2|\operatorname{curl}\mathbf{D}|^2\,dx \\
&\qquad\leqslant \biggl(\int_{\{x\in\Omega\colon \Delta(x)\leqslant\rho\}} |\mathbf{w}|^4\,dx\biggr)^{1/2}\biggl(\int_{\{x\in\Omega\colon \Delta(x)\leqslant\rho\}} |\operatorname{curl}\mathbf{D}|^4\,dx\biggr)^{1/2} \\
&\qquad\leqslant c\biggl(\int_\Omega |\nabla\mathbf{w}|^2\,dx\biggr)\|\mathbf{D}\|^2_{W^{2,2}(\{x\colon \Delta(x)\leqslant\rho\})}.
\end{aligned}$$

Hence, we can rewrite (2.8) as

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{A}\,dx \leqslant c\big(\delta^2 + \|\mathbf{D}\|^2_{W^{2,2}(\{x\colon \Delta(x)\leqslant\rho\})}\big)^{1/2}\int_\Omega |\nabla\mathbf{w}|^2\,dx. \tag{2.9}$$

For any $\varepsilon > 0$ we take $\delta = \varepsilon/(2c)$ and choose a sufficiently small $\rho > 0$ so that

$$\|\mathbf{D}\|_{W^2_2(\{x\colon \Delta(x)\leqslant\rho\})} \leqslant \frac{\varepsilon}{2c};$$

then we can deduce Hopf's inequality (2.2) from (2.9).

Now that we have at our disposal a vector field $\mathbf{A}(\,\cdot\,,\varepsilon) \in W^{1,2}(\Omega)$ such that $\mathbf{A}\big|_{\partial\Omega} = \mathbf{a}$ and (2.2) holds with $\varepsilon = \nu/2$, we can deduce from (2.1) the a priori estimate

$$\|\nabla \mathbf{w}\|^2_{L^2(\Omega)} \leqslant c\big(\|\nabla \mathbf{A}\|^2_{L^2(\Omega)} + \|\mathbf{A}\|^2_{L^4(\Omega)}\big), \tag{2.10}$$

which ensures that the problem (1.1)–(1.3), (1.6) is solvable.

Thus, we see that the condition (1.6) is sufficient for the existence (for any $\varepsilon > 0$) of a divergence-free extension $\mathbf{A}(x,\varepsilon)$ of $\mathbf{a}$ satisfying (2.2). But it turns out that (1.6) is also a necessary condition. The first counterexample demonstrating this was due to Takeshita [27] (1993). Here we present a simplified version of Takeshita's construction, which was presented in Galdi's book [28]. Let Ω be an annulus: $\Omega = \{x \in \mathbb{R}^2 \colon R_1 < |x| < R_2\}$. Suppose that

$$F = \int_{S_{R_2}} \mathbf{a}\cdot\mathbf{n}\, dS = -\int_{S_{R_1}} \mathbf{a}\cdot\mathbf{n}\, dS < 0. \tag{2.11}$$

Representing the vector field $\mathbf{A}$ in polar variables (r,θ) and using the fact that $\mathbf{A}$ is divergence-free, that is,

$$\frac{\partial(rA_r)}{\partial r} + \frac{\partial A_\theta}{\partial\theta} = 0,$$

we deduce from (2.11) that

$$r\int_0^{2\pi} A_r(r,\theta,\varepsilon)\, d\theta = F \qquad \forall\, r \in (R_1,R_2).$$

Let us take a vector field $\mathbf{w} = u(r)\mathbf{e}_\theta$ with $u \in C_0^\infty[R_1,R_2]$. Obviously, $\mathbf{w} \in H(\Omega)$, and for such $\mathbf{w}$ we have

$$\begin{aligned}\int_\Omega \mathbf{A}\cdot(\mathbf{w}\cdot\nabla)\mathbf{w}\, dx &= -\int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{A}\cdot\mathbf{w}\, dx\\ &= -\int_\Omega\left(\frac{1}{r}\,\frac{\partial A_\theta}{\partial\theta} + \frac{A_r}{r}\right)u^2(r)\, dx = -F\int_R^{R_2}\frac{u^2(r)}{r}\, dr.\end{aligned} \tag{2.12}$$

If (2.2) holds, then (2.12) implies that

$$-F\int_{R_1}^{R}\frac{u^2(r)}{r}\, dx \leqslant \varepsilon\int_\Omega |\nabla\mathbf{w}|^2\, dx$$

for any $\varepsilon > 0$. But this inequality is clearly impossible if $F < 0$.

More general counterexamples were constructed later by Farwig, Kozono, and Yanagisawa [29] and by Heywood [30]. In particular, Heywood gave a counterexample valid for $F > 0$ as well.

2.2. A method for proving an a priori bound by contradiction. If (1.6) fails to hold, that is, the fluxes F_i are non-zero,[1] then the method described above

[1] In view of the necessary condition (1.5), the total flux vanishes:

$$\int_{\partial\Omega}\mathbf{a}\cdot\mathbf{n}\, dS = \sum_{i=0}^N \int_{\Gamma_i}\mathbf{a}\cdot\mathbf{n}\, dS = \sum_{i=0}^N F_i = 0.$$

does not work. However, one can attempt to prove the required a priori bound by contradiction. This idea was first proposed by Leray [1] and has been subsequently used and modified by many authors (see, for instance, [2], [31]–[33], [17], [19]).

Let $\mathbf{A} \in W^{1,2}(\Omega)$ be some extension of the boundary field $\mathbf{a}$. We must show that the solutions $\mathbf{w}^{(\lambda)}$ of the operator equation (1.14) have norms bounded by a constant independent of $\lambda \in [0,1]$. Suppose not. Then there exist sequences $\{\lambda_k\}_{k\in\mathbb{N}} \subset [0,1]$ and $\{\mathbf{w}_k = \mathbf{w}^{(\lambda_k)}\}_{k\in\mathbb{N}} \in H(\Omega)$ such that

$$\begin{aligned}\nu \int_\Omega \nabla \mathbf{w}_k \cdot \nabla \boldsymbol{\eta}\, dx - \lambda_k \int_\Omega \big((\mathbf{w}_k + \mathbf{A}) \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{w}_k\, dx - \lambda_k \int_\Omega (\mathbf{w}_k \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A}\, dx \\ = -\lambda_k \nu \int_\Omega \nabla \mathbf{A} \cdot \nabla \boldsymbol{\eta}\, dx + \lambda_k \int_\Omega (\mathbf{A} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A}\, dx + \lambda_k \int_\Omega \mathbf{f} \cdot \boldsymbol{\eta}\, dx \quad \forall \boldsymbol{\eta} \in H(\Omega)\end{aligned} \tag{2.13}$$

and

$$\lim_{k\to\infty} \lambda_k = \lambda_0 \in [0,1], \qquad \lim_{k\to\infty} J_k = \lim_{k\to\infty} \|\mathbf{w}_k\|_{H(\Omega)} = \infty. \tag{2.14}$$

Let $\widehat{\mathbf{w}}_k = J_k^{-1}\mathbf{w}_k$. Since $\|\widehat{\mathbf{w}}_k\|_{H(\Omega)} = 1$, there exists a subsequence $\{\widehat{\mathbf{w}}_{k_l}\}$ converging weakly in $H(\Omega)$ to a vector field $\widehat{\mathbf{w}} \in H(\Omega)$. And since the embedding $H(\Omega) \hookrightarrow L^r(\Omega)$ is compact for all $r \in [1,\infty)$ if $n=2$ and for all $r \in [1,6)$ if $n=3$, the subsequence $\{\widehat{\mathbf{w}}_{k_l}\}$ converges strongly in $L^r(\Omega)$. We set $\boldsymbol{\eta} = J_{k_l}^{-1}\widehat{\mathbf{w}}_{k_l}$ in (2.13), and, taking the limit as $k_l \to \infty$ in the resulting equality, we get that

$$\nu = \lambda_0 \int_\Omega (\widehat{\mathbf{w}} \cdot \nabla)\widehat{\mathbf{w}} \cdot \mathbf{A}\, dx. \tag{2.15}$$

In particular, it follows from (2.15) that $\lambda_0 > 0$. Hence the λ_k are bounded away from zero.

Now we take $\boldsymbol{\eta} = J_{k_l}^{-2}\boldsymbol{\xi}$ in (2.13), where $\boldsymbol{\xi}$ is an arbitrary vector field in $H(\Omega)$. Again we take the limit as $k_l \to \infty$ and deduce the integral identity

$$\int_\Omega (\widehat{\mathbf{w}} \cdot \nabla)\widehat{\mathbf{w}} \cdot \boldsymbol{\xi}\, dx = 0 \qquad \forall \boldsymbol{\xi} \in H(\Omega). \tag{2.16}$$

Thus, $\widehat{\mathbf{w}} \in H(\Omega)$ is a generalized solution of the boundary-value problem for the Euler system

$$\begin{cases} (\widehat{\mathbf{w}} \cdot \nabla)\widehat{\mathbf{w}} + \nabla \widehat{p} = 0, & x \in \Omega, \\ \operatorname{div} \widehat{\mathbf{w}} = 0, & x \in \Omega, \\ \widehat{\mathbf{w}} = 0, & x \in \partial\Omega. \end{cases} \tag{2.17}$$

The function $\widehat{p}$ in (2.17) belongs to $W^{1,s}(\Omega)$, where $s \in [1,2)$ if $n=2$ and $s \in [1,3/2]$ if $n=3$. Since $\widehat{\mathbf{w}} = 0$ on $\partial\Omega$, we can use (2.17) to prove that the pressure $\widehat{p}$ takes constant values $\widehat{p}_j$ on the connected components Γ_j of $\partial\Omega$. The next result was established in [33] (Lemma 4) and [17] (Theorem 2.2) (see also [19], Remark 3.2).

Lemma 2.2. *For each solution* $(\mathbf{w}, p) \in (H(\Omega), W^{1,s}(\Omega))$ *of* (2.17) *there exist constants* $\widehat{p}_i \in \mathbb{R}$ *such that*

$$\widehat{p}(x)\big|_{\Gamma_i} = \widehat{p}_i, \qquad i = 0, 1, \dots, N. \tag{2.18}$$

Taking the inner product of the Euler system (2.17) with $\mathbf{A}$, we integrate the result by parts. Then by (2.18),

$$\int_\Omega (\widehat{\mathbf{w}}\cdot\nabla)\widehat{\mathbf{w}}\cdot\mathbf{A}\,dx = -\int_{\partial\Omega}\widehat{p}\,\mathbf{A}\cdot\mathbf{n}\,dS = -\sum_{i=0}^N \widehat{p}_i\int_{\Gamma_i}\mathbf{A}\cdot\mathbf{n}\,dS = -\sum_{i=0}^N \widehat{p}_i F_i. \tag{2.19}$$

If $N=1$ or $F_i=0$ for $i=0,1,\dots,N$ (the condition (1.5) is fulfilled), then (2.19) implies that

$$\int_\Omega (\widehat{\mathbf{w}}\cdot\nabla)\widehat{\mathbf{w}}\cdot\mathbf{A}\,dx = 0. \tag{2.20}$$

This contradicts (2.15). Hence, for $\lambda\in[0,1]$ our assumption fails and the norms of the solutions $\mathbf{w}^{(\lambda)}$ of the operator equation (1.14) are uniformly bounded, so that (1.13) has at least one solution by the Leray–Schauder theorem.

We could make the analogous conclusion in the case when all the constants $\widehat{p}_j$ are equal:

$$\widehat{p}_0 = \widehat{p}_1 = \dots = \widehat{p}_N. \tag{2.21}$$

Indeed, then by (1.5)

$$\sum_{i=0}^N \widehat{p}_i F_i = \widehat{p}_0\sum_{i=0}^N F_i = 0,$$

and (2.20) follows from (2.19) again. However, we cannot say that in the general case all the $\widehat{p}_i$ are equal: Amick [17] constructed a solution of (2.17) for which (2.21) fails to hold. Let $\Omega=\{x\in\mathbb{R}^2\colon 1<|x|<2\}$ be a plane annulus and let $\psi\in C^1([1,2])$, with $\psi'(1)=\psi'(2)=0$ and $\psi''\in L^2((1,2))$. Then the solution of the Euler problem (2.17) is given by

$$\widehat{\mathbf{w}}(x) = \left(\frac{x_2}{|x|}\psi'(|x|), -\frac{x_1}{|x|}\psi'(|x|)\right)\in H(\Omega), \qquad \widehat{p}(x) = \int_1^{|x|}\frac{|\psi'(s)|^2}{s}\,ds.$$

It is easy to see that

$$\widehat{p}(x)\big|_{|x|=1} = 0, \qquad \widehat{p}(x)\big|_{|x|=2} = \int_1^2 \frac{|\psi'(s)|^2}{s}\,ds > 0.$$

2.3. The symmetric planar flux problem. The global solvability of the flux problem without the assumption (1.6) was first established by Amick [17] in 1984. He considered planar flows, assuming that they were symmetric. Following Amick, we give two definitions.

Definition 2.1. A bounded domain $\Omega\subset\mathbb{R}^2$ is said to be *admissible* if it satisfies the following conditions:

(a) $\partial\Omega$ consists of $N+1\geqslant 2$ connected components Γ_i;
(b) Ω is symmetric relative to the line $\{x_2=0\}$ (see Fig. 2);
(c) each component Γ_i intersects the line $\{x_2=0\}$.

Definition 2.2. A function $\mathbf{h}=(h_1,h_2)$ mapping Ω or $\partial\Omega$ into $\mathbb{R}^2$ is said to be *symmetric relative to the line* $\{x_2=0\}$ if h_1 is an even function of x_2, while h_2 is an odd function of x_2.

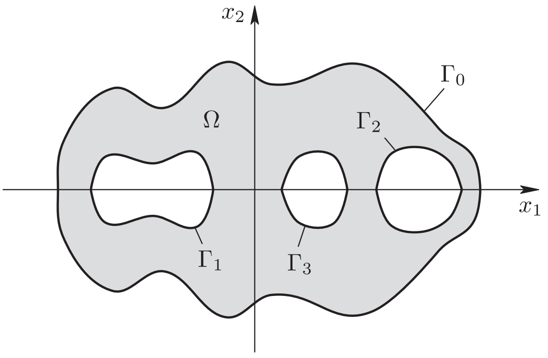

Figure 2. A symmetric plane domain

The planar Navier–Stokes equations are known to be invariant under reflections in the coordinate axes. This property allows one to look for symmetric solutions $(\mathbf{u}, p)$ of the given system in which the velocity $\mathbf{u}$ is symmetric relative to $\{x_2 = 0\}$ and the pressure p is an even function of x_2.

Theorem 2.1 (Amick [17]). *Let $\Omega \subset \mathbb{R}^2$ be an admissible domain with Lipschitz boundary $\partial\Omega$, and let $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ and $\mathbf{f} \in L^2(\Omega)$ be symmetric functions relative to $\{x_2 = 0\}$. Then for any $\nu > 0$ the problem (1.1)–(1.3) has a generalized solution $\mathbf{u} \in W^{1,2}(\Omega)$. In it the velocity vector $\mathbf{u}$ is symmetric relative to $\{x_2 = 0\}$ and the corresponding pressure p is an even function of x_2.*

Amick proved this result by contradiction. We reproduce his arguments showing that all the constants $\widehat{p}_i = \widehat{p}(x)\big|_{\Gamma_i}$ giving the pressure in the Euler problem (2.17) are equal, that is, (2.21) holds. As shown in § 2.2, this is sufficient for a generalized solution of the Navier–Stokes problem (1.1)–(1.3) to exist.

Let Γ_0 be the outer boundary component of $\partial\Omega$. We shall show that $\widehat{p}_0 = \widehat{p}_i$ for $i = 1, \dots, N$. The set $\{x\colon x_2 = 0\} \cap \Gamma_i$ consists of two points, $(\alpha_i, 0)$ and $(\beta_i, 0)$, where $\alpha_i < \beta_i$. In a neighbourhood of $(\alpha_0, 0)$ the boundary component Γ_0 can be represented as a graph $\Gamma_0 = \{(h_0(x_2), x_2)\colon x_2 \in (-\delta, \delta)\}$ for sufficiently small $\delta > 0$. Similarly, in a neighbourhood of $(\alpha_1, 0)$ the component Γ_1 has the form $\Gamma_1 = \{(h_1(x_2), x_2)\colon x_2 \in (-\delta, \delta)\}$. The functions h_i, $i = 0, 1$, are Lipschitz continuous. We set

$$B(\delta) = \{x\colon x_1 \in (h_0(x_2), h_1(x_2)),\ x_2 \in (0, \delta)\} \subset \Omega$$

for sufficiently small δ.

Let $\widehat{\Phi}(x) = \widehat{p}(x) + |\widehat{\mathbf{w}}(x)|^2/2$ denote the total head pressure in the Euler problem (2.17). It is immediate that

$$\frac{\partial}{\partial x_1}\widehat{\Phi} = -\widehat{w}_2\widehat{\omega},$$

where $\widehat{\omega} = \dfrac{\partial \widehat{w}_1}{\partial x_2} - \dfrac{\partial \widehat{w}_2}{\partial x_1}$ is the curl of the velocity vector $\widehat{\mathbf{w}}$. Integrating this equality over $B(\delta)$ and taking (2.18) into account, we find that

$$\int_0^\delta \big(\widehat{p}(h_1(x_2), x_2) - \widehat{p}(h_0(x_2), x_2)\big)\, dx_2 = \delta(\widehat{p}_1 - \widehat{p}_0) = -\int_{B(\delta)} \widehat{w}_2 \widehat{\omega}\, dx.$$

By symmetry, $\widehat{w}_2(x_1, 0) = 0$ (in the sense of traces), and therefore by Hardy's inequality

$$\begin{aligned}\int_{B(\delta)} \frac{|\widehat{w}_2(x_1,x_2)|^2}{x_2^2}\, dx &= \int_{-\infty}^{\infty}\int_0^\delta \frac{|\widehat{w}_2(x_1,x_2)|^2}{x_2^2}\, dx_2\, dx_1 \\ &\leqslant 4\int_{-\infty}^{\infty}\int_0^\delta \left|\frac{\partial \widehat{w}_2(x_1,x_2)}{\partial x_2}\right|^2 dx_2\, dx_1 \leqslant 4\int_{B(\delta)} |\nabla \widehat{\mathbf{w}}(x)|^2\, dx.\end{aligned}$$

Here we have assumed that $\widehat{w}_2$ is extended by zero to $\mathbb{R} \times (0, \delta) \setminus B(\delta)$. It follows from the last two relations that

$$\begin{aligned}|\widehat{p}_0 - \widehat{p}_1|^2 &\leqslant \frac{1}{\delta^2}\bigg(\int_{B(\delta)} x_2^2 \frac{|\widehat{w}_2(x)|^2}{x_2^2}\, dx\bigg)\bigg(\int_{B(\delta)} |\widehat{\omega}(x)|^2\, dx\bigg) \\ &\leqslant \frac{1}{\delta^2}\bigg(4\delta^2\int_{B(\delta)} |\nabla \widehat{\mathbf{w}}(x)|^2\, dx\bigg)\bigg(4\int_{B(\delta)} |\nabla \widehat{\mathbf{w}}(x)|^2\, dx\bigg) \\ &\leqslant 16\bigg(\int_{B(\delta)} |\nabla \widehat{\mathbf{w}}(x)|^2\, dx\bigg)^2 \to 0\end{aligned}$$

as $\delta \to 0$. Hence $\widehat{p}_0 = \widehat{p}_1$. We can prove similarly that

$$\widehat{p}_1 = \widehat{p}_2, \quad \dots, \quad \widehat{p}_{N-1} = \widehat{p}_N.$$

This completes the proof of (2.21).

Sazonov [10] in 1993 (and independently Fujita [14] in 1997) gave an effective proof of the solvability of the symmetric planar flux problem, by constructing a symmetric extension of the boundary field $\mathbf{a}$ which satisfies Hopf's inequality (2.2) for symmetric vector fields. Pukhnachev proved a similar result in the spatial problem with an axis of symmetry and a plane of symmetry orthogonal to the axis [15].

Apparently, Sazonov was not aware of Amick's paper [17], but he proved Theorem 2.1 in a simpler way, using the construction of a virtual drain [10]. The term itself is due to Fujita [14], who also established an a priori bound for the Dirichlet integral (1.15) in the symmetric planar flux problem. (Sazonov used an argument by contradiction.) Below we use ideas due to Sazonov and Fujita to construct such an extension. Let

$$H_S(\Omega) = \{\mathbf{u} \in H(\Omega),\ \mathbf{u} \text{ is symmetric}\}.$$

Lemma 2.3. *Let Ω be an admissible domain with Lipschitz boundary and let $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ be a function symmetric relative to the line $\{x_2 = 0\}$. Then for any*

$\varepsilon > 0$ the vector field $\mathbf{a}$ *has a symmetric divergence-free extension* $\mathbf{A} \in W^{1,2}(\Omega)$ *to* Ω *such that*

$$\left| \int_\Omega \mathbf{A} \cdot (\mathbf{u} \cdot \nabla)\mathbf{u}\, dx \right| \leqslant \varepsilon \|\nabla \mathbf{u}\|^2_{L^2(\Omega)} \qquad \forall\, \mathbf{u} \in H_S(\Omega). \tag{2.22}$$

Furthermore, the support of $\mathbf{A}$ *can lie in an arbitrarily narrow strip in* $\overline{\Omega}$ *which adjoins* $\partial\Omega$ *and the axis of symmetry* x_1*. Namely, for any* $\delta > 0$ *the support of* $\mathbf{A}$ *can be such that*

$$\operatorname{supp} \mathbf{A} \subseteq \omega(\delta) := \{x \in \overline{\Omega}\colon \operatorname{dist}(x, \partial\Omega \cup \{x_2 = 0\}) \leqslant \delta\}.$$

In order not to overburden the reader with technical details, we present the proof of Lemma 2.3 in the case when $\partial\Omega$ consists of two connected components, Γ_0 and Γ_1. We can assume without loss of generality that Ω does not contain the origin, which lies in the domain bounded by Γ_1. In view of (1.5), $F_0 = -F_1 = F$.

