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COMPENSATED COMPACTNESS AND HARDY SPACES

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ABSTRACT. — We prove that various nonlinear quantities (like the jacobian, "div-curl"...) identified by the compensated compactness theory belong, under natural conditions, to multidimensional Hardy spaces. We also explain how this regularity is related to various known facts from Harmonic Analysis (commutators with BMO multipliers, multi-linear analysis) and to weak convergence questions. Finally, we indicate a few applications of this fact.

Key-words. – Compensated compactness. Hardy spaces, weak convergence, bilinear forms, quadratic nonlinear terms, cancellations, rank condition, maximal functions.

RÉSUMÉ. — Nous montrons que diverses quantités non linéaires (comme le jacobien, le terme "div-rot"...) identifiées par la théorie de la compacité par compensation appartiennent, sous des conditions naturelles, aux espaces de Hardy multidimensionnels. Nous expliquons aussi comment cette régularité est reliée à divers faits connus en Analyse Harmonique (commutateurs avec des multiplicateurs dans BMO, analyse multilinéaire) et à des problèmes de convergence faible. Enfin, nous indiquons quelques applications de ce fait.

I. Introduction

This paper is devoted to the illustration of intrinsic links between the theory of compensated compactness initiated and developed by L. Tartar ([44], [45]) and F. Murat ([34], [35], [36]) – related results and (or) phenomena are to be found in J. Ball [4], Reshetnyak ([38], [39]) – and classical tools of Harmonic and Real Analysis such as Hardy spaces, commutators and operators estimates...

Before providing a more detailed background to these links, let us immediately present one example: let $u \in W^{1,N}(\mathbb{R}^N)^N$ (*i.e.* the usual Sobolev space of L^N functions having first derivatives in the sense of distributions in L^N) then its jacobian $J(u) = \det(\nabla u)$ belongs to the multidimensional Hardy space that we denote by $\mathscr{H}^1(\mathbb{R}^N)$. This space (introduced by E. Stein and G. Weiss [43]) can be characterized as follows (*see* C. Fefferman and E. Stein [22], R. Coifman and G. Weiss [14]...):

$$\mathscr{H}^1(\mathbb{R}^N) = \{ f \in L^1(\mathbb{R}^N) / \sup_{t \geq 0} |h_t * f| \in L^1(\mathbb{R}^N) \} \tag{1}$$

where $h_t = 1/t^N h(./t)$, $h \in C_0^\infty(\mathbb{R}^N)$, $h \geq 0$, Supp $h \in B(0, 1)$ (for instance).

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Notice of course that $J(u)$ belongs trivially to $L^1(\mathbb{R}^N)$. Thus, the "specific algebraic" structure of $J(u)$ allows to find a proper subspace of L^1 namely $\mathcal{H}^1$ which contains the range of the mapping $(u \mapsto J(u))$ from $W^{1,N}(\mathbb{R}^N)^N$ into $L^1(\mathbb{R}^N)$ – let us mention, by the way, that the exact determination of the range is an outstanding open problem on which we shall come back later on. We shall prove however that $\mathcal{H}^1$ is the minimal linear vector space containing this range.

In some sense, the above result (a typical example of one type of results shown in this paper) indicates an improvement of the trivial L^1 regularity. This improvement may be best appreciated when recalling Stein's lemma (*cf.* E. Stein [41]) about the structure of L^1_{loc} nonnegative functions f in $\mathcal{H}^1_{loc}$: $f \in \mathcal{H}^1_{loc}$ if and only if $f \log f \in L^1_{loc}$. Therefore, from the local variant of the above result, we recover the remarkable result of S. Müller [32]: let $u \in (W^{1,N}_{loc})^N$, assume that $J(u) \geqq 0$ then $J(u) \log J(u) \in L^1_{loc}$. In fact, this result was the main motivation for our work.

After presenting this typical example of our results and before explaining the general organization of the paper, we would like to make a few general comments. One way to look at the compensated compactness theory (see the aforementioned references) is to consider it as one consequence of the study of oscillations in nonlinear partial differential equations (arising from Continuum Mechanics, Physics or Differential Geometry...). It is far beyond the scope of this paper to discuss the reasons for such a study but we would like to mention at least that it is natural for the issue of the existence of global (generalized) solutions for many nonlinear systems of interest. In particular, one of the striking applications of the compensated-compactness theory has been the new developments on hyperbolic systems of conservation laws due to L. Tartar [44] and R. J. DiPerna ([18], [19]).

It is quite obvious that, in such a study, a fundamental role should be played by weakly continuous nonlinear quantities (or, to be more specific by nonlinear quantities which are sequentially continuous for sequences of functions having "certain natural" bounds). The compensated compactness theory has identified classes of such nonlinear quantities (as well as some general tools to determine them in a systematic fashion). The terminology stems from the fact that *compensations* arise in those nonlinear quantities such as $J(u) = \det(\nabla u)$, compensations which in turn allow the weak continuity (or the compactness).

This work has several ambitions and goals: one is to shed a new light on these phenomena, the other being to present a few extensions and applications made possible by our viewpoint.

Indeed, we shall show, without taking sequences, that these nonlinear quantities have an improved regularity (typically, $\mathcal{H}^1$ instead of L^1), which can be seen as a direct consequence of a *cancellation* property. This improved regularity has many applications to regularity results (slightly improved regularity for Leray solutions of three-dimensional Navier-Stokes equations for instance), to the embedding of these non-linear quantities into $\mathcal{H}^p$ for $p < 1$ [for instance, $J(u) \in \mathcal{H}^p$ if $u \in (W^{1,Np})^N$ and $p > N/(N+1)$] and of course to a "stronger weak convergence" when we deal with sequences.

We shall also try to show that the cancellation property is the fundamental one by showing how the "bilinear" machinery developed in R. Coifman and Y. Meyer [11] yields the equivalence for "natural" bi-linear operators between weak continuity, the embedding of the range into $\mathscr{H}^1$ and