In the domain $\Omega_+^{(+)} = \{x \in \Omega\colon x_1 > 0,\ x_2 > 0\}$ we take the cutoff function

$$\zeta_+(\theta, \delta) = \psi\left(\delta \log \frac{\delta}{\theta}\right), \tag{2.23}$$

where $\psi(t)$ is the function in (2.3), $\delta \in (0, 1)$ is sufficiently small, and (r, θ) are polar coordinates in the plane. Obviously,

$$\zeta_+(\theta, \delta) = \begin{cases} 1, & \theta \leqslant \delta e^{-1/\delta}, \\ 0, & \theta \geqslant \delta. \end{cases} \tag{2.24}$$

It is easy to see that $\zeta_+(\theta, \delta) = 1$ in a neighbourhood of the half-line $\{(x_1, x_2) \in \mathbb{R}^2 \setminus \Omega_1\colon x_1 > 0,\ x_2 = 0\}$. Let $\widehat{\zeta}_+(\theta, \delta)$ be the function defined in $\Omega_+ = \{x \in \Omega\colon x_2 > 0\}$ as the extension of $\zeta_+(\theta, \delta)$ to $\Omega_+^{(-)} = \{x \in \Omega_+\colon x_1 < 0\}$ as an odd function of x_1:

$$\widehat{\zeta}_+(\theta, \delta) = \begin{cases} \zeta_+(\theta, \delta), & \theta \in (0, \pi/2], \\ -\zeta_+(\pi - \theta, \delta), & \theta \in [\pi/2, \pi). \end{cases} \tag{2.25}$$

Since $\widehat{\zeta}_+(\theta, \delta) = 0$ on the half-line $\{x \in \mathbb{R}^2 \setminus \Omega_1\colon x_1 = 0,\ x_2 > 0\}$, the function $\widehat{\zeta}_+$ is smooth for $x \in \Omega_+$. Direct calculations show that

$$\left| \frac{d\widehat{\zeta}_+(\theta, \delta)}{d\theta} \right| \leqslant \frac{c_1 \delta}{\theta}, \quad \theta \in \left(0, \frac{\pi}{2}\right), \qquad \left| \frac{d\widehat{\zeta}_+(\theta, \delta)}{d\theta} \right| \leqslant \frac{c_1 \delta}{\pi - \theta}, \quad \theta \in \left(\frac{\pi}{2}, \pi\right), \tag{2.26}$$

$$\left| \frac{\partial\widehat{\zeta}_+(\theta, \delta)}{\partial x_k} \right| \leqslant \frac{c(\delta)}{r}, \qquad \left| \frac{\partial^2\widehat{\zeta}_+(\theta, \delta)}{\partial x_k\, \partial x_r} \right| \leqslant \frac{c(\delta)}{r^2}. \tag{2.27}$$

The constant c_1 in (2.26) is independent of δ, and $c(\delta)$ in (2.27) tends to ∞ as $\delta \to 0$.

Let

$$\mathbf{b}(x) = \nabla \log |x| = \left(\frac{x_1}{|x|^2}, \frac{x_2}{|x|^2}\right).$$

The vector field $\mathbf{b}$ is divergence-free and symmetric relative to the line $\{x_2 = 0\}$. In the polar coordinates it has the form

$$\mathbf{b}(r,\theta) = \big(b_r(r,\theta), b_\theta(r,\theta)\big) = \bigg(\frac{1}{r}\,, 0\bigg). \tag{2.28}$$

In Ω_+ we consider the vector field

$$\mathbf{B}_+(x,\delta) = \big(B_{+,r}(r,\theta,\delta), B_{+,\theta}(r,\theta,\delta)\big) = -\frac{F}{4}\bigg(\frac{1}{r}\,\frac{d\widehat{\zeta}_+(\theta,\delta)}{d\theta}\,, 0\bigg), \tag{2.29}$$

where

$$F = \int_{\Gamma_0} \mathbf{a}\cdot\mathbf{n}\,dS.$$

This field is divergence-free:

$$\operatorname{div}\mathbf{B}_+ = \frac{\partial B_{+,r}}{\partial r} + \frac{B_{+,r}}{r} = -\frac{F}{4}\,\frac{d\widehat{\zeta}_+}{d\theta}\operatorname{div}\mathbf{b}(x) = 0.$$

Moreover, it is easy to verify that

$$\int_{\Gamma_0\cap\mathbb{R}^2_+} \mathbf{B}_+\cdot\mathbf{n}\,d\Gamma = -\int_{\Gamma_1\cap\mathbb{R}^2_+} \mathbf{B}_+\cdot\mathbf{n}\,d\Gamma = \frac{F}{2} \tag{2.30}$$

(recall that $\mathbf{n}$ is the unit outward normal vector relative to Ω).

We claim that for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ such that the field $\mathbf{B}_+(x,\delta)$ satisfies the inequality

$$\bigg|\int_{\Omega_+} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{B}_+\,dx\bigg| \leqslant \frac{\varepsilon}{2}\int_{\Omega_+} |\nabla\mathbf{u}|^2\,dx \qquad \forall\,\mathbf{u}\in H_S(\Omega). \tag{2.31}$$

The integral on the left here can be decomposed into a sum:

$$\begin{aligned}\mathscr{J} &= \int_{\Omega_+} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{B}_+\,dx = -\frac{F}{4}\int_{\Omega_+} u_r\frac{\partial u_r}{\partial r}\,\frac{1}{r}\,\frac{d\widehat{\zeta}_+}{d\theta}\,dx \\ &\quad -\frac{F}{4}\int_{\Omega_+} u_\theta\bigg(\frac{\partial u_r}{\partial\theta} - u_\theta\bigg)\frac{1}{r^2}\,\frac{d\widehat{\zeta}_+}{d\theta}\,dx =: -\frac{F}{4}(\mathscr{J}_1 + \mathscr{J}_2).\end{aligned} \tag{2.32}$$

Without loss of generality we assume that $\mathbf{u}$ is extended by zero to $\mathbb{R}^2_+\setminus\Omega_+$. We consider $\mathscr{J}_1$. Let

$$B_R = \{x\colon |x| < R\} \supset \Omega, \qquad B_{+,R} = \{x\in B_R\colon x_2 > 0\}.$$

Then

$$\begin{aligned}\mathscr{J}_1 &= \int_{\Omega_+} u_r(x)\frac{\partial u_r(x)}{\partial r}\,\frac{1}{r}\,\frac{d\widehat{\zeta}_+(\theta,\delta)}{d\theta}\,dx = \int_{B_{+,R}} u_r(x)\frac{\partial u_r(x)}{\partial r}\,\frac{1}{r}\,\frac{d\widehat{\zeta}_+(\theta,\delta)}{d\theta}\,dx \\ &= \int_0^\pi \frac{d\widehat{\zeta}_+(\theta,\delta)}{d\theta}\bigg(\int_0^R u_r(r,\theta)\frac{\partial u_r(r,\theta)}{\partial r}\,dr\bigg)\,d\theta \\ &= \frac{1}{2}\int_0^\pi \frac{d\widehat{\zeta}_+(\theta,\delta)}{d\theta}|u_r(R,\theta)|^2\,d\theta = 0.\end{aligned}$$

By symmetry $u_\theta(x)\big|_{x_2=0} = 0$. Hence, it follows from (2.26) and from Hardy's and Friedrichs' inequalities that

$$\begin{aligned}
\int_{\Omega_+} \frac{|u_\theta(x)|^2}{r^2}\left|\frac{d\widehat{\zeta}_+(\theta,\delta)}{d\theta}\right|^2 dx &= \int_{B_{+,R}} \frac{|u_\theta(x)|^2}{r^2}\left|\frac{d\widehat{\zeta}_+(\theta,\delta)}{d\theta}\right|^2 dx \\
&\leqslant c\delta^2\left(\int_0^R \frac{dr}{r}\int_0^{\pi/2} \frac{|u_\theta(r,\theta)|^2}{\theta^2}\,d\theta + \int_0^R \frac{dr}{r}\int_{\pi/2}^{\pi} \frac{|u_\theta(r,\theta)|^2}{(\pi-\theta)^2}\,d\theta\right) \\
&\leqslant c\delta^2 \int_{B_{+,R}} \frac{1}{r^2}\left|\frac{\partial u_\theta(x)}{\partial\theta}\right|^2 dx \leqslant c\delta^2\left(\int_{\Omega_+} |\nabla \mathbf{u}(x)|^2\,dx + \int_{\Omega_+} \frac{|u_r(x)|^2}{r^2}\,dx\right) \\
&\leqslant c\delta^2\left(\int_{\Omega_+} |\nabla \mathbf{u}(x)|^2\,dx + \int_{\Omega_+} |u_r(x)|^2\,dx\right) \leqslant c\delta^2 \int_{\Omega_+} |\nabla \mathbf{u}(x)|^2\,dx.
\end{aligned}$$

Here we have also used the expression for the gradient $\nabla \mathbf{u}$ of a vector-valued function in polar coordinates:

$$\nabla \mathbf{u} = \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial u_r}{\partial \theta} - u_\theta\right) \\[2ex] \dfrac{\partial u_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial u_\theta}{\partial \theta} + u_r\right) \end{pmatrix}.$$

Similarly, $\mathscr{J}_2$ has the estimate

$$\begin{aligned}
|\mathscr{J}_2| &\leqslant \left(\int_{\Omega_+} \frac{|u_\theta(x)|^2}{r^2}\left|\frac{d\widehat{\zeta}_+(\theta,\delta)}{d\theta}\right|^2 dx\right)^{1/2}\left(\int_{\Omega_+} \frac{1}{r^2}\left|\frac{\partial u_r(x)}{\partial\theta} - u_\theta(x)\right|^2 dx\right)^{1/2} \\
&\leqslant c\delta \int_{\Omega_+} |\nabla \mathbf{u}(x)|^2\,dx.
\end{aligned}$$

Taking δ sufficiently small, we deduce (2.31) from the estimates for $\mathscr{J}_1$ and $\mathscr{J}_2$.

Next we extend $\mathbf{B}_+(x,\delta)$ to $\Omega_- = \{x \in \Omega \colon x_2 < 0\}$ as a symmetric field:

$$\mathbf{B}(x,\delta) = \begin{cases} \big(B_{+,1}(x_1,x_2,\delta), B_{+,2}(x_1,x_2,\delta)\big), & x \in \Omega_+, \\ \big(B_{+,1}(x_1,-x_2,\delta), -B_{+,2}(x_1,-x_2,\delta)\big), & x \in \Omega_-. \end{cases} \tag{2.33}$$

Clearly, $\operatorname{div} \mathbf{B} = 0$,

$$\int_{\Gamma_0} \mathbf{B}\cdot\mathbf{n}\,dS = -\int_{\Gamma_1} \mathbf{B}\cdot\mathbf{n}\,dS = F, \tag{2.34}$$

and $\mathbf{B}$ satisfies (2.31).

Let us now complete the proof of Lemma 2.3. We define a vector field $\widehat{\mathbf{a}}$ on the boundary of Ω:

$$\widehat{\mathbf{a}} = \mathbf{a} - \mathbf{B}\big|_{\partial\Omega}.$$

By (2.34) the flux of the field $\widehat{\mathbf{a}}$ across each connected component of the boundary of Ω is zero, so we can find a divergence-free extension $\widehat{\mathbf{B}} \in W^{1,2}(\Omega)$ of $\widehat{\mathbf{a}}$ to Ω such that Hopf's inequality holds:

$$\left|\int_\Omega \widehat{\mathbf{B}}\cdot(\mathbf{u}\cdot\nabla)\mathbf{u}\,dx\right| \leqslant \varepsilon\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} \qquad \forall\,\mathbf{u}\in H(\Omega). \tag{2.35}$$

Furthermore, by construction the connected components of the support of $\widehat{\mathbf{B}}$ lie in arbitrarily narrow strips adjoining the surfaces Γ_i, $i=0,1$ (see § 2.1). However, $\widehat{\mathbf{B}}$ is not necessarily symmetric. We symmetrize it by introducing another vector field $\widetilde{\mathbf{B}}$ with components

$$\widetilde{B}_1(x,\delta)=\frac{1}{2}\bigl(\widehat{B}_1(x_1,x_2,\delta)+\widehat{B}_1(x_1,-x_2,\delta)\bigr),$$
$$\widetilde{B}_2(x,\delta)=\frac{1}{2}\bigl(\widehat{B}_2(x_1,x_2,\delta)-\widehat{B}_2(x_1,-x_2,\delta)\bigr).$$

Obviously, $\widetilde{\mathbf{B}}$ inherits all the required properties of $\widehat{\mathbf{B}}$.

Let $\mathbf{A}$ be the vector field defined by

$$\mathbf{A}(x,\delta)=\mathbf{B}(x,\delta)+\widetilde{\mathbf{B}}(x,\delta).$$

By construction $\mathbf{A}(x,\delta)\big|_{\partial\Omega}=\mathbf{a}$, and $\mathbf{A}$ has all the properties in Lemma 2.3.

Using Hopf's inequality (2.22) for symmetric functions, we arrive at the a priori bound $\|\nabla\mathbf{w}\|_{L^2(\Omega)}\leqslant C$ with a positive constant C depending on ν and $\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}$. Hence, the flux problem (1.1)–(1.3) is solvable in the class of symmetric generalized solutions. Fujita [14] also proved that the pressure p can be recovered from the $\mathbf{w}$ obtained, and p (which is defined up to an additive constant) is an even function of x_2.

Another way of proving that a symmetric planar flux problem is solvable was proposed by Morimoto [16]. It is based on the use of the stream function $\psi(x_1,x_2)$ of the planar flow, which is determined by the equations

$$\frac{\partial\psi}{\partial x_2}=u_1, \qquad \frac{\partial\psi}{\partial x_1}=-u_2. \tag{2.36}$$

We note that when (1.6) fails to hold, the stream function determined by a given divergence-free velocity vector field $\mathbf{u}=(u_1,u_2)$ is multivalued. However, if the domain $\Omega_+=\{x\in\Omega\colon x_2>0\}$ has a connected boundary, then the stream function can be recovered from (2.36) uniquely up to an additive constant.

As for classical solutions, the methods developed in [2] enable one to prove their existence, provided that the problem data belong to Hölder classes: $\partial\Omega\in C^{2+\alpha}$, $\mathbf{a}\in C^{2+\alpha}(\partial\Omega)$. In that case $\mathbf{u}\in C^{2+\alpha}(\overline{\Omega})$ and $\nabla p\in C^{\alpha}(\overline{\Omega})$, with $0<\alpha<1$.

2.4. Local results. The above results on solvability of the flux problem were proved either under the assumption that all the fluxes F_i across the connected components Γ_i of $\partial\Omega$ are zero (the condition (1.6)) or under the assumption that the domain is symmetric. The first steps in the analysis of (1.1)–(1.3) without the assumption (1.6) were made in 1961 by Fujita [7] and Finn [8]. Fujita proved the solvability of (1.1)–(1.3) for small fluxes $|F_i|$. Finn [8] looked at a generalization of the flow problem. Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with connected boundary $\partial\Omega$. The classical statement of the flow problem for the Navier–Stokes equations [34] asks about a solution $\mathbf{u}$, p of the system (1.1), (1.2) in $\mathbb{R}^3\setminus\Omega$ satisfying the conditions

$$\mathbf{u} = 0, \qquad x \in \partial\Omega, \tag{2.37}$$

$$\mathbf{u} \to \mathbf{u}_\infty, \qquad x \to \infty, \tag{2.38}$$

where $\mathbf{u}_\infty$ is a given non-zero constant vector. The solvability of the flow problem without the assumption that the data are small was first established by Leray [1]. Subsequently, the problem was investigated by many authors (see [28]). Finn's generalization consisted in replacing the homogeneous condition (2.37) by the general condition (1.3). Let F be the flux of the velocity across of the surface $\partial\Omega$. It was proved in [8] that the problem (1.1)–(1.3), (2.38) has a solution if $|F| < c_*\nu$, where $c_* > 0$ is sufficiently small.

We now return to the case of a bounded domain Ω of the form (1.4). For small fluxes the problem (1.1)–(1.3) has been investigated by many authors (for instance, [9], [11]–[13]). In particular, Borchers and Pileckas [11] studied the flux problem in the domain Ω between concentric spheres (or circles for $n = 2$) with radii R_1 and $R_2 > R_1$. They found effective bounds for the range of values of the velocity flux F for which this problem is solvable. Other versions of effective bounds for the values of the fluxes of the velocity vector across the boundary components were proposed in [12] and [13].

We give a simple proof of the solvability of (1.1)–(1.3) for small fluxes, based on deriving an a priori bound by a contradiction argument (see [13]). Let χ_i be the solution of the following Dirichlet problem for the Laplace operator in the exterior domain:

$$\begin{aligned} \Delta\chi_i &= 0, & x &\in \mathbb{R}^3 \setminus \Omega_i,\\ \chi_i &= 1, & x &\in \Gamma_i,\\ \chi_i(x) &\to 0, & |x| &\to \infty. \end{aligned}$$

The maximum principle gives us that $0 \leqslant \chi_i(x) \leqslant 1$ for $x \in \mathbb{R}^3 \setminus \Omega_i$. By the harmonic capacity of the compact set Ω_i we mean the quantity

$$\operatorname{cap}(\Omega_i) = \int_{\Gamma_i} \frac{\partial\chi_i}{\partial\mathbf{n}}\, dS = \int_{\mathbb{R}^n\setminus\Omega_i} |\nabla\chi_i(x)|^2\, dx.$$

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of the form (1.4) with Lipschitz boundary $\partial\Omega$. If $\mathbf{f} \in L^2(\Omega)$, the boundary values $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ satisfy (1.5), and*

$$\sum_{i=0}^{N} \frac{|F_i|}{\operatorname{cap}(\Omega_i)} < \nu, \tag{2.39}$$

then the problem (1.1)–(1.3) has at least one generalized solution $\mathbf{u} \in W^{1,2}(\Omega)$.

Proof. We have already mentioned that to prove the solvability of (1.1)–(1.3) it is sufficient to have an a priori bound for a solution. In § 2.2 we showed that if there is no such bound, then there exists a pair $(\widehat{\mathbf{w}}, \widehat{p}) \in H(\Omega) \times W^{1,s}(\Omega)$ with $s \in [1, 3/2]$ that satisfies the Euler system

$$\begin{cases} (\widehat{\mathbf{w}} \cdot \nabla)\widehat{\mathbf{w}} + \nabla \widehat{p} = 0, & x \in \Omega, \\ \operatorname{div} \widehat{\mathbf{w}} = 0, & x \in \Omega, \\ \widehat{\mathbf{w}} = 0, & x \in \partial\Omega. \end{cases} \tag{2.40}$$

Furthermore, $\|\nabla \widehat{\mathbf{w}}\|_{L^2(\Omega)} \leqslant 1$ and we have

$$\nu = \lambda_0 \int_\Omega (\widehat{\mathbf{w}} \cdot \nabla)\widehat{\mathbf{w}} \cdot \mathbf{A}\, dx, \qquad \lambda_0 \in (0, 1], \tag{2.41}$$

and

$$\int_\Omega (\widehat{\mathbf{w}} \cdot \nabla)\widehat{\mathbf{w}} \cdot \mathbf{A}\, dx = -\sum_{i=0}^N \widehat{p}_i F_i, \tag{2.42}$$

where

$$\widehat{p}(x)\big|_{\Gamma_i} = \widehat{p}_i, \quad \widehat{p}_i \in \mathbb{R}, \qquad i = 0, 1, \ldots, N.$$

Taking the inner product of (2.40) with $\nabla \chi_i$, we integrate over $\mathbb{R}^3 \setminus \Omega_i$, assuming that $\widehat{\mathbf{w}}$ is extended by zero to $\mathbb{R}^3 \setminus \Omega$, and we set $\widehat{p}(x)\big|_{\mathbb{R}^3 \setminus \Omega_0} = \widehat{p}_0$ and $\widehat{p}(x)\big|_{\Omega_i} = \widehat{p}_i$, $i = 1, \ldots, N$. Since $\operatorname{cap}(\Omega_i)$ is positive, integration by parts gives us that

$$\begin{aligned} |\widehat{p}_i| \operatorname{cap}(\Omega_i) = |\widehat{p}_i \operatorname{cap}(\Omega_i)| &= \left| \int_{\mathbb{R}^n \setminus \Omega_i} (\widehat{\mathbf{w}} \cdot \nabla)\widehat{\mathbf{w}} \cdot \nabla \chi_i\, dx \right| \\ &= \left| \int_{\mathbb{R}^n \setminus \Omega_i} \nabla \widehat{\mathbf{w}} \cdot \nabla \widehat{\mathbf{w}} \chi_i\, dx \right| \leqslant \|\nabla \widehat{\mathbf{w}}\|^2_{L^2(\Omega)} = 1. \end{aligned}$$

Using (2.42) and (2.39), we deduce the inequality

$$\left| \int_\Omega (\widehat{\mathbf{w}} \cdot \nabla)\widehat{\mathbf{w}} \cdot \mathbf{A}\, dx \right| = \left| \sum_{i=0}^N \widehat{p}_i F_i \right| \leqslant \sum_{i=0}^N \frac{|F_i|}{\operatorname{cap}(\Omega_i)} < \nu,$$

which contradicts (2.41). This contradiction shows that at least one solution of (1.1)–(1.3) must exist. □

2.5. Flows close to potential flows. Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain of the form (1.4). It is well known that the system (1.1), (1.2) with a potential field of the external forces $\mathbf{f}$ has a family of solutions in which the velocity field is a potential field, that is, $\mathbf{u} = \nabla \varphi$ for some harmonic function φ, and the pressure can be found from the Bernoulli integral: $p + |\nabla \varphi|^2/2 = \text{const}$. Taking φ to be a linear combination of fundamental solutions of the Laplace equation with singularities in Ω_i, we obtain a class of exact solutions of the flux problem for the Navier–Stokes equations with certain special boundary conditions.

Fujita and Morimoto [35] investigated the problem (1.1)–(1.3) in a three- or two-dimensional domain Ω with two smooth boundary components Γ_0 and Γ_1. With the boundary conditions in the form

$$\mathbf{u} = \mu \nabla \varphi + \widetilde{\mathbf{a}}_i, \qquad x \in \Gamma_i, \quad i = 0, 1, \tag{2.43}$$

where $\mu \in \mathbb{R}$, φ is a fundamental solution of the Laplace equation, and the functions $\widetilde{\mathbf{a}}_i$ satisfy (1.6), they showed the existence of a countable set $M \in \mathbb{R}$ such that if $\mu \notin M$ and the corresponding norms of the functions $\widetilde{\mathbf{a}}_i$ are small, then (1.1)–(1.3) is solvable. Moreover, if $\Omega \subset \mathbb{R}^2$ is a circular annulus, then M is empty.

Russo and Starita [36] relaxed the smoothness conditions imposed on the surfaces (curves) Γ_i and the functions $\widetilde{\mathbf{a}}_i$. In particular, they proved the following theorem [35].

Theorem 2.3. *Let* $\mathbf{f} = 0$, *assume that the boundary values* $\mathbf{a}$ *have a representation* (2.43), *where* φ *is a fundamental solution of the Laplace equation, and that the functions* $\widetilde{\mathbf{a}}_i \in W^{1/2,2}(\Gamma_i)$ *satisfy*

$$\int_{\Gamma_i} \widetilde{\mathbf{a}}_i \cdot \mathbf{n}\, dS = 0, \qquad i = 0, 1.$$

Then there exists a discrete countable set $M \subset \mathbb{R}$ *and a positive constant* ε_* *such that for any* $\mu \in \mathbb{R} \setminus M$ *the problem* (1.1)–(1.3) *has at least one generalized solution* $\mathbf{u} \in W^{1,2}(\Omega)$ *if*

$$\|\widetilde{\mathbf{a}}_0\|_{W^{1/2,2}(\Gamma_0)} + \|\widetilde{\mathbf{a}}_1\|_{W^{1/2,2}(\Gamma_1)} \leqslant \varepsilon_*. \tag{2.44}$$

Proof. This theorem has a rather simple yet beautiful proof, so we present the main points of the proof, without giving details. We seek a solution $\mathbf{u}$ in the form of a sum $\mathbf{u} = \mathbf{w} + \mu\nabla\varphi + \mathbf{A}$, where $\mathbf{w} \in H(\Omega)$ and $\mathbf{A}$ satisfies the boundary condition $\mathbf{A}\big|_{\Gamma_i} = \widetilde{\mathbf{a}}_i$ for $i = 0, 1$ and is a generalized solution of the homogeneous linear Stokes problem, that is, $\mathbf{A} \in W^{1,2}(\Omega)$ satisfies the integral identity

$$\nu \int_{\Omega} \nabla\mathbf{A} \cdot \nabla\boldsymbol{\eta}\, dx = 0 \qquad \forall\, \boldsymbol{\eta} \in H(\Omega)$$

and the inequality

$$\|\nabla\mathbf{A}\|_{L^2(\Omega)} \leqslant c\big(\|\widetilde{\mathbf{a}}_0\|_{W^{1/2,2}(\Gamma_0)} + \|\widetilde{\mathbf{a}}_1\|_{W^{1/2,2}(\Gamma_1)}\big)$$

(see, for instance, [2]).

We find the vector field $\mathbf{w} \in H(\Omega)$ using the integral identity

$$\begin{aligned} \nu \int_{\Omega} \nabla\mathbf{w} \cdot \nabla\boldsymbol{\eta}\, dx &+ \mu \int_{\Omega} \big((\nabla\varphi \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\nabla\varphi\big) \cdot \boldsymbol{\eta}\, dx \\ &+ \int_{\Omega} \big((\mathbf{A} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{A}\big) \cdot \boldsymbol{\eta}\, dx + \int_{\Omega} (\mathbf{w} \cdot \nabla)\mathbf{w} \cdot \boldsymbol{\eta}\, dx \\ &= -\mu \int_{\Omega} \big((\nabla\varphi \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\nabla\varphi\big) \cdot \boldsymbol{\eta}\, dx - \int_{\Omega} (\mathbf{A} \cdot \nabla)\mathbf{A} \cdot \boldsymbol{\eta}\, dx, \end{aligned} \tag{2.45}$$

which holds for all $\boldsymbol{\eta} \in H(\Omega)$. Here we have used the fact that the gradient of the harmonic function φ and the corresponding pressure (found from the Bernoulli integral) satisfy the Navier–Stokes system.

We claim that the integral identity (2.45) is equivalent to an operator equation. For each $\mathbf{b} \in W^{1,2}(\Omega)$ satisfying $\operatorname{div} \mathbf{b} = 0$ we define a linear operator $L(\mathbf{b})$ from $H(\Omega)$ to $H(\Omega)^*$ by

$$\langle L(\mathbf{b})\mathbf{w}, \boldsymbol{\eta} \rangle = \int_{\Omega} \big(\mathbf{b} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{b}\big) \cdot \boldsymbol{\eta}\, dx \qquad \forall\, \mathbf{w}, \boldsymbol{\eta} \in H(\Omega),$$

where $\langle\,\cdot\,,\cdot\,\rangle$ denotes the pairing of $H(\Omega)$ and $H(\Omega)^*$. Using standard embedding theorems, we can show that $L(\mathbf{b})$ is a compact operator.

Next we introduce the operator G which associates with any $\mathbf{F}\in H(\Omega)^*$ the generalized solution $\mathbf{v}\in H(\Omega)$ of the Stokes system:

$$\int_\Omega \nabla\mathbf{v}\cdot\nabla\boldsymbol{\eta}\,dx=\langle\mathbf{F},\boldsymbol{\eta}\rangle \qquad \forall\,\boldsymbol{\eta}\in H(\Omega).$$

It is well known that G is a bounded linear operator (see, for instance, [2]).

Using the operators G, $L(\nabla\varphi)$, and $L(\mathbf{A})$, we can write the integral identity (2.45) as an operator equation in $H(\Omega)$:

$$\mathbf{w}+\frac{\mu}{\nu}GL(\nabla\varphi)\mathbf{w}=-\frac{1}{\nu}GL(\mathbf{A})\mathbf{w}-\frac{1}{\nu}G(\mathbf{w}\cdot\nabla)\mathbf{w}+G\Phi, \tag{2.46}$$

where

$$\Phi=-\frac{\mu}{\nu}L(\nabla\varphi)\mathbf{A}-\frac{1}{\nu}(\mathbf{A}\cdot\nabla)\mathbf{A}.$$

Since $K=-(1/\nu)GL(\nabla\varphi)$ is a compact operator on $H(\Omega)$, its spectrum $\sigma(K):=M$ is a discrete countable set which can have the limit point $\lambda=0$. If $1/\mu\notin M$, then there exists a bounded inverse $(I-\mu K)^{-1}$, and we can rewrite (2.46) in the form

$$\mathbf{w}=-\frac{1}{\nu}(I-\mu K)^{-1}\biggl(-\frac{1}{\nu}GL(\mathbf{A})\mathbf{w}-\frac{1}{\nu}G(\mathbf{w}\cdot\nabla)\mathbf{w}+G\Phi\biggr)=:\mathfrak{A}\mathbf{w}. \tag{2.47}$$

For a sufficiently small ε_* the condition (2.44) means that the operator $\mathfrak{A}$ takes some ball in the space into itself and is contracting on this ball. Hence, equation (2.47) has a solution. □

2.6. The flux problem in a circular annulus. The cycle of papers [37]–[40] is an investigation of flows in a circular annulus $\Omega=\{x\in\mathbb{R}^2\colon R_1<|x|<R_0\}$. Morimoto [37], [38] looked at this problem in the case when the values $\mathbf{a}_i$ of the velocity vector field on the boundary components have the form

$$\mathbf{a}_i=\mu R_i^{-1}\mathbf{e}_r+b_i\mathbf{e}_\theta, \qquad x\in\Gamma_i=\{x\in\mathbb{R}^2\colon |x|=R_i\}, \quad i=0,1, \tag{2.48}$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit vectors in the polar coordinate system (r,θ) and μ, b_0, and b_1 are constants. The problem (1.1)–(1.3), (2.48) has an exact solution $\mathbf{u}=\mathbf{u}(r)$, $p=p(r)$, which can be described by explicit formulae. If $\mu=0$, then it becomes the well-known Couette solution [34]. It was proved in [38] that if $|\mu|$ and $|b_1-b_0|$ are sufficiently small and $\mu\neq-2\nu$, then this exact solution of (1.1)–(1.3), (2.48) is unique. Uniqueness also holds for $\mu=-2\nu$ if $|\mu|$, $|b_0|$, and $|b_1|$ are sufficiently small. Moreover, if the viscosity ν is sufficiently large, then the solution of (1.1)–(1.3), (2.48) is exponentially stable.

Now we look at a flow in an annulus with boundary conditions of the more general form

$$\begin{gathered}\mathbf{a}_i=[\mu R_i^{-1}+\varphi_i(\theta)]\mathbf{e}_r+[\omega_iR_i+\psi_i(\theta)]\mathbf{e}_\theta,\\ x\in\Gamma_i=\{x\in\mathbb{R}^2\colon |x|=R_i\}, \qquad i=0,1,\end{gathered} \tag{2.49}$$

where the φ_i and ψ_i are 2π-periodic functions in $W^{1/2,2}(\mathbb{R})$. The problem (1.1)–(1.3), (2.49) was considered by Morimoto and Ukai [39]. Their main result was as follows.

Theorem 2.4. *Assume that*

$$|\omega_1 - \omega_0| \frac{R_0^2 R_1^2}{R_1^2 - R_0^2} \left(\log \frac{R_1}{R_0} \right)^2 < 2\nu. \tag{2.50}$$

Then there exists a finite or countable set M such that for any $\mu \in \mathbb{R} \setminus M$ the problem (1.1)–(1.3), (2.49) *is solvable for sufficiently small φ_i and ψ_i $(i = 0, 1)$.*

Note that under the assumptions of this theorem the value of $|\mu|$ can be arbitrarily large in comparison with ν. It could be interesting to single out a class of conditions (2.49) for which M is an empty set. Fujita, Morimoto, and Okamoto [40] showed that it is empty for $\omega_1 = \omega_0$. In this case (2.50) automatically holds. The special case when $b_1 = b_0 = 0$ in (2.48) corresponds to a purely radial flow with the velocity field $u_r = \mu r^{-1}$, $u_\theta = 0$. As shown in [40], the radial flow in a circular annulus is isolated in the class of steady-state solutions of the Navier–Stokes equations, whatever the Reynolds number μ/ν and the ratio R_0/R_1 of the radii are. (We remark that it is only for small values of μ/ν that we can prove uniqueness of the radial solution of the flow problem in an annulus.) If $b_1 R_1 = b_0 R_0$, then (1.1)–(1.3), (2.48) has the self-similar solution $u_r = \mu r^{-1}$, $u_\theta = \lambda r^{-1}$, where $\lambda = b_0 R_0$. A numerical analysis of the problem of non-stationary perturbations of the self-similar solution showed [40] that when the Reynolds number attains a certain critical value, which depends on the ratios λ/ν and R_0/R_1, a time-periodic solution splits off, that is, a Hopf bifurcation occurs.

2.7. Axially symmetric problems. The planar symmetric problem considered in [10], [14], [16], [17] has an axially symmetric analogue [15]. In what follows we let

$$r = (x_1^2 + x_2^2)^{1/2}, \quad \theta = \arctan \frac{x_2}{x_1}, \quad z = x_3$$

denote the cylindrical coordinates, and u_r, u_θ, u_z the projections of the velocity vector $\mathbf{u}$ on the r-, θ-, and z-axes. In the cylindrical coordinates the equations of a stationary motion of a viscous incompressible fluid have the following form (see [34]):

$$\begin{aligned}
&u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \\
&\qquad = -\frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2} \right), \\
&u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} \\
&\qquad = -\frac{1}{r} \frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right), \\
&u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} = -\frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right), \\
&\frac{1}{r} \frac{\partial (r v_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0.
\end{aligned} \tag{2.51}$$

A motion is said to be axially symmetric without swirl if $u_\theta = 0$ and the functions u_r, u_z, and p are independent of θ. Under these assumptions (2.51) becomes the following system:

$$\begin{aligned}
&u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} = -\frac{\partial p}{\partial r} + \nu\biggl(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\,\frac{\partial u_r}{\partial r} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2}\biggr),\\
&u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z} = -\frac{\partial p}{\partial z} + \nu\biggl(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\,\frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2}\biggr),\\
&\frac{1}{r}\,\frac{\partial (rv_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0.
\end{aligned} \tag{2.52}$$

Definition 2.3. A bounded domain $\Omega \subset \mathbb{R}^3$ is said to be *admissible* if the following conditions hold:

(a) $\partial\Omega \in C^\infty$;

(b) $\partial\Omega$ consists of $N+1 \geqslant 2$ connected components Γ_i;

(c) Ω has the axis of symmetry $r=0$ and the plane of symmetry $z=0$;

(d) each Γ_i intersects the plane $z=0$.

A function $\mathbf{h} = (h_r, h_\theta, h_z)$ from Ω or $\partial\Omega$ to $\mathbb{R}^3$ is said to be *axially symmetric without swirl* if $h_\theta = 0$, while h_r and h_z are independent of θ. A function $\mathbf{h} = (h_r, h_\theta, h_z)$ from Ω or $\partial\Omega$ to $\mathbb{R}^3$ is said to be *symmetric relative to the plane* $z=0$ if h_r and h_θ are even functions of z, while h_z is an odd function of z.

Definition 2.4. A vector $\mathbf{a}$ is said to be *admissible* if the following conditions are satisfied:

(a) $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$;

(b) $\mathbf{a}$ is axially symmetric without swirl and symmetric relative to the plane $z=0$.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^3$ be an admissible domain and $\mathbf{a}$ an admissible vector in the sense of Definitions 2.3 and 2.4. Then for any $\nu > 0$ the problem (1.1)–(1.3) has a generalized solution $\mathbf{u} \in W^{1,2}(\Omega)$. The velocity vector $\mathbf{u}$ is axially symmetric without swirl and symmetric relative to the plane $z=0$. Furthermore, $\|\mathbf{u}\|_{W^{1,2}(\Omega)} \leqslant C$ with a positive constant C depending on ν and $\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}$.*

The proof of Theorem 2.4 was given in [15]. It is based on the construction of an axially symmetric analogue of a virtual drain.

Along with the boundary-value problem (1.1)–(1.3), the statement of the problem in 'vorticity–stream function' terms is used extensively in the axially symmetric case (see, for instance, [41] and the references there). The stream function ψ of an axially symmetric flow and the vorticity ω of the flow are introduced by the equalities

$$\frac{\partial\psi}{\partial r} = -ru_z, \qquad \frac{\partial\psi}{\partial z} = ru_r, \qquad \omega = \frac{\partial u_z}{\partial r} - \frac{\partial u_r}{\partial z}\,. \tag{2.53}$$

It is often more convenient to work with the reduced vorticity $\lambda = r^{-1}\omega$. In these terms the equations of a steady axially symmetric flow of a viscous incompressible fluid have the form

$$E\psi = -r^2\lambda, \qquad \mathbf{u}\cdot\nabla\lambda = \nu L\lambda, \tag{2.54}$$

where $\mathbf{u} = (u_r, u_z)$, $\nabla = (\partial_r, \partial_z)$, and E and L are linear elliptic operators defined by

$$E = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \qquad L = \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

The elliptic system (2.54) has a much simpler structure than the original system (2.52), and moreover, it contains fewer unknown functions. Therefore, the variables ψ, ω have turned out to be very convenient for the numerical solution of axially symmetric problems. So far no author has investigated the flux problem for this system. In [42] such a problem was considered in an axially symmetric domain Q of the spherical layer type.

Let S_1 and S_0 denote the outer and inner boundary components of Q. For simplicity assume that $S_1, S_0 \in C^\infty$. The meridional section of Q will be denoted by Ω. It has the form of a horseshoe, with boundary formed by two arcs Σ_1 and Σ_0 that are the meridional sections of S_1 and S_0, respectively, and two intervals Λ_1 and Λ_0 of the z-axis (see Fig. 3).

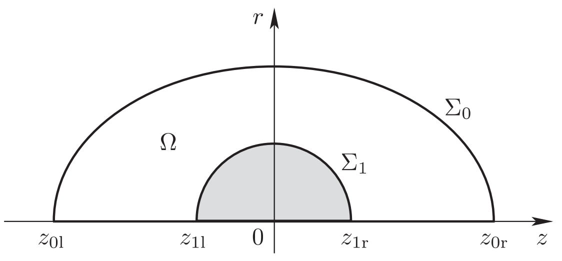

Figure 3. The meridional section of the flow region Q

For the system (2.54) we impose the boundary conditions

$$\psi = a_i(s), \quad \lambda = b_i(s), \qquad (r, z) \in \Sigma_i, \quad i = 0, 1, \tag{2.55}$$

corresponding to the values of the normal component of the velocity and the vorticity given on the boundary of the domain. For a parameter s on the curve Σ_i it is convenient to take the arc length measured from the left-hand endpoint of the interval Λ_i of the z-axis $(i = 0, 1)$.

If the function a_i takes equal values at the endpoints of Σ_i (which we denote by $z_{i\mathrm{l}}$ and $z_{i\mathrm{r}}$), then (1.6) is satisfied: the flux of the fluid across each surface S_i $(i = 0, 1)$ is equal to zero. Then we can prove the solvability of the problem (2.54), (2.55) for sufficiently smooth input data using standard arguments. We are interested in the case when this condition is violated:

$$a_1(l_1) - a_1(0) = a_0(l_0) - a_0(0) \neq 0.$$

Here l_i denotes the length of Σ_i. The coinciding quantities in the last relation are equal to $(2\pi)^{-1}F$, where F is the flux across S_1. It was proved in [42] that the problem (2.54), (2.55) is solvable without constraints on the magnitude of $|F|$.

Note that for a fixed velocity field the stream function is determined up to an additive constant. Making the natural assumption that $u_r \to 0$ as $r \to 0$, we conclude that the intervals Λ_1 and Λ_0 of the z-axis are streamlines. Without loss of generality we can set $\psi = 0$ on Λ_1 (that is, $a_1(0) = a_0(0) = 0$). Then ψ takes the value $a_1(l_1) = a_0(l_0) = (2\pi)^{-1}F$ on Λ_0.

Let us transform the problem (2.54), (2.55) into a problem with homogeneous boundary conditions. To do this we construct a function χ as a solution of the following boundary-value problem:

$$\begin{aligned} E\chi &= 0, && (r,z) \in \Omega; \\ \chi &= a_i(s), && (r,z) \in \Sigma_i, \qquad i = 0,1; \\ \chi &= 0, && (r,z) \in \Lambda_1; \\ \chi &= (2\pi)^{-1}q, && (r,z) \in \Lambda_0. \end{aligned} \tag{2.56}$$

For simplicity assume that a_i and b_i $(i = 0,1)$ are functions in the Hölder classes $C^{2+\alpha}([0,l_i])$, $0 < \alpha < 1$. Also assume that, as $s \to 0$,

$$\begin{gathered} a_1 = a_{21}s^2 + O(s^{2+\alpha}), \qquad a_0 = (2\pi)^{-1}q + a_{20}s^2 + O(s^{2+\alpha}), \\ b_i = b_{1i} + b_{2i}s^2 + O(s^{2+\alpha}), \qquad i = 0,1. \end{gathered} \tag{2.57}$$

Under these assumptions the problem (2.56) has a classical solution, which is moreover unique and satisfies the estimates

$$\frac{1}{r}\frac{\partial\chi}{\partial r} = O(r), \quad \frac{1}{r}\frac{\partial\chi}{\partial z} = O(r), \qquad (r,z) \in \overline{\Omega}, \quad r \to 0. \tag{2.58}$$

The function χ extends the boundary values of the stream function inside Ω. To extend the boundary values of the reduced vorticity, we introduce local curvilinear orthogonal coordinates s and n associated with the curves Σ_1 and Σ_0. We have already introduced the s-variable, and as n we take the distance from a point $(r,z) \in \Delta_i \subset \overline{\Omega}$ to the curve Σ_i $(i = 0,1)$. Here Δ_i is a curved strip of width δ which adjoins Σ_i. We take δ small enough so that the point on Σ_i which is closest to (r,z) is well defined. Such curvilinear (von Mises) variables are well known in boundary layer theory [34]. Next we define a function $\vartheta(x)$ by

$$\begin{gathered} \vartheta = b_i(s)\xi(n,\varepsilon), \qquad (r,z) \in \overline{\Delta}_i, \quad i = 0,1, \\ \vartheta = 0, \qquad (r,z) \in \overline{\Omega} \setminus \overline{\Delta}_1 \cup \overline{\Delta}_2, \end{gathered} \tag{2.59}$$

where $\xi(n,\varepsilon)$ is the Hopf cutoff function. From (2.56) and the properties of the Hopf function we conclude that:

$$\begin{gathered} \vartheta \in C^{2+\alpha}(\overline{\Omega}); \quad \vartheta = b_i(s), \quad (r,z) \in \Sigma_i; \\ \left|\frac{\partial\vartheta}{\partial n}\right| \leqslant \frac{c\varepsilon}{n}; \quad \frac{\partial\vartheta}{\partial r} = 0, \quad (r,z) \in \Lambda_i; \end{gathered} \tag{2.60}$$

here $\varepsilon > 0$ can be arbitrarily small.

In the problem (2.54), (2.55) we go over to the new unknown functions

$$\phi = \psi - \chi, \qquad \mu = \lambda - \vartheta.$$

The functions ϕ and μ form a solution of the following boundary-value problem:

$$\begin{cases} E\phi = -r^2(\mu+\vartheta), & (r,z)\in\Omega, \\ \mathbf{U}\cdot\nabla\mu + \mathbf{W}\cdot\nabla\mu + \mathbf{U}\cdot\nabla\vartheta + \mathbf{W}\cdot\nabla\vartheta = \nu L\mu + \nu L\vartheta, & (r,z)\in\Omega, \end{cases} \tag{2.61}$$

$$\phi = 0, \quad (r,z)\in\partial\Omega; \quad \mu = 0, \quad (r,z)\in\Sigma_i; \quad \frac{\partial\mu}{\partial r} = 0, \quad (r,z)\in\Lambda_i; \quad i=0,1. \tag{2.62}$$

Here we have set

$$\mathbf{U} = \left(-\frac{1}{r}\frac{\partial\phi}{\partial z}, \frac{1}{r}\frac{\partial\phi}{\partial r}\right), \qquad \mathbf{W} = \left(-\frac{1}{r}\frac{\partial\chi}{\partial z}, \frac{1}{r}\frac{\partial\chi}{\partial r}\right).$$

By (2.55), (2.56), and (2.62) the vector-valued functions $\mathbf{U}$ and $\mathbf{W}$ satisfy the conditions

$$\begin{aligned} &\mathbf{U}\cdot\mathbf{n}_i = 0, \quad \mathbf{W}\cdot\mathbf{n}_i = 0, \qquad (r,z)\in\Sigma_i \quad (i=0,1), \\ &\mathbf{U}\cdot\mathbf{n}_0 = 0, \quad \mathbf{W}\cdot\mathbf{n}_0 = 0, \qquad (r,z)\in\Lambda_i \quad (i=0,1), \end{aligned} \tag{2.63}$$

where the $\mathbf{n}_i$ denote the outward unit normals to the curves Σ_i, $i=0,1$ (relative to Ω), and $\mathbf{n}_0 = (-1,0)$ is the normal to the straight parts Λ_1 and Λ_0 of the boundary of Ω.

The equations of the system (2.61) are degenerate on the line $r=0$, and we require weighted function classes for investigating the solutions of it. Let Ω_- be the domain in the (r,z)-plane obtained by reflecting $\Omega \equiv \Omega_+$ in the z-axis, and let

$$\Omega_* = \Omega_+ \cup \Omega_- \cup \Gamma_1 \cup \Gamma_0.$$

Now let $H^l(\Omega;r)$ denote the closure of the set of functions $\varphi \in C_0^\infty(\Omega_*)$ in the norm of the Sobolev space of functions having in the domain Ω^* derivatives of order up to l that are square-integrable with weight r over this domain. The subspaces of even and odd functions $\varphi \in H^l(\Omega;r)$ will be denoted by $H^l_{\mathrm{e}}(\Omega;r)$ and $H^l_{\mathrm{o}}(\Omega;r)$, respectively. The space $L^2(\Omega;r)$ coincides with $H^0(\Omega;r)$. The spaces $L^2_{\mathrm{e}}(\Omega;r)$ and $L^2_{\mathrm{o}}(\Omega;r)$ are defined similarly.

Then we have the following results.

Lemma 2.4. *Suppose that $\rho \in L^2_{\mathrm{e}}(\Omega;r)$, $\partial\rho/\partial r \in L^2_{\mathrm{o}}(\Omega;r)$, and $\partial\rho/\partial z \in L^2_{\mathrm{e}}(\Omega;r)$. Then the problem*

$$E\phi = -r^2\rho \ \textit{ for } (r,z)\in\Omega, \quad \phi = 0 \ \textit{ for } (r,z)\in\partial\Omega \tag{2.64}$$

has a unique solution $\phi \in H^3_{\mathrm{e}}(\Omega;r)$ with $r^{-1}\partial\phi/\partial r \in H^2_{\mathrm{e}}(\Omega;r)$, and the following estimates hold:

$$\begin{aligned} &\left\|\frac{1}{r}\frac{\partial\phi}{\partial r}\right\|_{L^2(\Omega;r)} \leqslant c\|\rho\|_{L^2(\Omega;r)}, \qquad \left\|\frac{1}{r}\frac{\partial\phi}{\partial z}\right\|_{L^2(\Omega;r)} \leqslant c\|\rho\|_{L^2(\Omega;r)}, \\ &\left\|\frac{1}{r}\frac{\partial\phi}{\partial r}\right\|_{H^2(\Omega;r)} \leqslant c\|\rho\|_{H^1(\Omega;r)}, \qquad \|\phi\|_{H^3(\Omega;r)} \leqslant c\|\rho\|_{H^1(\Omega;r)}. \end{aligned} \tag{2.65}$$

Lemma 2.5. *Let* $a_i, b_i \in C^{2+\alpha}([0, l_i])$, $0 < \alpha < 1$, $i = 0, 1$, *and assume the conditions* (2.57). *Then a generalized solution* ϕ, μ *of the problem* (2.61), (2.62) *has the estimate*

$$\|\phi\|_{H^3(\Omega;r)} + \|\mu\|_{H^1(\Omega;r)} \leqslant C \tag{2.66}$$

with a constant $C = C(\nu, F, \Omega, a_i, b_i) > 0$.

The following theorem was established in [42] by using the a priori bounds in Lemmas 2.4 and 2.5.

Theorem 2.6. *If the conditions of Lemma* 2.4 *hold, then the problem* (2.61), (2.62) *has a solution* $\phi \in H^3_{\mathrm{e}}(\Omega; r)$, $\mu \in H^1_{\mathrm{e}}(\Omega; r)$.

We do not give the proof here; it follows the classical scheme as presented in [2].

In conclusion we note that we can replace the smoothness condition $\partial\Omega \in C^\infty$ of the boundary of the flow region by the condition $\partial\Omega \in C^{2+\alpha}$, $0 < \alpha < 1$. Theorem 2.6 ensures that the problem (2.61), (2.62) will have at least one generalized solution. However, the smoothness and matching conditions imposed above on the functions a_i and b_i ($i = 0, 1$) enable us to prove that in fact this solution is a classical solution.

For simplicity we confined ourselves above to the case when the boundary of an axially symmetric flow region has two connected components. However, the line of argument will be the same if the boundary consists of arbitrarily many connected components: it is only important that each of these surfaces of revolution intersect the rotation axis in two points. No restrictions are imposed on the fluxes across the boundary components.

Now we consider the Galerkin approximations to the solution of (2.61), (2.62). They also have an a priori bound similar to (2.66), so we can justify the convergence of the Galerkin method to the solution of the axially symmetric flux problem in the 'vorticity-stream function' variables.

3. An existence theorem in the general planar case

In this section we look at the boundary-value problem for the Navier–Stokes system of equations (1.1)–(1.3) in a bounded domain $\Omega \subset \mathbb{R}^2$ with C^2-smooth boundary $\partial\Omega = \bigcup_{j=0}^{N} \Gamma_j$ consisting of $N+1$ disjoint connected components Γ_j, that is,

$$\Omega = \Omega_0 \setminus \left(\bigcup_{j=1}^{N} \overline{\Omega}_j \right), \qquad \overline{\Omega}_j \subset \Omega_0, \quad j = 1, \dots, N, \tag{3.1}$$

where $\Gamma_j = \partial\Omega_j$. We assume that $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ and $\mathbf{f} \in W^{1,2}(\Omega)$. Without loss of generality we can also assume that $\mathbf{f} = \nabla^\perp b$ with $b \in W^{2,2}(\Omega)$, where $(x, y)^\perp = (-y, x)$.[2]

[2] According to the Helmholtz–Weyl decomposition, each vector-valued function $\mathbf{f} \in W^{1,2}(\Omega)$ on a bounded domain $\Omega \subset \mathbb{R}^n$ with C^2-boundary, $n = 2, 3$, can be represented as a sum $\mathbf{f} = \operatorname{curl} \mathbf{b} + \nabla\varphi$ for $n = 3$ or $\mathbf{f} = \nabla^\perp b + \nabla\varphi$ for $n = 2$, where $\mathbf{b}, b, \varphi \in W^{2,2}(\Omega)$, and the gradient component can be included in the term containing the pressure (see, for instance, [2]).

3.1. Auxiliary results.

3.1.1. *Analogues of the Morse–Sard theorem and the Luzin N-property for Sobolev functions in $W^{2,1}(\mathbb{R}^2)$.* We start by recalling several classical differentiability properties of Sobolev functions. In treating Sobolev functions we shall always assume that we have chosen their 'best representatives'. For $w \in L^1_{\mathrm{loc}}(\Omega)$ the best representative w^* is defined as follows:

$$w^*(x) = \begin{cases} \lim_{r\to 0} \fint_{B_r(x)} w(z)\,dz & \text{if the limit exists and is finite,} \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\fint_{B_r(x)} w(z)\,dz = \frac{1}{\operatorname{meas}(B_r(x))} \int_{B_r(x)} w(z)\,dz$$

and $B_r(x) = \{y\colon |y-x| < r\}$ is a ball with radius r and centre x. It is well known that if $w \in W^{1,q}_{\mathrm{loc}}(\Omega)$ and $\Omega \subset \mathbb{R}^2$, then $\mathfrak{H}^1$-almost all points $x \in \Omega$ are Lebesgue points of w, so that the above limits exist $\mathfrak{H}^1$-almost everywhere. Here and throughout we let $\mathfrak{H}^1$ denote the one-dimensional Hausdorff measure: $\mathfrak{H}^1(F) = \lim_{t\to 0+} \mathfrak{H}^1_t(F)$, where

$$\mathfrak{H}^1_t(F) = \inf\biggl\{ \sum_{i=1}^{\infty} \operatorname{diam} F_i\colon\ \operatorname{diam} F_i \leqslant t,\ F \subset \bigcup_{i=1}^{\infty} F_i \biggr\}.$$

Lemma 3.1 (see [43], Proposition 1). *If $\psi \in W^{2,1}(\mathbb{R}^2)$, then ψ is a continuous function, and there is a set $A_\psi \subset \mathbb{R}^2$ such that $\mathfrak{H}^1(A_\psi) = 0$ and ψ is differentiable (in the classical sense) at all points $x \in \mathbb{R}^2 \setminus A_\psi$. Furthermore, its classical derivative coincides with the value of $\nabla\psi(x)$, and*

$$\lim_{r\to 0} \fint_{B_r(x)} |\nabla\psi(z) - \nabla\psi(x)|^2\,dz = 0$$

(so that x is a Lebesgue point of $\nabla\psi(\,\cdot\,)$).

The next result was obtained by Bourgain, Korobkov, and Kristensen [25] (see also [44] and [45] for several dimensions).

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. If $\psi \in W^{2,1}(\Omega)$, then the following hold.*

(i) $\mathfrak{H}^1(\{\psi(x)\colon x \in \overline{\Omega} \setminus A_\psi \ \&\ \nabla\psi(x) = 0\}) = 0$.

(ii) *For any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathfrak{H}^1(\psi(U)) < \varepsilon$ for each set $U \subset \overline{\Omega}$ whenever $\mathfrak{H}^1_\infty(U) < \delta$.*

(iii) *For any $\varepsilon > 0$ there exist an open set $V \subset \mathbb{R}$ with $\mathfrak{H}^1(V) < \varepsilon$ and a function $g \in C^1(\mathbb{R}^2)$ such that if $x \in \overline{\Omega}$ and $\psi(x) \notin V$, then $x \notin A_\psi$, $\psi(x) = g(x)$, and $\nabla\psi(x) = \nabla g(x) \neq 0$.*

(iv) *For $\mathfrak{H}^1$-almost all $y \in \psi(\overline{\Omega}) \subset \mathbb{R}$ the inverse image $\psi^{-1}(y)$ is the union of finitely many C^1-smooth curves S_j, $j = 1, \dots, N(y)$. Each S_j is either a cycle in Ω (so that $S_j \subset \Omega$ is homeomorphic to the unit circle $\mathbb{S}^1$) or a simple arc with endpoints on $\partial\Omega$ (and then S_j is transversal to $\partial\Omega$).*

3.1.2. *Some topological facts.* We need several definitions and results from general topology. By a *continuum* we mean a compact connected set, where connectedness is treated in the usual sense of general topology. A subset of a topological space is called an *arc* if it is homeomorphic to the closed unit interval $[0,1]$.

We start by recalling several results from Kronrod's classical paper [46] on level sets of continuous functions. Let $Q=[0,1]\times[0,1]$ be a square in $\mathbb{R}^2$, f a continuous function on Q, and $E_t=\{x\in Q\colon f(x)=t\}$ a level set of f. By the connected component K of E_t containing a point x_0 we mean the maximal connected subset of E_t containing x_0. Let T_f be the set of connected components of level sets of f. We take this set in its natural topology, where a system of neighbourhoods is defined as follows. For a component $C\in T_f$ and an open set $U\supset C$ the set $\{B\in T_f\colon B\subset U\}$ is called a neighbourhood of C. Accordingly, convergence in T_f is defined as follows: $T_f\ni C_i\to C$ if and only if $\sup_{x\in C_i}\operatorname{dist}(x,C)\to 0$. It was proved in [46] that T_f is a one-dimensional topological tree. (A locally connected continuum T is called a *topological tree* if it contains no curve homeomorphic to a circle, or equivalently, if any two points in T can be joined by a unique arc.) Hence T has topological dimension 1. Endpoints of this tree[3] are points $C\in T_f$ which do not separate Q, that is, the difference $Q\setminus C$ is connected. The branch points of this tree are the components $C\in T_f$ such that $Q\setminus C$ has more than 2 connected components (see [46], Theorem 5). According to Lemma 1 in [46] (see also [47] and [48]) T_f can have only countably many branch points. The main property of the tree is that any two points in it can be joined by a unique arc. Thus, we have the following lemma.

Lemma 3.2 (see Lemma 13 in [46]). *If $f\in C(Q)$, then for any pair of distinct elements $A\in T_f$ and $B\in T_f$ there is a unique arc $J=J(A,B)\subset T_f$ joining A with B. Moreover, for each interior point C of this arc A and B lie in different connected components of the set $T_f\setminus\{C\}$.*

Remark 3.1. The assertion of Lemma 3.2 also holds for level sets of a continuous function $f\colon\overline{\Omega}\to\mathbb{R}$, where Ω is a bounded multiply connected domain of type (3.1), provided that $f\equiv\xi_j=\text{const}$ on each inner boundary component Γ_j, $j=1,\dots,N$. In fact, we can extend f to the whole of $\overline{\Omega}_0$ by setting $f(x)=\xi_j$ for $x\in\overline{\Omega}_j$, $j=1,\dots,N$. The extended function f is continuous on the set $\overline{\Omega}_0$, which is homeomorphic to the unit square $Q=[0,1]^2$.

3.2. Leray's argument by contradiction. If (1.5) holds for the boundary values $\mathbf{a}\in W^{3/2,2}(\partial\Omega)$, then the boundary function $\mathbf{a}$ has a divergence-free extension $\mathbf{A}\in W^{2,2}(\Omega)$ (see, for instance, [2], [28]). Using this fact and some standard results [2], we can find a generalized solution $\mathbf{U}\in W^{2,2}(\Omega)$ of the linear Stokes problem such that

$$\mathbf{U}-\mathbf{A}\in H(\Omega)\cap W^{2,2}(\Omega)$$

and

$$\nu\int_\Omega \nabla\mathbf{U}\cdot\nabla\boldsymbol{\eta}\,dx=\int_\Omega \mathbf{f}\cdot\boldsymbol{\eta}\,dx\qquad\forall\,\boldsymbol{\eta}\in H(\Omega). \tag{3.2}$$

[3] A point in a continuum K is called an *endpoint* (a *branch point*) of K if it has topological index 1 (topological index at least 3, respectively). For a topological tree T this definition is equivalent to the following one: $C\in T$ is an endpoint (a *branch point*) of T if $T\setminus\{C\}$ is connected (if $T\setminus\{C\}$ has three or more connected components, respectively).

Moreover,

$$\|\mathbf{U}\|_{W^{2,2}(\Omega)} \leqslant c(\|\mathbf{a}\|_{W^{3/2,2}(\partial\Omega)} + \|\mathbf{f}\|_{L^2(\Omega)}). \tag{3.3}$$

Obviously, $\mathbf{U}$ extends the boundary value $\mathbf{a}$ into Ω. Hence, it follows from the definition (1.12) that a generalized solution $\mathbf{u} = \mathbf{w}+\mathbf{U}$, $\mathbf{w} \in H(\Omega)$, of the boundary-value problem (1.1)–(1.3) for the system of Navier–Stokes equations satisfies the integral identity

$$\begin{aligned} \nu \int_\Omega \nabla \mathbf{w} \cdot \nabla \boldsymbol{\eta}\, dx - \int_\Omega ((\mathbf{w}+\mathbf{U})\cdot\nabla)\boldsymbol{\eta}\cdot \mathbf{w}\, dx - \int_\Omega (\mathbf{w}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{U}\, dx \\ = \int_\Omega (\mathbf{U}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{U}\, dx \qquad \forall\, \boldsymbol{\eta} \in H(\Omega). \end{aligned} \tag{3.4}$$

In § 2.2 we described Leray's method for establishing an a priori bound, which is based on an argument by contradiction. The following lemma is an important element of the proof.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of the form (3.1) with C^2-smooth boundary $\partial\Omega$, let $\mathbf{f} = \nabla^\perp b$ with $b \in W^{2,2}(\Omega)$, and let $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ satisfy (1.5). If the system (1.1)–(1.3) does not have generalized solutions, then there exists a pair of functions $\widehat{\mathbf{w}}$, $\widehat{p}$ with the following properties.*

(E) $\widehat{\mathbf{w}} \in W^{1,2}(\Omega)$, $\widehat{p} \in W^{1,q}(\Omega)$, $q \in (1,2)$, *and the pair $(\widehat{\mathbf{w}}, \widehat{p})$ satisfies the Euler system of equations*

$$(\widehat{\mathbf{w}}\cdot\nabla)\widehat{\mathbf{w}} + \nabla\widehat{p} = 0, \qquad x \in \Omega, \tag{3.5a}$$

$$\operatorname{div}\widehat{\mathbf{w}} = 0, \qquad x \in \Omega, \tag{3.5b}$$

$$\widehat{\mathbf{w}} = 0, \qquad x \in \partial\Omega. \tag{3.5c}$$

(E-NS) *The conditions* (E) *hold and there exist sequences of functions $\mathbf{u}_k \in W^{1,2}(\Omega)$ and $p_k \in W^{1,q}(\Omega)$ and sequences of numbers $\nu_k \to 0+$ and $\lambda_k \to \lambda_0 > 0$ such that the norms $\|\mathbf{u}_k\|_{W^{1,2}(\Omega)}$ and $\|p_k\|_{W^{1,q}(\Omega)}$ are uniformly bounded for each $q \in [1,2)$, the pairs $(\mathbf{u}_k, p_k)$ satisfy the system of equations*

$$-\nu_k \Delta \mathbf{u}_k + (\mathbf{u}_k\cdot\nabla)\mathbf{u}_k + \nabla p_k = \mathbf{f}_k, \qquad x \in \Omega, \tag{3.6a}$$

$$\operatorname{div}\mathbf{u}_k = 0, \qquad x \in \Omega, \tag{3.6b}$$

$$\mathbf{u}_k = \mathbf{a}_k, \qquad x \in \partial\Omega, \tag{3.6c}$$

with $\mathbf{f}_k = \dfrac{\lambda_k \nu_k^2}{\nu^2}\mathbf{f}$ and $\mathbf{a}_k = \dfrac{\lambda_k\nu_k}{\nu}\mathbf{a}$, and

$$\|\nabla \mathbf{u}_k\|_{L^2(\Omega)} \to 1, \quad \mathbf{u}_k \rightharpoonup \widehat{\mathbf{w}} \quad \text{in} \quad W^{1,2}(\Omega),$$
$$p_k \rightharpoonup \widehat{p} \quad \text{in} \quad W^{1,q}(\Omega) \quad \forall\, q \in [1,2).$$

In addition, $\mathbf{u}_k \in W^{3,2}_{\mathrm{loc}}(\Omega)$ and $p_k \in W^{2,2}_{\mathrm{loc}}(\Omega)$.

We shall assume in what follows that the conditions (E-NS) are satisfied. As we showed in § 2.2, if all the fluxes F_i are zero (see (1.6)), then the conditions (E-NS) lead to a contradiction, thereby proving that (1.1)–(1.3) is solvable. In this section our goal is to demonstrate that these conditions also lead to a contradiction in the general case when the boundary data $\mathbf{a}$ satisfy only the necessary condition (1.5). This will justify the existence Theorem 1.1.

Some results on the Euler equations. Here we collect results on the properties obtained in papers before [22] for the limiting solution $(\widehat{\mathbf{w}}, \widehat{p})$ of (3.5).

As already mentioned in Lemma 2.2, it follows from the conditions (E) that there exist constants $\widehat{p}_0, \dots, \widehat{p}_N$ such that

$$\widehat{p}(x) \equiv \widehat{p}_j \quad \text{for } \mathfrak{H}^1\text{-almost all } x \in \Gamma_j. \tag{3.7}$$

Furthermore, it follows from (2.15) and (2.19) that

$$-\frac{\nu}{\lambda_0} = \sum_{j=0}^{N} \widehat{p}_j \int_{\Gamma_j} \mathbf{a} \cdot \mathbf{n} \, ds = \sum_{j=0}^{N} \widehat{p}_j \mathscr{F}_j. \tag{3.8}$$

Assuming that (1.6) holds and that all the fluxes F_i, $i = 0, 1, \dots, N$, are zero, we now get the desired contradiction from the relation (3.8). For the same reasons we arrive at a contradiction when each flux F_j is 'sufficiently small'. Thus, in these cases the proof of the existence theorem is complete. However, the case when $F_j \neq 0$ is more delicate and the argument must be more refined.

It follows from (3.5b) and (3.5c) that there exists a stream function $\psi \in W^{2,2}(\Omega)$ such that

$$\nabla \psi \equiv \widehat{\mathbf{w}}^{\perp} \quad \text{in } \overline{\Omega}. \tag{3.9}$$

(Recall that $(a, b)^{\perp} = (-b, a)$ by our definition.)

Let $\widehat{\Phi}$ denote the total head pressure corresponding to the solution $(\widehat{\mathbf{w}}, \widehat{p})$: $\widehat{\Phi} = \widehat{p} + |\widehat{\mathbf{w}}|^2/2$. The next formula is an immediate consequence of (3.5):

$$\nabla \widehat{\Phi} \equiv \widehat{\omega} \widehat{\mathbf{w}}^{\perp} = \widehat{\omega} \nabla \psi \quad \text{in } \Omega, \tag{3.10}$$

where $\widehat{\omega}$ denotes the corresponding vorticity

$$\widehat{\omega} = \frac{\partial \widehat{w}_1}{\partial x_2} - \frac{\partial \widehat{w}_2}{\partial x_1} = \Delta \psi.$$

In our case streamlines coincide with level sets of ψ, so from (3.10) we immediately deduce the classical Bernoulli law for smooth ψ and $\widehat{\Phi}$:

The total head pressure is constant on each streamline.

However, the Sobolev case is more delicate: the stream function $\psi \in W^{2,1}(\Omega)$ is now not C^1-smooth and the total head pressure $\widehat{\Phi}$ belongs to the space $W^{1,q}(\Omega)$ with $q < 2$, in which functions are not necessarily continuous but are 'well defined' outside some 'bad' subset of zero length (one-dimensional Hausdorff measure) (see, for instance, [49], Theorem 1 in § 4.8 or Theorem 2 in § 4.9.2). Thus, Bernoulli's law for solutions in Sobolev spaces must be formulated 'modulo' a negligible 'bad' subset $A_{\widehat{\mathbf{w}}}$ with zero Hausdorff $\mathfrak{H}^1$-measure. Such a version of Bernoulli's law was established in [18], Theorem 1 (see also [19], Theorem 3.2, where more details of the proof were given).

Theorem 3.2 (Bernoulli's law). *Assume the conditions (E). Then there is a set $A_{\widehat{\mathbf{w}}}$ with $\mathfrak{H}^1(A_{\widehat{\mathbf{w}}}) = 0$ such that each point $x \in \overline{\Omega} \setminus A_{\widehat{\mathbf{w}}}$ is a Lebesgue point*[4] *of the*

[4]To define a Lebesgue point on $\partial\Omega$ we take the natural extensions of the functions $\widehat{\mathbf{w}}$ and $\widehat{\Phi}$ to the whole of $\mathbb{R}^2$ by the constants $\mathbf{0}$ and $\widehat{p}_j$, respectively.

functions $\widehat{\mathbf{w}}$ *and* $\widehat{\Phi}$, *and the following property holds for compact connected subsets* $K \subset \overline{\Omega}$: *if*

$$\psi\big|_K = \text{const}, \tag{3.11}$$

then

$$\widehat{\Phi}(x_1) = \widehat{\Phi}(x_2) \quad \textit{for all } x_1, x_2 \in K \setminus A_{\widehat{\mathbf{w}}}. \tag{3.12}$$

Lemma 3.4. *If the conditions (E) hold, then there exist constants* $\xi_0, \dots, \xi_N \in \mathbb{R}$ *such that* $\psi(x) \equiv \xi_j$ *on the component* Γ_j, $j = 0, 1, \dots, N$.

Proof. This can easily be deduced from the fact that, extending $\widehat{\mathbf{w}}$ by zero outside Ω, we obtain a function in $H(\mathbb{R}^2) \subset W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$, so that the stream function $\psi \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^2)$ is 'well defined' in the whole of the plane $\mathbb{R}^2$, with $\nabla\psi = 0$ in $\mathbb{R}^2 \setminus \Omega$. □

For $x \in \overline{\Omega}$ let K_x denote the connected component of the level set $\{z \in \overline{\Omega} \colon \psi(z) = \psi(x)\}$ which contains x. By Lemma 3.4, $K_x \cap \partial\Omega = \varnothing$ for each $y \in \psi(\overline{\Omega}) \setminus \{\xi_0, \dots, \xi_N\}$ and for any $x \in \psi^{-1}(y)$. Thus, Theorem 3.1 (ii), (iv) means that for almost all $y \in \psi(\overline{\Omega})$ and each $x \in \psi^{-1}(y)$ we have $K_x \cap A_{\widehat{\mathbf{w}}} = \varnothing$, and hence the component $K_x \subset \Omega$ is a C^1-curve homeomorphic to a circle. We call such curves K_x *admissible cycles.*

The following lemma was proved in [19], Lemma 3.3.

Lemma 3.5. *If the conditions (E-NS) are satisfied, then there exists a subsequence* Φ_{k_l} *such that* $\Phi_{k_l}\big|_S$ *converges uniformly to* $\widehat{\Phi}\big|_S$, $\Phi_{k_l}\big|_S \rightrightarrows \widehat{\Phi}\big|_S$, *on almost all*[5] *admissible cycles* S.

In connection with Lemma 3.5 we remark that Amick [17] proved the uniform convergence $\Phi_k \rightrightarrows \widehat{\Phi}$ on almost all circles. However, his method can easily be modified to prove uniform convergence on almost all level sets of any C^1-smooth function with non-vanishing gradient. Such a modification was carried out in the proof of Lemma 3.3 in [19].

In what follows we assume (without loss of generality) that the sequence Φ_{k_l} in Lemma 3.5 coincides with Φ_k. We shall call the admissible cycles S in the lemma *regular cycles.*

3.3. Arriving at a contradiction. We look at two cases.

(a) $\widehat{\Phi}$ attains its maximum on $\partial\Omega$:

$$\max_{j=0,\dots,N} \widehat{p}_j = \operatorname*{ess\,sup}_{x\in\Omega} \widehat{\Phi}(x). \tag{3.13}$$

(b) $\widehat{\Phi}$ does not attain its maximum on $\partial\Omega$ (the case when $\operatorname{ess\,sup}_{x\in\Omega} \widehat{\Phi}(x) = +\infty$ is also possible):

$$\max_{j=0,\dots,N} \widehat{p}_j < \operatorname*{ess\,sup}_{x\in\Omega} \widehat{\Phi}(x). \tag{3.14}$$

[5]By almost all cycles we mean the cycles in the inverse images $\psi^{-1}(y)$ of almost all $y \in \psi(\overline{\Omega})$.

3.3.1. $\widehat{\Phi}$ *attains a maximum on* $\partial\Omega$. Assume that (3.13) holds. By adding a constant to the pressure we can assume without loss of generality that

$$\max_{j=0,\ldots,N} \widehat{p}_j = \operatorname*{ess\,sup}_{x\in\Omega} \widehat{\Phi}(x) = 0. \tag{3.15}$$

In particular,

$$\widehat{\Phi}(x) \leqslant 0 \quad \text{in } \Omega. \tag{3.16}$$

If $\widehat{p}_0 = \widehat{p}_1 = \cdots = \widehat{p}_N$, then since the total flux vanishes (1.5), we immediately obtain the desired contradiction by (3.8). Thus, we assume in what follows that

$$\min_{j=0,\ldots,N} \widehat{p}_j < 0. \tag{3.17}$$

We change (if needed) the numbering of the boundary components $\Gamma_0, \Gamma_1, \ldots, \Gamma_N$ so that

$$\widehat{p}_j < 0, \qquad j = 0, 1, \ldots, M, \tag{3.18}$$

$$\widehat{p}_{M+1} = \cdots = \widehat{p}_N = 0. \tag{3.19}$$

First of all we explain heuristically the central idea of the proof. It is well known that each function Φ_k satisfies the linear elliptic equation

$$\Delta\Phi_k = \omega_k^2 + \frac{1}{\nu_k}\operatorname{div}(\Phi_k \mathbf{u}_k) - \frac{1}{\nu_k}\mathbf{f}_k \cdot \mathbf{u}_k. \tag{3.20}$$

If $\mathbf{f}_k = 0$, then by Hopf's maximum principle, for a subdomain $\Omega' \Subset \Omega$ with C^2-boundary $\partial\Omega'$ the maximum of Φ_k in Ω' is attained on $\partial\Omega'$, and if $x_* \in \partial\Omega'$ is a maximum point, then the normal derivative of Φ_k at x_* is strictly positive. However, it is not sufficient to apply this property as such: we instead use certain 'integral analogues' of it which will bring us to a contradiction with the co-area formula (see (3.36), (3.37)). For $i \in \mathbb{N}$ and sufficiently large $k \geqslant k(i)$ we construct a set $E_i \subset \Omega$ formed by level curves of Φ_k such that $\Phi_k\big|_{E_i} \to 0$ as $i \to \infty$ and E_i separates the boundary component Γ_N (on which $\widehat{\Phi} = 0$) from the boundary components Γ_j with $j = 0, 1, \ldots, M$ (on which $\widehat{\Phi} < 0$). On the one hand, each of these level curves has length bounded below by a positive constant (because they separate the boundary components), and by the co-area formula this gives a lower bound for the integral $\int_{E_i} |\nabla\Phi_k|$. On the other hand, using the elliptic equation (3.20) for Φ_k, the convergence $\mathbf{f}_k \to 0$, and the boundary condition (3.6c), we can find an upper bound for $\int_{E_i} |\nabla\Phi_k|^2$ (see Lemma 3.6 below), and this latter bound will asymptotically be in contradiction with the previous bound.

For a multiply connected domain of general form the proof is essentially the same as for an annular domain (when $\partial\Omega = \Gamma_0 \cup \Gamma_1$). The proof is analytic in nature, and the inessential differences concern only well-known geometric properties of level sets of continuous functions of two variables.

Using Kronrod's results (see § 3.1.2), we can construct a decreasing sequence of domains $V_{i+1} \Subset V_i \subset \Omega$ with the properties

$$\partial V_i = A_i^0 \cup \cdots \cup A_i^M \cup \Gamma_K \cup \cdots \cup \Gamma_N, \tag{3.21}$$

where $K \in \{M+1, \dots, N\}$ is a fixed integer and each set A_i^j with $j = 0, 1, \dots, M$ and $i \in \mathbb{N}$ is a regular cycle separating Γ_j from Γ_N such that

$$\widehat{\Phi}(A_i^j) = -t_i = 2^{-i} t_0 \tag{3.22}$$

(see Fig. 4; some details of the construction are explained, for instance, in § 5).

By the definition of a regular cycle (see the comments to Lemma 3.5) we have the uniform convergence $\Phi_k\big|_{A_i^j} \rightrightarrows \widehat{\Phi}(A_i^j) = -t_i$ as $k \to \infty$. Thus, for each $i \in \mathbb{N}$ there is a k_i such that for all $k \geqslant k_i$,

$$\Phi_k\big|_{A_i^j} < -\frac{7}{8} t_i, \quad \Phi_k\big|_{A_{i+1}^j} > -\frac{5}{8} t_i \qquad \forall\, j = 0, 1, \dots, M. \tag{3.23}$$

Then

$$\forall\, t \in \left[\frac{5}{8} t_i, \frac{7}{8} t_i\right] \quad \forall\, k \geqslant k_i \quad \Phi_k\big|_{A_i^j} < -t, \quad \Phi_k\big|_{A_{i+1}^j} > -t \quad \forall\, j = 0, 1, \dots, M. \tag{3.24}$$

For $k \geqslant k_i$, $j = 0, 1, \dots, M$, and $t \in [5t_i/8, 7t_i/8]$ let $W_{ik}^j(t)$ denote the connected component of the open set $\{x \in V_i \setminus \overline{V}_{i+1} \colon \Phi_k(x) > -t\}$ such that $\partial W_{ik}^j(t) \supset A_{i+1}^j$, and let

$$W_{ik}(t) = \bigcup_{j=0}^{M} W_{ik}^j(t), \qquad S_{ik}(t) = (\partial W_{ik}(t)) \cap V_i \setminus \overline{V}_{i+1}.$$

Clearly, $\Phi_k \equiv -t$ on $S_{ik}(t)$. By construction

$$\partial W_{ik}(t) = S_{ik}(t) \cup A_{i+1}^0 \cup \dots \cup A_{i+1}^M \tag{3.25}$$

(see Fig. 4). In view of the properties (E-NS), each function Φ_k belongs to the Sobolev class $W^{2,2}_{\mathrm{loc}}(\Omega)$, and hence the Morse–Sard theorem for Sobolev functions (see Theorem 3.1) gives us that for almost all $t \in [5t_i/8, 7t_i/8]$ the level set $S_{ik}(t)$ consists of finitely many C^1-smooth cycles, and Φ_k is differentiable (in the classical sense) at each point $x \in S_{ik}(t)$ such that $\nabla \Phi_k(x) \neq 0$. We call values $t \in [5t_i/8, 7t_i/8]$ with this property (k, i)-*regular* values. By construction

$$\int_{S_{ik}(t)} \nabla \Phi_k \cdot \mathbf{n}\, ds = -\int_{S_{ik}(t)} |\nabla \Phi_k|\, ds < 0, \tag{3.26}$$

where $\mathbf{n}$ is the unit outward normal vector to $\partial W_{ik}(t)$ (relative to $W_{ik}(t)$).

The following estimate is the key step.

Lemma 3.6. *For each $i \in \mathbb{N}$ there is an index $k(i) \in \mathbb{N}$ such that*

$$\int_{S_{ik}(t)} |\nabla \Phi_k|\, ds < \mathscr{F} t \tag{3.27}$$

for all $k \geqslant k(i)$ and almost all $t \in [5t_i/8, 7t_i/8]$, where the constant $\mathscr{F}$ is independent of t, k, and i.

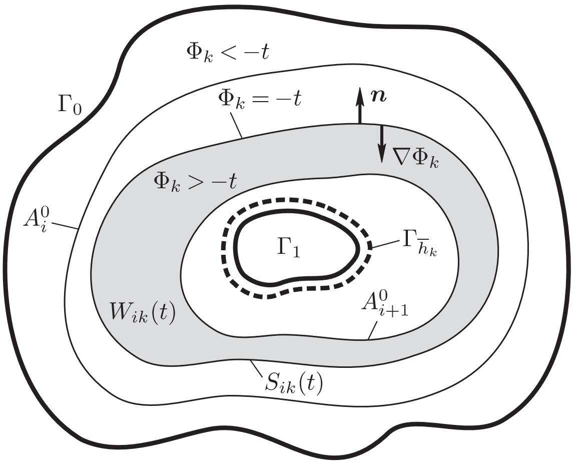

Figure 4. The case of an annular domain ($N = 1$)

We discuss only the main steps of the proof. For $h > 0$ let

$$\Gamma_h = \{x \in \Omega\colon \operatorname{dist}(x, \Gamma_K \cup \cdots \cup \Gamma_N) = h)\},$$
$$\Omega_h = \{x \in \Omega\colon \operatorname{dist}(x, \Gamma_K \cup \cdots \cup \Gamma_N) < h)\}.$$

In the first step of the proof we find for sufficiently large k a small number $\overline{h}_k$ such that $\Omega_{\bar{h}_k} \subset V_{i+2}$ and

$$\int_{\Gamma_{\overline{h}_k}} \Phi_k^2 \, ds < \sigma^2, \tag{3.28}$$

$$\left| \int_{\Gamma_{\overline{h}_k}} \nabla\Phi_k \cdot \mathbf{n} \, ds \right| = \left| \int_{\Gamma_{\overline{h}_k}} \omega_k \mathbf{u}_k^{\perp} \cdot \mathbf{n} \, ds \right| < \varepsilon, \tag{3.29}$$

$$\int_{\Gamma_{\overline{h}_k}} |\mathbf{u}_k|^2 \, ds < C_\varepsilon \nu_k^2, \tag{3.30}$$

where σ and ε are arbitrary (but fixed) small numbers, and the constant C_ε is *independent of k and σ*. We can find such a value of $\overline{h}_k$ on the basis of the assumption (E-NS), by using the weak convergence $\Phi_k \rightharpoonup \widehat{\Phi}$ and $\mathbf{u}_k \rightharpoonup \widehat{\mathbf{w}}$ and the boundary conditions $\|\mathbf{u}_k\|_{L^2(\partial\Omega)} = O(\nu_k)$ for $\nu_k \to 0$ and $\widehat{\mathbf{w}} \equiv 0$ and $\widehat{\Phi} \equiv 0$ on $\Gamma_K \cup \cdots \cup \Gamma_N$ (see (3.6c) and (3.5c)).

Now for a (k, i)-regular value $t \in [5t_i/8, 7t_i/8]$ we take the domain

$$\Omega_{i\overline{h}_k}(t) = W_{ik}(t) \cup \overline{V}_{i+1} \setminus \overline{\Omega}_{\overline{h}_k}.$$

By construction $\partial\Omega_{i\overline{h}_k}(t) = \Gamma_{\overline{h}_k} \cup S_{ik}(t)$ (see Fig. 4). Integrating the identity

$$\Delta\Phi_k = \omega_k^2 + \frac{1}{\nu_k} \operatorname{div}(\Phi_k \mathbf{u}_k) - \frac{1}{\nu_k} \mathbf{f}_k \cdot \mathbf{u}_k \tag{3.31}$$

over $\Omega_{i\bar{h}_k}(t)$, we get that

$$\begin{aligned}\int_{S_{ik}(t)} \nabla\Phi_k\cdot\mathbf{n}\,ds+\int_{\Gamma_{\bar{h}_k}}\nabla\Phi_k\cdot\mathbf{n}\,ds&=\int_{\Omega_{i\bar{h}_k}(t)}\omega_k^2\,dx-\frac{1}{\nu_k}\int_{\Omega_{i\bar{h}_k}(t)}\mathbf{f}_k\cdot\mathbf{u}_k\,dx\\&\quad+\frac{1}{\nu_k}\int_{S_{ik}(t)}\Phi_k\mathbf{u}_k\cdot\mathbf{n}\,ds+\frac{1}{\nu_k}\int_{\Gamma_{\bar{h}_k}}\Phi_k\mathbf{u}_k\cdot\mathbf{n}\,ds\\&=\int_{\Omega_{i\bar{h}_k}(t)}\omega_k^2\,dx-\frac{1}{\nu_k}\int_{\Omega_{i\bar{h}_k}(t)}\mathbf{f}_k\cdot\mathbf{u}_k\,dx-t\lambda_k\overline{\mathscr{F}}+\frac{1}{\nu_k}\int_{\Gamma_{\bar{h}_k}}\Phi_k\mathbf{u}_k\cdot\mathbf{n}\,ds,\end{aligned}\tag{3.32}$$

where $\overline{\mathscr{F}}=(F_1+\cdots+F_M)/\nu$. In view of (3.26) and (3.29),

$$\begin{aligned}\int_{S_{ik}(t)}|\nabla\Phi_k|\,ds\leqslant t\mathscr{F}+\varepsilon+\frac{1}{\nu_k}\int_{\Omega_{i\bar{h}_k}(t)}\mathbf{f}_k\cdot\mathbf{u}_k\,dx-\int_{\Omega_{i\bar{h}_k}(t)}\omega_k^2\,dx\\+\frac{1}{\nu_k}\Big(\int_{\Gamma_{\bar{h}_k}}\Phi_k^2\,ds\Big)^{1/2}\Big(\int_{\Gamma_{\bar{h}_k}}|\mathbf{u}_k|^2\,ds\Big)^{1/2},\end{aligned}\tag{3.33}$$

where $\mathscr{F}=|\overline{\mathscr{F}}|$. By definition,

$$\frac{1}{\nu_k}\|\mathbf{f}_k\|_{L^2(\Omega)}=\frac{\lambda_k\nu_k}{\nu^2}\|\mathbf{f}\|_{L^2(\Omega)}\to 0\quad\text{as } k\to\infty.$$

Consequently,

$$\Big|\frac{1}{\nu_k}\int_{\Omega_{i\bar{h}_k}(t)}\mathbf{f}_k\cdot\mathbf{u}_k\,dx\Big|\leqslant\varepsilon$$

for sufficiently large k. Using the inequalities (3.28) and (3.30), we find that

$$\int_{S_{ik}(t)}|\nabla\Phi_k|\,ds\leqslant t\mathscr{F}+2\varepsilon+\sigma\sqrt{C_\varepsilon}-\int_{\Omega_{i\bar{h}_k}(t)}\omega_k^2\,dx,\tag{3.34}$$

where C_ε is independent of k and σ. We can show that for sufficiently large k the last term satisfies the inequality

$$\int_{\Omega_{i\bar{h}_k}(t)}\omega_k^2\,dx\geqslant\varepsilon_i\tag{3.35}$$

with a parameter $\varepsilon_i>0$ which is independent of k (otherwise $\omega(x)\equiv 0$, and accordingly $\Phi(x)\equiv\text{const}$ on the set $V_{i+1}\setminus V_{i+2}\subset\Omega_{i\bar{h}_k}(t)$, which is impossible by construction). Taking $\varepsilon=\varepsilon_i/6$, $\sigma=\varepsilon_i/(3\sqrt{C_\varepsilon})$, and a sufficiently large k, we get from (3.35) that

$$2\varepsilon+\sigma\sqrt{C_\varepsilon}-\int_{\Omega_{i\bar{h}_k}(t)}\omega_k^2\,dx\leqslant 0.$$

In combination with (3.34) this inequality yields the required estimate (3.27).

We can now obtain the desired contradiction using the co-area formula.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of type (3.1) with C^2-smooth boundary $\partial\Omega$, let $\mathbf{f} \in W^{1,2}(\Omega)$, and assume that the boundary data $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ satisfy the condition of zero total flux (1.5). Then the assumptions (E-NS) and (3.13) lead to a contradiction.*

Proof. For $i \in \mathbb{N}$ and $k \geqslant k(i)$ (see Lemma 3.6) let

$$E_i = \bigcup_{t\in[5t_i/8,7t_i/8]} S_{ik}(t).$$

By the co-area formula (see, for instance, [50]), for each integrable function $g\colon E_i \to \mathbb{R}$ we have

$$\int_{E_i} g|\nabla\Phi_k|\,dx = \int_{5t_i/8}^{7t_i/8}\int_{S_{ik}(t)} g(x)\,d\mathfrak{H}^1(x)\,dt. \tag{3.36}$$

In particular, for $g = |\nabla\Phi_k|$ we get from (3.27) that

$$\int_{E_i} |\nabla\Phi_k|^2\,dx = \int_{5t_i/8}^{7t_i/8}\int_{S_{ik}(t)} |\nabla\Phi_k|(x)\,d\mathfrak{H}^1(x)\,dt \leqslant \int_{5t_i/8}^{7t_i/8} \mathscr{F} t\,dt = \mathscr{F}' t_i^2, \tag{3.37}$$

where the parameter $\mathscr{F}' = 3\mathscr{F}/16$ is independent of i. Setting $g = 1$ in (3.36) and using Hölder's inequality, we now find that

$$\begin{aligned}\int_{5t_i/8}^{7t_i/8} \mathfrak{H}^1\bigl(S_{ik}(t)\bigr)\,dt &= \int_{E_i} |\nabla\Phi_k|\,dx \leqslant \biggl(\int_{E_i} |\nabla\Phi_k|^2\,dx\biggr)^{1/2}\bigl(\operatorname{meas}(E_i)\bigr)^{1/2} \\ &\leqslant \sqrt{\mathscr{F}'}\,t_i\bigl(\operatorname{meas}(E_i)\bigr)^{1/2}.\end{aligned} \tag{3.38}$$

By construction, for almost all $t \in [5t_i/8, 7t_i/8]$ the set $S_{ik}(t)$ is a finite union of smooth cycles, and $S_{ik}(t)$ separates A_i^j from A_{i+1}^j for $j = 0, 1, \dots, M$. Thus, each set $S_{ik}(t)$ separates the component Γ_j from Γ_N. In particular,

$$\mathfrak{H}^1(S_{ik}(t)) \geqslant \min\bigl(\operatorname{diam}\Gamma_j, \operatorname{diam}\Gamma_N\bigr).$$

Hence, the left-hand integral in (3.38) is greater than Ct_i, where the positive constant C is independent of i. On the other hand, it is obvious that

$$\operatorname{meas}(E_i) \leqslant \operatorname{meas}\bigl(V_i \setminus V_{i+1}\bigr) \to 0 \quad \text{as} \quad i \to \infty,$$

a contradiction. $\square$

3.3.2. $\widehat{\Phi}$ *does not attain a maximum on* $\partial\Omega$. In this subsection we look at the case (b), when the assumption (3.14) holds (here $\operatorname{ess\,sup}_{x\in\Omega} \widehat{\Phi}(x)$ can $= +\infty$). Let $\sigma = \max_{j=0,\dots,N} \widehat{p}_j$.

First, [22] establishes the following technical result.

Lemma 3.8. *There exists a regular cycle $F \subset \Omega$ such that $\widehat{\Phi}(F) > \sigma$.*

The rest of the proof in the case (3.14) under consideration is similar to the argument in § 3.3.1 for the case (3.13) already covered. The differences are as follows: now $M = N$, the set F plays the role of the component Γ_N in the above argument, and the calculations become even simpler because F lies strictly inside Ω. Namely, for some $t<\Phi(F)$ and sufficiently large k we can construct a subdomain $\Omega_k(t)\Subset\Omega$ such that

$$F\Subset\Omega_k(t),\qquad \partial\Omega_k(t)=S_k(t),\qquad \Phi_k\big|_{S_k(t)}\equiv t,$$

and $S_k(t)$ is a finite union of C^1-smooth cycles such that the gradient of Φ_k at points of $S_k(t)$ is directed strictly inside $\Omega_k(t)$, that is,

$$\int_{S_k(t)}\nabla\Phi_k\cdot\mathbf{n}\,ds=-\int_{S_k(t)}|\nabla\Phi_k|\,ds<0, \tag{3.39}$$

where $\mathbf{n}$ is the unit outward normal vector to $S_k(t)$ (relative to $\Omega_k(t)$). Integrating the identity (3.31) over $\Omega_k(t)$, we get that

$$\begin{aligned}0>\int_{S_k(t)}\nabla\Phi_k\cdot\mathbf{n}\,ds&=\int_{\Omega_k(t)}\omega_k^2\,dx+\frac{1}{\nu_k}\int_{S_k(t)}\Phi_k\mathbf{u}_k\cdot\mathbf{n}\,ds-\frac{1}{\nu_k}\int_{\Omega_k(t)}\mathbf{f}_k\cdot\mathbf{u}_k\,dx\\&=\int_{\Omega_k(t)}\omega_k^2\,dx+\frac{t}{\nu_k}\int_{S_k(t)}\mathbf{u}_k\cdot\mathbf{n}\,ds-\frac{\nu_k}{\nu^2}\int_{\Omega_k(t)}\mathbf{f}\cdot\mathbf{u}_k\,dx\\&=\int_{\Omega_k(t)}\omega_k^2\,dx-\frac{\nu_k}{\nu^2}\int_{\Omega_k(t)}\mathbf{f}\cdot\mathbf{u}_k\,dx.\end{aligned} \tag{3.40}$$

As previously, we can show that the first term on the right-hand side of the last formula satisfies $\int_{\Omega_k(t)}\omega_k^2\,dx\geqslant\varepsilon_t$ with a positive constant ε_t independent of k, while the second term $\frac{\nu_k}{\nu^2}\int_{\Omega_k(t)}\mathbf{f}\cdot\mathbf{u}_k\,dx$ tends to zero as $k\to\infty$, which is a contradiction. (The reader can find details in [22], § 3.3.2.) Thus, we have established the following result.

Lemma 3.9. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain of type (3.1) with C^2-smooth boundary $\partial\Omega$, let $\mathbf{f}\in W^{1,2}(\Omega)$, and assume that the boundary data $\mathbf{a}\in W^{3/2,2}(\partial\Omega)$ satisfy the condition of zero total flux (1.5). Then the assumptions (E-NS) and (3.14) lead to a contradiction.*

Proof of Theorem 1.1. Assume that the hypotheses of Theorem 1.1 hold, but its assertion fails to hold. Then by Lemma 3.3 there exist functions $\widehat{\mathbf{w}}$, $\widehat{p}$ and a sequence $(\mathbf{u}_k,p_k)$ satisfying the set of conditions (E-NS). However, by Lemmas 3.9 and 3.7 these assumptions lead to a contradiction. □

4. The axially symmetric case

We start by refining some notation. Let O_{x_1}, O_{x_2}, O_{x_3} be the coordinate axes in $\mathbb{R}^3$ and let

$$\theta=\arctan\frac{x_2}{x_1},\qquad r=(x_1^2+x_2^2)^{1/2},\quad z=x_3$$

be the cylindrical system of coordinates. Denote by v_θ, v_r, v_z the projections of a vector $\mathbf{v}$ on the θ-, r-, and z-axes.

Recall that a scalar function f is *axially symmetric* if it does not depend on θ. In turn, a vector-valued function $\mathbf{h}=(h_r,h_\theta,h_z)$ is said to be *axially symmetric* if h_r, h_θ, and h_z are independent of θ. A vector-valued function $\mathbf{h}'=(h_r,h_\theta,h_z)$ is said to be *axially symmetric without swirl* if $h_\theta = 0$ and h_r and h_z are independent of θ.

The central result in this section is as follows.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded axially symmetric domain (see, for instance, Fig. 5) with C^2-smooth boundary $\partial\Omega$ consisting of $N+1$ connected components Γ_i. If $\mathbf{f} \in W^{1,2}(\Omega)$ and $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ are axially symmetric and the boundary values $\mathbf{a}$ satisfy the condition (1.5) of zero total flux, then the system (1.1)–(1.3) has at least one generalized axially symmetric solution. Moreover, if $\mathbf{f}$ and $\mathbf{a}$ are axially symmetric functions without swirl, then (1.1)–(1.3) also has at least one generalized axially symmetric solution without swirl.*

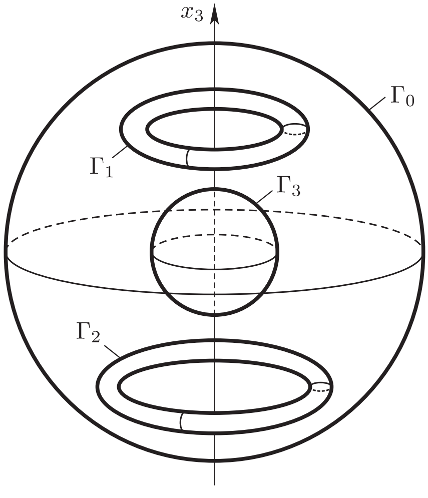

Figure 5. An axially symmetric domain ($N = 3$)

Using Leray's argument by contradiction (whose main idea was presented in § 3.2 in the planar case; the reader can find details reflecting the peculiarities of the axially symmetric case, for instance, in [23]), we can prove the following.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded axially symmetric domain with a C^2-smooth boundary $\partial\Omega$ consisting of $N+1$ connected components Γ_i, and let $\mathbf{f} = \operatorname{curl} \mathbf{b}$, $\mathbf{b} \in W^{2,2}(\Omega)$, and $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ be axially symmetric vector-valued functions such that the boundary data $\mathbf{a}$ satisfy the condition (1.5) of zero total flux. If the assertion of Theorem 4.1 is false, then there exist $\widehat{\mathbf{w}}$ and $\widehat{p}$ with the following properties.*

(E-AX) *The functions $\widehat{\mathbf{w}} \in \mathring{W}^{1,2}(\Omega)$ and $\widehat{p} \in W^{1,3/2}(\Omega)$ are axially symmetric and satisfy the Euler system of equations (3.5).*

(E-NS-AX) *The condition (E-AX) is satisfied and there exist sequences of axially symmetric functions $\mathbf{u}_k \in W^{1,2}(\Omega)$ and $p_k \in W^{1,3/2}(\Omega)$ and sequences of numbers $\nu_k \to 0+$ and $\lambda_k \to \lambda_0 > 0$ such that the norms $\|\mathbf{u}_k\|_{W^{1,2}(\Omega)}$ and $\|p_k\|_{W^{1,3/2}(\Omega)}$ are uniformly bounded, the pairs $(\mathbf{u}_k, p_k)$ satisfy the system of equations (3.6) with*

$\mathbf{f}_k = \dfrac{\lambda_k \nu_k^2}{\nu^2}\mathbf{f}$ *and* $\mathbf{a}_k = \dfrac{\lambda_k \nu_k}{\nu}\mathbf{a}$, *and*

$$\|\nabla \mathbf{u}_k\|_{L^2(\Omega)} \to 1, \qquad \mathbf{u}_k \rightharpoonup \widehat{\mathbf{w}} \quad \textit{in } W^{1,2}(\Omega), \qquad p_k \rightharpoonup \widehat{p} \quad \textit{in } W^{1,3/2}(\Omega). \tag{4.1}$$

Furthermore, $\mathbf{u}_k \in W^{3,2}_{\mathrm{loc}}(\Omega)$ *and* $p_k \in W^{2,2}_{\mathrm{loc}}(\Omega)$.

As in the previous section, to prove the existence theorem (Theorem 4.1) we must show that the conditions (E-NS-AX) lead to a contradiction.

We define the sets $P_+ = \{(0, x_2, x_3) \colon x_2 > 0,\ x_3 \in \mathbb{R}\}$ and $\mathscr{D} = \Omega \cap P_+$. In the half-plane P_+ the variables x_2, x_3 obviously coincide with r, z.

For a set $A \subset \mathbb{R}^3$ let $\breve{A} := A \cap P_+$, and for a set $B \subset P_+$ let $\widetilde{B}$ denote the subset of $\mathbb{R}^3$ obtained by rotating B about the axis O_z.

The following properties are easy to see.

(S_1) $\mathscr{D}$ is a bounded plane domain with Lipschitz boundary. Moreover, each set $\breve{\Gamma}_j$, $j = 0, 1, \dots, N$, is connected. In other words, $\{\breve{\Gamma}_j \colon j = 0, 1, \dots, N\}$ is the family of all connected components of the set $P_+ \cap \partial\mathscr{D}$.

Thus, $\widehat{\mathbf{w}}$ and $\widehat{p}$ satisfy the following system in $\mathscr{D}$:

$$\begin{aligned} \frac{\partial \widehat{p}}{\partial r} - \frac{(\widehat{w}_\theta)^2}{r} + \widehat{w}_r \frac{\partial \widehat{w}_r}{\partial r} + \widehat{w}_z \frac{\partial \widehat{w}_r}{\partial z} &= 0, \\ \frac{\partial \widehat{p}}{\partial z} + \widehat{w}_r \frac{\partial \widehat{w}_z}{\partial r} + \widehat{w}_z \frac{\partial \widehat{w}_z}{\partial z} &= 0, \\ \frac{\widehat{w}_\theta \widehat{w}_r}{r} + \widehat{w}_r \frac{\partial \widehat{w}_\theta}{\partial r} + \widehat{w}_z \frac{\partial \widehat{w}_\theta}{\partial z} &= 0, \\ \frac{\partial (r\widehat{w}_r)}{\partial r} + \frac{\partial (r\widehat{w}_z)}{\partial z} &= 0 \end{aligned} \tag{4.2}$$

(these equations hold for almost all $x \in \mathscr{D}$) and

$$\widehat{\mathbf{w}}(x) = 0 \quad \text{for } \mathfrak{H}^1\text{-almost all } x \in P_+ \cap \partial\mathscr{D}. \tag{4.3}$$

We have the following integral estimates: $\widehat{\mathbf{w}} \in W^{1,2}_{\mathrm{loc}}(\mathscr{D})$,

$$\int_{\mathscr{D}} r |\nabla \widehat{\mathbf{w}}(r,z)|^2 \, dr\, dz < \infty, \tag{4.4}$$

and by the Sobolev embedding theorem for three-dimensional domains, $\widehat{\mathbf{w}} \in L^6(\Omega)$, that is,

$$\int_{\mathscr{D}} r |\widehat{\mathbf{w}}(r,z)|^6 \, dr\, dz < \infty. \tag{4.5}$$

In addition, the condition $\nabla \widehat{p} \in L^{3/2}(\Omega)$ can be written as

$$\int_{\mathscr{D}} r |\nabla \widehat{p}(r,z)|^{3/2} \, dr\, dz < \infty. \tag{4.6}$$

4.1. Some results on the Euler equations. Here we have collected results from papers before [22] for the limiting solution $(\widehat{\mathbf{w}},\widehat{p})$ of the system (4.2), (4.3).

The following result was proved in [33] (Lemma 4) and [17] (Theorem 2.2).

Theorem 4.2. *If the conditions* (E-AX) *are satisfied, then*

$$\forall j\in\{0,1,\dots,N\}\quad \exists\widehat{p}_j\in\mathbb{R}:\quad \widehat{p}(x)\equiv\widehat{p}_j\quad \text{for } \mathfrak{H}^2\text{-almost all } x\in\Gamma_j. \tag{4.7}$$

In particular, by axial symmetry

$$\widehat{p}(x)\equiv\widehat{p}_j\quad \text{for } \mathfrak{H}^1\text{-almost all } x\in\breve{\Gamma}_j. \tag{4.8}$$

We need a weak version of Bernoulli's law for Sobolev solutions $(\widehat{\mathbf{w}},\widehat{p})$ of the Euler equations (4.2) (see Theorem 4.3 below).

It follows from the last equality in (4.2) and from (4.4) that there exists a stream function $\psi\in W^{2,2}_{\mathrm{loc}}(\mathscr{D})$ such that

$$\frac{\partial\psi}{\partial r}=-r\widehat{w}_z,\qquad \frac{\partial\psi}{\partial z}=r\widehat{w}_r. \tag{4.9}$$

It is easy to see that ψ is continuous at points of the set

$$\overline{\mathscr{D}}\setminus O_z=\overline{\mathscr{D}}\setminus\{(0,z)\colon z\in\mathbb{R}\}.$$

Lemma 4.2 (cf. Lemma 3.4). *If the conditions* (E-AX) *are satisfied, then there exist constants* $\xi_0,\xi_1,\dots,\xi_N\in\mathbb{R}$ *such that* $\psi(x)\equiv\xi_j$ *on each curve* $\breve{\Gamma}_j$, $j=0,1,\dots,N$.

Let $\widehat{\Phi}=\widehat{p}+|\widehat{\mathbf{w}}|^2/2$ denote the total head pressure corresponding to the solution $(\widehat{\mathbf{w}},\widehat{p})$.

Theorem 4.3 (Bernoulli's law [23]). *If the conditions* (E-AX) *are satisfied, then there exists a set* $A_{\widehat{\mathbf{w}}}$ *with* $\mathfrak{H}^1(A_{\widehat{\mathbf{w}}})=0$ *such that each* $x\in\overline{\mathscr{D}}\setminus(O_z\cup A_{\widehat{\mathbf{w}}})$ *is a Lebesgue point*[6] *of* $\widehat{\mathbf{w}}$ *and* $\widehat{\Phi}$, *and the following property holds for each compact connected set* $K\subset\overline{\mathscr{D}}\setminus O_z$: *if*

$$\psi\big|_K=\mathrm{const}, \tag{4.10}$$

then

$$\widehat{\Phi}(x_1)=\widehat{\Phi}(x_2)\quad \text{for all } x_1,x_2\in K\setminus A_{\widehat{\mathbf{w}}}. \tag{4.11}$$

4.2. Arriving at a contradiction. Assume that

$$\begin{aligned}\Gamma_j\cap O_{x_3}&\neq\varnothing, &\qquad j&=0,1,\dots,M',\\ \Gamma_j\cap O_{x_3}&=\varnothing, &\qquad j&=M'+1,\dots,N.\end{aligned}$$

The following result was proved in [23].

Theorem 4.4. *If the conditions* (E-AX) *are satisfied, then* $\widehat{p}_0=\dots=\widehat{p}_{M'}$, *where the* $\widehat{p}_j$ *are the constants in Theorem* 4.2.

[6] In the definition of a Lebesgue point on $(\partial\mathscr{D})\setminus(O_z\cup A_{\widehat{\mathbf{w}}})$ we take the natural extensions of the functions $\widehat{\mathbf{w}}$ and $\widehat{\Phi}$ to the whole of P_+ by the constants $\mathbf{0}$ and $\widehat{p}_j$, respectively.

This is connected with the fact that the symmetry axis can be approximated by streamlines, and the total head pressure is constant on streamlines (see Theorem 4.4).

Now we look at three possible cases.

(a) $\widehat{\Phi}$ attains its maximum on a boundary component intersecting the axis of symmetry:

$$\widehat{p}_0 = \max_{j=0,\dots,N} \widehat{p}_j = \operatorname*{ess\,sup}_{x\in\Omega} \widehat{\Phi}(x). \tag{4.12}$$

(b) $\widehat{\Phi}$ attains its maximum on a boundary component disjoint from the axis of symmetry:

$$\widehat{p}_0 < \widehat{p}_N = \max_{j=0,\dots,N} \widehat{p}_j = \operatorname*{ess\,sup}_{x\in\Omega} \widehat{\Phi}(x). \tag{4.13}$$

(c) $\widehat{\Phi}$ does not attain its maximum on $\partial\Omega$:

$$\max_{j=0,\dots,N} \widehat{p}_j < \operatorname*{ess\,sup}_{x\in\Omega} \widehat{\Phi}(x). \tag{4.14}$$

We consider the case (4.12). By adding a constant to the pressure p we can assume without loss of generality that

$$\widehat{p}_0 = \operatorname*{ess\,sup}_{x\in\Omega} \widehat{\Phi}(x) = 0. \tag{4.15}$$

Since $\widehat{p}_0 = \widehat{p}_1 = \dots = \widehat{p}_N$ is impossible, we have $\widehat{p}_j < 0$ for some $j \in \{M'+1, \dots, N\}$ (recall that $\widehat{p}_0 = \dots = \widehat{p}_{M'} = 0$ by Theorem 4.4).

Now we can obtain a contradiction by repeating the arguments in [23] and [20]. From (3.5a) and (3.5b),

$$\begin{aligned} 0 &= x \cdot \nabla \widehat{p}(x) + x \cdot \big(\widehat{\mathbf{w}}(x) \cdot \nabla\big)\widehat{\mathbf{w}}(x) \\ &= \operatorname{div}\big[x\,\widehat{p}(x) + \big(\widehat{\mathbf{w}}(x)\cdot x\big)\widehat{\mathbf{w}}(x)\big] - \widehat{p}(x)\operatorname{div} x - |\widehat{\mathbf{w}}(x)|^2 \\ &= \operatorname{div}\big[x\,\widehat{p}(x) + \big(\widehat{\mathbf{w}}(x)\cdot x\big)\widehat{\mathbf{w}}(x)\big] - 3\widehat{\Phi}(x) + \frac{1}{2}|\widehat{\mathbf{w}}(x)|^2. \end{aligned} \tag{4.16}$$

Integrating over Ω, we have

$$\begin{aligned} 0 \geqslant \int_\Omega \Big[3\widehat{\Phi}(x) - \frac{1}{2}|\widehat{\mathbf{w}}(x)|^2\Big]\,dx &= \int_{\partial\Omega} \widehat{p}(x)\big(x\cdot\mathbf{n}\big)\,ds = \sum_{j=0}^{N} \widehat{p}_j \int_{\Gamma_j} \big(x\cdot\mathbf{n}\big)\,ds \\ = -\sum_{j=1}^{N} \widehat{p}_j \int_{\Omega_j} \operatorname{div} x\,dx = -3\sum_{j=1}^{N} \widehat{p}_j |\Omega_j| > 0. \end{aligned}$$

This contradiction completes the proof in the case (4.12).

In the second case, (4.13), we surround the maximum component Γ_N by a regular cycle separating Γ_N from the axis of symmetry O_z such that $\widehat{\Phi} \equiv c \in (\widehat{p}_0, \widehat{p}_N)$ on this cycle. We thereby eliminate a neighbourhood of the singular line O_z from consideration. Now we can reduce the proof to the planar case we examined in § 3.3.1. We can also make a similar reduction in the third case, (4.14). We have therefore shown that the assumptions (E-NS-AX) lead to a contradiction in all three possible cases (4.12)–(4.14). This completes the proof of Theorem 4.1.

5. Supplement to §§ 3 and 4

In this section we describe the construction of subdomains V_i with the property (3.21). By Remark 3.1 and Lemma 3.4, Kronrod's results hold for the stream function ψ. On the Kronrod tree T_ψ (see § 3.1.2) we define the total head pressure as follows. Let $K \in T_\psi$ be such that $\operatorname{diam} K > 0$, take an arbitrary point $x \in K \setminus A_{\widehat{\mathbf{w}}}$, and let $\widehat{\Phi}(K) = \widehat{\Phi}(x)$. This definition is consistent by Bernoulli's law (see Theorem 3.2). We have the following result.

Lemma 5.1 [22]. *Let $A, B \in T_\psi$, with $\operatorname{diam} A > 0$ and $\operatorname{diam} B > 0$. Let $[A,B] \subset T_\psi$ be the arc joining A and B (see Lemma 3.2). Then the restriction $\widehat{\Phi}\big|_{[A,B]}$ is a continuous function.*

We note that the proof of the lemma is based on Bernoulli's law (see Theorem 3.2) and the well-known quasi-continuity[7] of Sobolev functions (see, for instance, [49], Theorem 1 in § 4.8 and Theorem 2 in § 4.9.2).

We say that a subset $\mathscr{Z}$ of T_ψ has T-measure zero if $\mathfrak{H}^1(\{\psi(C)\colon C \in \mathscr{Z}\}) = 0$. The function $\widehat{\Phi}\big|_{T_\psi}$ turns out to have an analogue of the Luzin N-property.

Lemma 5.2 [22]. *Let $A, B \in T_\psi$, with $\operatorname{diam} A > 0$ and $\operatorname{diam} B > 0$. If a set $\mathscr{Z} \subset [A,B]$ has T-measure zero, then $\mathfrak{H}^1(\{\widehat{\Phi}(C)\colon C \in \mathscr{Z}\}) = 0$.*

The corresponding proof is based on the co-area formula (see, for instance, [50]).

From Lemmas 3.5 and 5.2 we obtain the following result.

Corollary 5.1. *If $A, B \in T_\psi$, $\operatorname{diam} A > 0$, and $\operatorname{diam} B > 0$, then*

$$\mathfrak{H}^1\bigl(\{\widehat{\Phi}(C)\colon C \in [A,B] \textit{ and } C \textit{ is not a regular cycle}\}\bigr) = 0.$$

Let $B_0, B_1, \dots, B_N$ be elements of T_ψ such that $B_j \supset \Gamma_j$, $j = 0,1,\dots,N$. By Lemma 3.4 each element $C \in [B_i, B_j] \setminus \{B_i, B_j\}$ is a connected component of a level set of ψ such that B_i and B_j lie in different connected components of $\mathbb{R}^2 \setminus C$.

We set

$$\alpha = \max_{j=0,\dots,M} \min_{C \in [B_j, B_N]} \widehat{\Phi}(C)$$

and note that $\alpha < 0$ by (3.18). Now we take a sequence of positive numbers $t_i \in (0, -\alpha)$, $i \in \mathbb{N}$, with $t_{i+1} = t_i/2$ such that the implication

$$\widehat{\Phi}(C) = -t_i \quad \Rightarrow \quad C \text{ is a regular cycle}$$

holds for each $j = 0,1,\dots,M$ and all $C \in [B_j, B_N]$. Corollary 5.1 ensures that there is such a sequence.

We introduce the natural ordering on the arc $[B_j, B_N]$, namely, we write $C' < C''$ if C'' is closer to B_N than C' (this means that C' and B_N belong to different connected components of the set $T_\psi \setminus \{C''\}$). For $j = 0,1,\dots,M$ and $i \in \mathbb{N}$ let

$$A_i^j = \max\{C \in [B_j, B_N] \colon \widehat{\Phi}(C) = -t_i\}.$$

[7] The property of quasi-continuity of a function $\widehat{\Phi}$ means the following: for any $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^2$ such that $\mathfrak{H}^1_\infty(U) < \varepsilon$ and the restriction $\widehat{\Phi}\big|_{\overline{\Omega} \setminus U}$ is continuous.

In other words, A_i^j is the element of $\{C \in [B_j, B_N] \colon \widehat{\Phi}(C) = -t_i\}$ closest to Γ_N.

By construction each set A_i^j is a regular cycle (see Fig. 4 for the case of an annular domain, with $N = 1$). Some of these cycles A_i^j may coincide, that is, it is possible that $A_i^{j_1} = A_i^{j_2}$ (when the arcs $[B_{j_1}, B_N]$ and $[B_{j_2}, B_N]$ of the Kronrod tree have a non-trivial intersection), but this a priori possibility does not affect our argument. By construction the cycles A_i^j are either disjoint or coincide, that is, if $A_i^{j_1} \neq A_i^{j_2}$, then $A_i^{j_1} \cap A_i^{j_2} = \varnothing$.

By the definition of a regular cycle (see the comments on Lemma 3.5) each A_i^j is a C^1-curve homeomorphic to the unit circle. Furthermore, $A_i^j \subset \Omega$, and in particular, for each $i \in \mathbb{N}$ the compact set $\bigcup_{j=0}^M A_i^j$ is separated from $\partial\Omega$ and

$$\operatorname{dist}\biggl(\bigcup_{j=0}^{M} A_i^j, \partial\Omega\biggr) > 0.$$

Then for each i and for sufficiently small $h > 0$ (how small depends on i) we have the inclusion

$$\{x \in \Omega \colon \operatorname{dist}(x, \Gamma_N) < h\} \subset \Omega \setminus \biggl(\bigcup_{j=0}^{M} A_i^j\biggr).$$

Of course, for small h the set $\{x \in \Omega \colon \operatorname{dist}(x, \Gamma_N) < h\}$ is connected (it is homeomorphic to an open annulus). Hence, for small h this set lies in a connected component of the open set $\Omega \setminus \bigl(\bigcup_{j=0}^M A_i^j\bigr)$. Let V_i be this component. In particular, $\Gamma_N \subset \partial V_i$ and

$$\Omega \cap \partial V_i = A_i^0 \cup \dots \cup A_i^M. \tag{5.1}$$

By construction the sequence of domains V_i is decreasing: $V_i \supset V_{i+1}$. Therefore, the sequence of sets $(\partial\Omega) \cap (\partial V_i)$ is non-increasing:

$$(\partial\Omega) \cap (\partial V_i) \supseteq (\partial\Omega) \cap (\partial V_{i+1}). \tag{5.2}$$

Each set $(\partial\Omega) \cap (\partial V_i)$ consists of several components Γ_l with $l > M$ (because the arcs $\bigcup_{j=0}^M A_i^j$ separate Γ_N from $\Gamma_0, \dots, \Gamma_M$ but not necessarily from the other Γ_l). Since there are only finitely many components Γ_l, we conclude from the monotonicity (5.2) that the set $(\partial\Omega) \cap (\partial V_i)$ is independent of i for large i. Thus, we can assume without loss of generality that $(\partial\Omega) \cap (\partial V_i) = \Gamma_K \cup \dots \cup \Gamma_N$, where $K \in \{M+1, \dots, N\}$. Consequently,

$$\partial V_i = A_i^0 \cup \dots \cup A_i^M \cup \Gamma_K \cup \dots \cup \Gamma_N. \tag{5.3}$$

6. Solutions with singularities in the flow region

6.1. Planar flows. In this section we deal with planar solutions and axially symmetric solutions of the flux problem which are generated by sources or sinks in the flow region. We start with a planar problem. Let Γ be a smooth Jordan curve in the plane which bounds a domain Ω and encloses the origin O. There is a source or sink with intensity F at O. We must find a solution of (1.1), (1.2) in $\Omega \setminus O$ which satisfies the conditions (1.3) and

$$u_r = \frac{F}{2\pi r}[1 + o(1)], \quad u_\varphi = o(1), \qquad r \to 0, \tag{6.1}$$

where u_r and u_φ are the radial and the tangential components of the velocity. Assume that the origin is the only singular point of the velocity field. Then it follows from (1.2) and the conditions (1.3) and (6.1) that

$$\int_\Gamma \mathbf{a} \cdot \mathbf{n}\, dS = F \tag{6.2}$$

($\mathbf{n}$ is the unit outward normal vector to Γ).

The problem (1.1)–(1.3), (6.1) is the limiting case of the problem of a steady flow of a viscous incompressible fluid in a curvilinear annulus with flux conditions at the boundary components. It was first considered in [51], where its solvability for small values of the parameter $|F|/\nu$ was announced. Details of the proof were presented in [52]. Another proof was given by Russo and Tartaglione [53]. A specific feature of the problem (1.1)–(1.3), (6.1) is the fact that the Dirichlet integral of the unknown vector-valued function $\mathbf{u}(x)$ is infinite in view of the representation (6.1). Moreover, a solution of this problem has infinite energy. However, there are hopes that in the regularized problem obtained by separating out the singular components of the functions $\mathbf{u}$ and p we can obtain an a priori estimate ensuring its solvability.

The equations of planar motion in the polar coordinate system (r, φ) are obtained from (2.51), where we must replace θ by φ, let $u_z = 0$, and assume that u_r, u_φ, and p are independent of z. Let us introduce the new unknown functions

$$w_r = u_r - \frac{F}{2\pi r}\,, \qquad w_\varphi = u_\varphi, \qquad \overline{p} = p + \frac{F^2}{8\pi^2 r^2}\,. \tag{6.3}$$

By (2.51), the vector $\mathbf{w}$ and the function $\overline{p}$ satisfy the system of equations

$$\begin{aligned}
&w_r \frac{\partial w_r}{\partial r} + \frac{1}{r} w_\varphi \frac{\partial w_r}{\partial \varphi} - \frac{1}{r} w_\varphi^2 + \frac{F}{2\pi r}\frac{\partial w_r}{\partial r} - \frac{F}{2\pi r^2} w_r \\
&\qquad = -\frac{\partial \overline{p}}{\partial r} + \nu \biggl(\frac{\partial^2 w_r}{\partial r^2} + \frac{1}{r}\frac{\partial w_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_r}{\partial \varphi^2} - \frac{1}{r^2} w_r - \frac{2}{r^2}\frac{\partial w_\varphi}{\partial \varphi} \biggr), \\
&w_r \frac{\partial w_\varphi}{\partial r} + \frac{1}{r} w_\varphi \frac{\partial w_\varphi}{\partial \varphi} + \frac{1}{r} w_r w_\varphi + \frac{F}{2\pi r}\frac{\partial w_\varphi}{\partial r} + \frac{F}{2\pi r^2} w_\varphi \\
&\qquad = -\frac{1}{r}\frac{\partial \overline{p}}{\partial \varphi} + \nu \biggl(\frac{\partial^2 w_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial w_\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_\varphi}{\partial \varphi^2} - \frac{1}{r^2} w_\varphi + \frac{2}{r^2}\frac{\partial w_r}{\partial \varphi} \biggr), \\
&\frac{\partial w_r}{\partial r} + \frac{1}{r} w_r + \frac{1}{r}\frac{\partial w_\varphi}{\partial \varphi} = 0.
\end{aligned} \tag{6.4}$$

In the new terms the condition (1.3) can be rewritten as

$$\mathbf{w} = \widetilde{\mathbf{a}}(x), \qquad x \in \Gamma, \tag{6.5}$$

where the vector $\widetilde{\mathbf{a}}$ has components $\widetilde{a}_r = a_r - F/(2\pi r)$ and $\widetilde{a}_\varphi = a_\varphi$. By (6.2) this vector satisfies the zero flux condition:

$$\int_\Gamma \widetilde{\mathbf{a}} \cdot \mathbf{n}\, dS = 0. \tag{6.6}$$

Using (6.6), we can reformulate the problem (6.4), (6.5) in terms of the stream function $\psi(r,\varphi)$, which is connected with the components of $\mathbf{w}$ by the relations

$$w_r = \frac{1}{r}\frac{\partial\psi}{\partial\varphi}, \qquad w_\varphi = -\frac{\partial\psi}{\partial r}. \tag{6.7}$$

(Passing to the stream function before separating out the singular component of the velocity field would not help, because the stream function of a point source in the planar problem is multivalued.) The stream function satisfies the equation

$$\nu\Delta^2\psi - \frac{1}{r}\frac{\partial(\Delta\psi,\psi)}{\partial(r,\varphi)} - \frac{F}{2\pi r}\frac{\partial\Delta\psi}{\partial r} = 0. \tag{6.8}$$

We can rewrite the boundary condition (6.5) in terms of the stream function as

$$\psi = a(x), \quad \frac{\partial\psi}{\partial n} = b(x), \qquad x\in\Gamma. \tag{6.9}$$

The functions a and b can be expressed in terms of the components of the vector $\widetilde{\mathbf{a}}$. If these components are in the class $W^{1/2,2}(\Gamma)$, then we have

$$a\in W^{3/2,2}(\Gamma), \qquad b\in W^{1/2,2}(\Gamma). \tag{6.10}$$

We make these assumptions in what follows.

In the problem (6.8), (6.9) we now pass to a new unknown function $\chi=\psi-f$ such that

$$f = a, \quad \frac{\partial f}{\partial n} = b, \qquad x\in\Gamma. \tag{6.11}$$

We shall specify our choice of f below. The function χ is a solution of the boundary-value problem

$$\nu\Delta^2\chi - \frac{1}{r}\frac{\partial(\Delta\chi,\chi)}{\partial(r,\varphi)} - \frac{1}{r}\frac{\partial(\Delta\chi,f)}{\partial(r,\varphi)} - \frac{1}{r}\frac{\partial(\Delta f,\chi)}{\partial(r,\varphi)} - \frac{F}{2\pi r}\frac{\partial\Delta\chi}{\partial r} = g, \quad x\in\Omega\setminus\{0\}, \tag{6.12}$$

$$\chi = 0, \quad \frac{\partial\chi}{\partial n} = 0, \qquad x\in\Gamma, \tag{6.13}$$

where

$$g = -\nu\Delta^2 f + \frac{1}{r}\frac{\partial(\Delta f,f)}{\partial(r,\varphi)} + \frac{F}{2\pi r}\frac{\partial\Delta f}{\partial r}. \tag{6.14}$$

Let $0<\gamma<\operatorname{dist}(\Gamma,\{0\})$, and denote by Ω_γ the domain bounded by the curve Γ and the circle $C_\gamma=\{x\colon r=\gamma\}$. Let $\mathring{H}^2(\Omega_\gamma;r)$ be the Hilbert space equal to the closure of the set of functions in $C_0^\infty(\Omega_\gamma)$ with respect to the norm

$$\|\eta\|^2_{\mathring{H}^2(\Omega_\gamma;r)} = \int_{\Omega_\gamma}\biggl(\eta_{rr}^2 + \frac{2}{r^2}\eta_{r\varphi}^2 + \frac{1}{r^4}\eta_{\varphi\varphi}^2 + \frac{1}{r^2}\eta_r^2\biggr) r\,dr\,d\varphi.$$

Letting γ approach zero, we obtain a Hilbert space $\mathring{V}(\Omega;r)$ with norm defined by the same equality as above, but with Ω as the domain of integration. The functions $\eta \in \mathring{V}(\Omega; r)$ are continuous in $\overline{\Omega}$ and vanish for $r = 0$, and the inequality $|\eta| \leqslant C_0 r^\alpha \|\eta\|_{\mathring{H}^2(\Omega;r)}$ holds for $(r, \varphi) \in \overline{\Omega}$ and $0 < \alpha < 1$, with a positive constant C_0 depending on Ω and the exponent α.

We call a function $\chi \in \mathring{V}(\Omega; r)$ a generalized solution of the problem (6.12), (6.13) if for any $\eta \in \mathring{V}(\Omega; r)$

$$\begin{aligned} \nu \int_\Omega \Big(\chi_{rr}\eta_{rr} + \frac{2}{r^2}\chi_{r\varphi}\eta_{r\varphi} + \frac{1}{r^4}\chi_{\varphi\varphi}\eta_{\varphi\varphi} + \frac{1}{r^2}\chi_r\eta_r \Big) r\,dr\,d\varphi \\ + \int_\Omega \Big[\frac{\Delta\chi}{r} \frac{\partial(\chi,\eta)}{\partial(r,\varphi)} + \frac{\Delta\chi}{r} \frac{\partial(f,\eta)}{\partial(r,\varphi)} + \frac{\Delta f}{r} \frac{\partial(\chi,\eta)}{\partial(r,\varphi)} + \frac{F\Delta\chi}{2\pi r}\eta_r \Big] r\,dr\,d\varphi \\ = \int_\Omega g\eta r\,dr\,d\varphi. \end{aligned} \tag{6.15}$$

The next statement is an analogue of Hopf's lemma (see Lemma 2.1).

Lemma 6.1. *Let $\Gamma \in C^\infty$ be a Jordan curve enclosing the origin. Let a and b be functions on this curve which satisfy (6.10). Then for any $\varepsilon > 0$ there exists a function $f \in W^{2,2}(\Omega)$ such that the conditions (6.11) hold and*

$$\Big| \int_\Omega \Delta\chi \frac{\partial(f,\chi)}{\partial(r,\varphi)}\,dx \Big| \leqslant \varepsilon \|\chi\|^2_{\mathring{H}^2(\Omega;r)} \qquad \forall\, \chi \in \mathring{V}(\Omega; r). \tag{6.16}$$

The proof of Lemma 6.1 was presented in [52].

Lemma 6.2. *Assume the hypotheses of Lemma 6.1. If F satisfies*

$$|F| < 2\pi\nu, \tag{6.17}$$

then each generalized solution χ of the problem (6.12), (6.13) has the estimate

$$\|\chi\|_{\mathring{H}^2(\Omega;r)} \leqslant C_* \tag{6.18}$$

with a constant $C_* = C_*(\nu, F, \Omega, a, b)$.

Proof. Let $\eta = \chi$ in the identity (6.15). Then it takes the form

$$\begin{aligned} \int_\Omega \Big[\nu \Big(\chi_{rr}^2 + \frac{2}{r^2}\chi_{r\varphi}^2 + \frac{1}{r^4}\chi_{\varphi\varphi}^2 + \frac{1}{r^2}\chi_r^2 \Big) + \frac{F}{2\pi}\Big(\frac{\chi_r^2}{r^2} - \frac{\chi_\varphi^2}{r^4} \Big) + \frac{\Delta\chi}{r} \frac{\partial(f,\chi)}{\partial(r,\varphi)} \Big] r\,dr\,d\varphi \\ = \int_\Omega g\chi r\,dr\,d\varphi. \end{aligned}$$

Choosing f given by Lemma 6.1 and taking account of the representation (6.14) of g in terms of f and its derivatives, we then have

$$\begin{aligned} \int_\Omega \Big[\nu \Big(\chi_{rr}^2 + \frac{2}{r^2}\chi_{r\varphi}^2 + \frac{1}{r^4}\chi_{\varphi\varphi}^2 + \frac{1}{r^2}\chi_r^2 \Big) - \frac{|F|}{2\pi}\Big(\frac{\chi_r^2}{r^2} + \frac{\chi_\varphi^2}{r^4} \Big) \Big] r\,dr\,d\varphi \\ \leqslant \varepsilon \|\chi\|^2_{\mathring{H}^2(\Omega;r)} + \big(\nu\|f\|_{W^{2,2}(\Omega)} + c\|f\|^2_{W^{2,2}(\Omega)} \big) \|\chi\|_{\mathring{H}^2(\Omega;r)}, \end{aligned}$$

where the constant c depends only on Ω. To finish the proof of the lemma it suffices to set

$$\varepsilon = \frac{\nu}{2} - \frac{|F|}{4\pi},$$

which yields the required a priori bound for the norm of χ:

$$\|\chi\|_{\mathring{H}^2(\Omega;r)} \leqslant 4\pi(2\pi\nu - |F|)^{-1}\big(\nu\|f\|_{W^{2,2}(\Omega)} + c\|f\|^2_{W^{2,2}(\Omega)}\big) = C_* \tag{6.19}$$

(here we bear in mind that the derivative of a periodic function has zero mean value over the period). $\square$

Theorem 6.1. *Under the conditions of Lemmas 6.1 and 6.2 the problem (6.12), (6.13) has at least one generalized solution $\chi \in \mathring{V}(\Omega;r)$, which has the estimate (6.19).*

The proof of Theorem 6.1 is based on the natural regularization of the problem (6.12), (6.13) (see [52]). We give a brief sketch of the proof. Consider the following auxiliary problem: find a solution of (6.12) in Ω_γ satisfying (6.13) and

$$\psi = 0, \quad \frac{\partial\psi}{\partial r} = 0, \qquad x \in C_\gamma. \tag{6.20}$$

In the problem (6.12), (6.13), (6.20) we pass to a new unknown function $\chi^{(\gamma)} = \psi - f^{(\gamma)}$, where $f^{(\gamma)}$ is a function such that

$$\begin{aligned} &f^{(\gamma)} = a(x), && \frac{\partial f^{(\gamma)}}{\partial n} = b(x), && x \in \Gamma,\\ &f^{(\gamma)} = 0, && \frac{\partial f^{(\gamma)}}{\partial r} = 0, && x \in C_\gamma. \end{aligned} \tag{6.21}$$

The function $\chi^{(\gamma)}$ is a solution of the equation

$$\begin{aligned} \nu\Delta^2\chi^{(\gamma)} - \frac{1}{r}\,\frac{\partial(\Delta\chi^{(\gamma)},\chi^{(\gamma)})}{\partial(r,\varphi)} - \frac{1}{r}\,\frac{\partial(\Delta\chi^{(\gamma)},f^{(\gamma)})}{\partial(r,\varphi)}&\\ - \frac{1}{r}\,\frac{\partial(\Delta f^{(\gamma)},\chi^{(\gamma)})}{\partial(r,\varphi)} - \frac{F}{2\pi r}\,\frac{\partial\Delta\chi^{(\gamma)}}{\partial r} &= g^{(\gamma)} \end{aligned} \tag{6.22}$$

and satisfies the boundary conditions

$$\begin{aligned} &\chi^{(\gamma)} = 0, && \frac{\partial\chi^{(\gamma)}}{\partial n} = 0, && x \in \Gamma,\\ &\chi^{(\gamma)} = 0, && \frac{\partial\chi^{(\gamma)}}{\partial r} = 0, && x \in C_\gamma, \end{aligned} \tag{6.23}$$

where

$$g^{(\gamma)} = -\nu\Delta^2 f^{(\gamma)} + \frac{1}{r}\,\frac{\partial(\Delta f^{(\gamma)},f^{(\gamma)})}{\partial(r,\varphi)} + \frac{F}{2\pi r}\,\frac{\partial\Delta f^{(\gamma)}}{\partial r}\,. \tag{6.24}$$

We call $\chi^{(\gamma)} \in \mathring{H}^2(\Omega_\gamma; r)$ a generalized solution of the problem (6.22), (6.23) if for any $\eta^{(\gamma)} \in \mathring{H}^2(\Omega_\gamma; r)$

$$\begin{aligned} \nu \int_{\Omega_\gamma} \biggl(\chi^{(\gamma)}_{rr} \eta^{(\gamma)}_{rr} + \frac{2}{r^2} \chi^{(\gamma)}_{r\varphi} \eta^{(\gamma)}_{r\varphi} + \frac{1}{r^4} \chi^{(\gamma)}_{\varphi\varphi} \eta^{(\gamma)}_{\varphi\varphi} + \frac{1}{r^2} \chi^{(\gamma)}_{r} \chi^{(\gamma)}_{r} \biggr) r\,dr\,d\varphi \\ + \int_{\Omega_\gamma} \biggl[\frac{\Delta\chi^{(\gamma)}}{r} \frac{\partial(\chi^{(\gamma)}, \eta^{(\gamma)})}{\partial(r,\varphi)} + \frac{\Delta\chi^{(\gamma)}}{r} \frac{\partial(f^{(\gamma)}, \eta^{(\gamma)})}{\partial(r,\varphi)} \\ + \frac{\Delta f^{(\gamma)}}{r} \frac{\partial(\chi^{(\gamma)}, \eta^{(\gamma)})}{\partial(r,\varphi)} + \frac{F\Delta\chi^{(\gamma)}}{2\pi r} \eta^{(\gamma)}_r \biggr] r\,dr\,d\varphi \\ = \int_{\Omega_\gamma} g^{(\gamma)} \eta^{(\gamma)} r\,dr\,d\varphi. \end{aligned} \tag{6.25}$$

Let us now consider the family of problems (6.22), (6.23) with $\gamma \in (0, \gamma_0]$. For each of them we can prove its solvability using well-known methods (see, for instance, [2] and [7]). The proof is based on analogues of Lemmas 6.1 and 6.2. For each $\gamma \in (0, \gamma_0]$ and for any $\varepsilon > 0$ there exists an $f^{(\gamma)} \in \mathring{H}^2(\Omega_\gamma; r)$ satisfying (6.21) such that

$$\biggl| \int_{\Omega_\gamma} \Delta\chi^{(\gamma)} \frac{\partial(f^{(\gamma)}, \chi^{(\gamma)})}{\partial(r,\varphi)}\,dr\,d\varphi \biggr| \leqslant \varepsilon \|\chi^{(\gamma)}\|^2_{\mathring{H}^2(\Omega_\gamma; r)} \tag{6.26}$$

for any $\chi^{(\gamma)} \in \mathring{H}^2(\Omega_\gamma; r)$. The proof of (6.26) repeats the proof of Lemma 6.1 almost word for word. It is important to see that we can take the same ε in (6.26) for all $\gamma \in (0, \gamma_0]$.

Next we prove an assertion similar to Lemma 6.2. Assume the hypotheses of Lemma 6.1 and let F satisfy (6.17). Then each generalized solution $\chi^{(\gamma)}$ of (6.22), (6.23) has a bound

$$\|\chi^{(\gamma)}\|_{\mathring{H}^2(\Omega_\gamma; r)} \leqslant C \tag{6.27}$$

with a constant C independent of γ for $0 < \gamma < \gamma_0$. Now it remains to pass to the limit as $\gamma \to 0$ in the problem (6.22), (6.20). To do this we extend $\chi^{(\gamma)}$ and $f^{(\gamma)}$ to $\Omega \setminus \Omega_\gamma$ by zero, keeping the same notation for them. The extended functions are defined in the whole of Ω and have finite norms in $\mathring{H}^2(\Omega; r)$. By (6.27), the norms of the functions $\chi^{(\gamma)}$ (for $0 < \gamma \leqslant \gamma_0$) are bounded in this space. The family of functions $\chi^{(\gamma)}$, $\gamma \in (0, \gamma_0]$, has a weak limit χ as $\gamma \to 0$. The limit function belongs to $\mathring{V}(\Omega; r)$. The set of functions $\chi^{(\gamma)}$, $\gamma \in [0, \gamma_0]$, is weakly compact in $\mathring{H}^2(\Omega; r)$ and compact in $W^{1,4}(\Omega)$, so we can carry out a passage to the limit in (6.25) as $\gamma \to 0$. The limit function χ satisfies (6.15), which completes the proof of Theorem 6.1.

6.2. Axially symmetric flows. Now let us consider the axially symmetric problem in a bounded domain $Q \subset \mathbb{R}^3$ whose boundary is a surface of revolution $S \in C^\infty$. We denote the meridional section of Q by Ω. The boundary of Ω is formed by an arc Γ and a line segment $\Lambda = \{r, z \colon r = 0,\ z_1 < z < z_2\}$ which contains sources or sinks distributed with constant linear density F (see Fig. 6). We must find a solution $\mathbf{u} = (u_r, u_z)$, p of the system (2.51) in Ω which satisfies

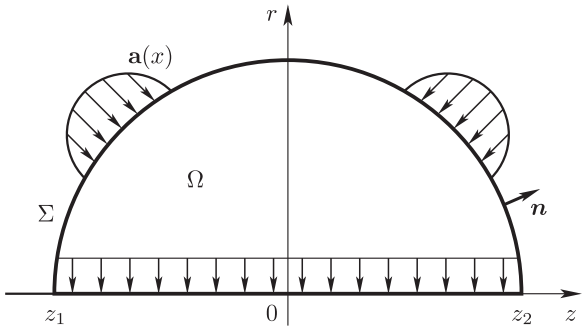

Figure 6. Meridional section of the domain Q with sinks on the axis of symmetry

the conditions

$$u_r = \frac{F}{2\pi r} + a_r, \quad u_z = a_z, \qquad (r, z) \in \Gamma, \tag{6.28}$$

$$u_r = \frac{F}{2\pi r}[1 + o(1)], \quad u_z = o(1), \qquad r \to 0, \quad z \in (z_1, z_2), \tag{6.29}$$

$$\int_\Gamma \mathbf{a} \cdot \mathbf{n}\, dS = 0. \tag{6.30}$$

Assume that the vector field $\mathbf{a}$ has a divergence-free extension $\mathbf{b}$ to Ω with finite Dirichlet integral. We introduce new unknown functions

$$w_r = u_r - \frac{q}{2\pi r} - b_r, \qquad w_z = u_z - b_z, \qquad \overline{p} = p + \frac{q^2}{8\pi^2 r^2}\,. \tag{6.31}$$

The vector $\mathbf{w} = (w_r, w_z)$ and the function $\overline{p}$ satisfy in Ω the system of equations

$$\begin{aligned}
&w_r \frac{\partial w_r}{\partial r} + w_z \frac{\partial w_r}{\partial z} + \frac{F}{2\pi r}\frac{\partial w_r}{\partial r} - \frac{F}{2\pi r^2} w_r + b_r \frac{\partial w_r}{\partial r} + b_z \frac{\partial w_r}{\partial z} + w_r \frac{\partial b_r}{\partial r} + b_z \frac{\partial w_r}{\partial z}\\
&\qquad = -\frac{\partial \overline{p}}{\partial r} + \nu\left(\frac{\partial^2 w_r}{\partial r^2} + \frac{1}{r}\frac{\partial w_r}{\partial r} + \frac{\partial^2 w_r}{\partial z^2} - \frac{w_r}{r^2}\right) + g_r,\\
&u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} + \frac{F}{2\pi r}\frac{\partial u_z}{\partial r} + b_r \frac{\partial u_z}{\partial r} + b_z \frac{\partial u_z}{\partial z} + u_r \frac{\partial b_z}{\partial r} + b_z \frac{\partial u_z}{\partial z}\\
&\qquad\quad - \frac{\partial \overline{p}}{\partial z} + \nu\left(\frac{\partial^2 w_r}{\partial r^2} + \frac{1}{r}\frac{\partial w_r}{\partial r} + \frac{\partial^2 w_r}{\partial z^2}\right) + g_z,\\
&\frac{1}{r}\frac{\partial (r w_r)}{\partial r} + \frac{\partial w_z}{\partial z} = 0
\end{aligned} \tag{6.32}$$

and the boundary condition

$$\mathbf{w} = 0, \qquad (r, z) \in \Gamma, \tag{6.33}$$

where g_r and g_z are the components of the vector

$$\mathbf{g} = \nu\Delta\mathbf{b} - \mathbf{b}\cdot\nabla\mathbf{b} - \frac{F}{2\pi}\nabla\left(\frac{\mathbf{b}}{r}\right).$$

Consider now the function space $H(\Omega;r)$ equal to the closure of the set of divergence-free axially symmetric vector-valued functions $\mathbf{v}\in C_0^\infty(\Omega)$ with respect to the norm defined by the Dirichlet integral

$$\|\mathbf{v}\|^2_{H(\Omega,r)}=\int_\Omega\biggl[\biggl(\frac{\partial v_r}{\partial r}\biggr)^2+\biggl(\frac{\partial v_r}{\partial z}\biggr)^2+\biggl(\frac{\partial v_z}{\partial r}\biggr)^2+\biggl(\frac{\partial v_z}{\partial z}\biggr)^2+\frac{v_r^2}{r^2}\biggr]r\,dr\,dz.$$

We call a vector-valued function $\mathbf{w}\in H(\Omega;r)$ a generalized solution of the problem (6.32), (6.33) if the integral identity

$$\int_\Omega\biggl[\nu\biggl(\nabla\mathbf{w}\cdot\nabla\boldsymbol{\eta}+\frac{u_rh_r}{r^2}\biggr)-\mathbf{w}\cdot(\mathbf{w}\cdot\nabla)\boldsymbol{\eta}-\mathbf{w}\cdot(\mathbf{b}\cdot\nabla)\boldsymbol{\eta}-\mathbf{b}\cdot(\mathbf{w}\cdot\nabla)\boldsymbol{\eta}$$
$$+\frac{F}{2\pi r}\biggl(\frac{\partial\mathbf{u}}{\partial r}\cdot\boldsymbol{\eta}-\frac{u_rh_r}{r}\biggr)\biggr]r\,dr\,dz=\int_\Omega\mathbf{g}\cdot\boldsymbol{\eta}r\,dr\,dz$$

holds for any $\boldsymbol{\eta}\in H(\Omega;r)$. Setting $\boldsymbol{\eta}=\mathbf{u}$ here, we get that

$$\int_\Omega\biggl[\nu\biggl(\nabla\mathbf{w}\cdot\nabla\mathbf{w}+\frac{u_r^2}{r^2}\biggr)-\frac{Fu_r^2}{2\pi r^2}-\mathbf{b}\cdot(\mathbf{w}\cdot\nabla)\mathbf{w}\biggr]r\,dr\,dz=\int_\Omega\mathbf{g}\cdot\mathbf{w}r\,dr\,dz. \quad (6.34)$$

We now observe that for any $\varepsilon>0$ we can construct an extension $\mathbf{b}$ of an arbitrary vector field $\mathbf{a}$ to Ω such that

$$\biggl|\int_\Omega\mathbf{b}\cdot(\mathbf{w}\cdot\nabla)\mathbf{w}r\,dr\,dz\biggr|\leqslant\varepsilon\|\mathbf{w}\|^2_{H(\Omega;r)}\qquad\forall\,\mathbf{w}\in H(\Omega;r). \quad (6.35)$$

Such an extension was constructed in [52] in terms of the stream function of an axially symmetric flow. Taking $\varepsilon=(4\pi)^{-1}(2\pi\nu-F)$ and assuming that

$$F<2\pi\nu, \quad (6.36)$$

we obtain from (6.34) and (6.35) the a priori estimate

$$\|\mathbf{w}\|_{H(\Omega;r)}\leqslant C_1\bigl(\nu\|\mathbf{b}\|_{H(\Omega;r)}+C_2\|\mathbf{b}\|^2_{H(\Omega;r)}\bigr), \quad (6.37)$$

where C_2 depends only on Ω and where we can take C_1 equal to $4\pi(2\pi\nu-F)^{-1}$ for $0\leqslant F<2\pi\nu$ and to $2/\nu$ for $F\leqslant 0$. The estimate (6.37) is crucial for the proof of the next theorem [52].

Theorem 6.2. *Let* $\mathbf{b}$ *be a divergence-free extension, with finite Dirichlet integral, of the vector field* $\mathbf{a}$ *in* (6.28) *to the domain* Ω. *If* F *satisfies* (6.36), *then the problem* (6.32), (6.33) *has a generalized solution* $\mathbf{w}\in H(\Omega;r)$, *and the norm of* $\mathbf{w}$ *has the estimate* (6.37).

Remarkably, we impose here a one-sided restriction on the parameter F: the distribution of sinks can have an arbitrary linear density. On the other hand, the solvability of the planar problem with a source or sink has been proved only when the absolute value of F satisfies (6.17) [51], [52]. By contrast, there are no restrictions on the norm of the function $\mathbf{a}$ in both problems.

7. Conclusion

The methods presented above cannot be generalized to three-dimensional problems. The Leray problem has been solved only in axially symmetric domains and remains open for an arbitrary three-dimensional domain with multiple boundary components. We believe that, as a first step in this direction, one can look at the flux problem with additional symmetry relative to a plane or relative to two mutually orthogonal planes (without the condition of axial symmetry). The main difficulties arising in the analysis of such a problem can be understood in the case when the flow region is a spherical layer. The idea of considering symmetric solutions of the flux problem (which is due to Amick [17], who realized it for planar flows) makes it possible to reduce the flux problem to a problem in a domain with simply connected boundary by 'cutting' the original domain along the line of symmetry. The following conjecture appears plausible. Assume that the flow has two planes of symmetry, which intersect all the boundary components of the flow region. Then (under the natural assumptions of smoothness for the input data) the flux problem has at least one solution.

The planar symmetric exterior problem (1.1)–(1.3) and the three-dimensional axially symmetric exterior problem were investigated in [21] and [24]. In both cases the solvability of the problem was established without restrictions on the fluxes of the boundary velocity field. However, the Leray problem in an arbitrary exterior planar or three-dimensional domain is still open. It seems that essentially new ideas are needed for its solution.

In [54], [13], [55] the stationary problem (1.1)–(1.3) was considered in non-compact domains Ω with multiple boundary components. In these papers the solvability of (1.1)–(1.3) was established under the assumption that the fluxes of the boundary vector field across the inner boundary components are small, while the fluxes of the field across the outer boundary components can be arbitrary. In [54] and [13] solutions with a finite Dirichlet integral were found, and in [55] the Dirichlet integral could be either finite or infinite, depending on the geometry of the outlets of Ω at infinity. It could be interesting to extend these results to the case of arbitrary fluxes of the boundary field across the inner boundary components.

Another interesting problem arises in the study of boundary layers adjoining parts with intensive inflow of a fluid across a permeable boundary of the flow region. The simplest version of the statement of such a problem is as follows. Consider a planar flow in a curvilinear annulus Ω with outer boundary Γ_0 and inner boundary Γ_1. Assume that the normal components of the velocity do not vanish on Γ_0 and Γ_1, and let F_0 and F_1 denote the corresponding velocity fluxes (clearly, $F_1 = -F_0$). Then we can construct an asymptotic solution of the flux problem for $Re = |F_1|/\nu \to \infty$ by using a modification of the classical Vishik–Lyusternik method [56]. By contrast to boundary layer theory in the flow problem [34], here the boundary layer has thickness of order Re^{-1} for large values of Re. The boundary layer is localized in a neighbourhood of Γ_1 for $F_1 > 0$ or in a neighbourhood of Γ_0 for $F_1 < 0$. In general, it is a highly non-trivial problem to justify taking the formal asymptotics to be the leading term as $Re \to \infty$ of the solution of the flux problem. However, there are some prospects for such a justification in the symmetric planar problem, where effective a priori estimates are known [14]. A natural approach to the justification will be to linearize the problem on an approximate solution and then to use Kantorovich's theorem on the convergence of Newton's method. Unfortunately, at this point we do not have enough information about the spectral properties of the linearized non-selfadjoint operator in the flux problem for large values of Re, because the asymptotic solution depends only implicitly on the parameter Re.

A methodical investigation of boundary-value problems for the Navier–Stokes equations with singularities in the flow region was begun less than a decade ago, although examples of exact (for instance, self-similar) singular solutions of these equations were well known [34]. In the planar problem with source or sink which was considered in [52] and [53], the singularity is characterized by the dimensionless parameter $|F|/\nu = Re$, which can be called the local Reynolds number. The methods proposed for proving solvability of the planar problem do not allow one to treat the case of large Reynolds numbers. The general case apparently requires completely new ideas, but we hope that under the additional assumption of a line of symmetry of the flow this can be proved using methods already available.

Results in [52] and [53] can be extended in several directions. Let us again consider planar flows with singularities, but now not in the interior but on the boundary Γ of the flow region. Specifically, there are two points A^+ and A^- on Γ where a source and a sink with intensities F and $-F$, respectively, are located. On the rest of the boundary we impose the no-slip conditions. Is this problem for the system (1.1), (1.2) solvable for small values of $|F|/\nu = Re$? As a first step, we can consider the case when the parts of the curve Σ near A^+ and A^- are straight line segments intersecting at an angle β. Then the singular part of the solution is described by the solution of the problem of a planar diffuser flow [34] localized near the corner points. This solution is unique for small Re and $\beta < \beta^* \approx 2.25$ [57]. After separating out this singularity of the velocity field, we arrive at a problem of the same type as one in § 6, though complicated by the presence of boundary corners. In view of Kondrat'ev's results in [58] there are hopes that the Dirichlet integral of the regular component of the velocity field is finite when the angle β and the Reynolds number Re are sufficiently small.

It could also be interesting to discuss planar flows with singularities on the boundary of the domain in the case when the source or the sink is located at a corner with zero opening angle.

Problems with singularities of higher orders than the ones in [52] and [53] seem significantly more complicated. One example is the axially symmetric problem with a point source or sink in the flow region. Even the appropriate function space is not so obvious here. This is because after linearizing the equations (1.1), (1.2) on the flow produced by the source (or sink) there occur terms containing first-order derivatives which are not subordinate to the Stokes operator in Sobolev or Kondratiev spaces. Furthermore, in the three-dimensional case the power F of the point source does not have the same dimension as the viscosity, and introducing a local Reynolds number is not productive.

As mentioned in the Introduction, complications in the analysis of solutions of the Navier–Stokes equations in domains with boundary consisting of several components are characteristic for stationary problems: no such complications occur in initial-boundary value problems. However, there also exist non-stationary problems without initial conditions in which the presence of several boundary components essentially complicates the analysis. Such are problems with time-periodic values of the velocity vector field prescribed on the boundary of the flow region. In this case one would expect a time-periodic solution to exist. An existence theorem for periodic solutions of the Navier–Stokes equations with homogeneous boundary conditions in the case of periodic external forces was proved by Yudovich in [59]. In a recent paper [60] Kobayashi studied the flux problem in a symmetric plane domain with symmetric and time-periodic velocity vector prescribed on the boundary of the domain. He proved that a time-periodic solution exists for any values of the partial fluxes. It would be interesting to look at this problem in an arbitrary bounded plane domain, in a bounded axially symmetric three-dimensional domain, and in various classes of unbounded domains.

It also seems reasonable to consider the problem with a pulsating source or sink in a plane domain (an analogue of the problem treated in § 6.1). If the power of the source (or sink) depends periodically on the time and the corresponding Reynolds number is not very large, then we can expect such a problem to be solvable in the class of periodic solutions.

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Received 20/AUG/14
Translated by N. KRUZHILIN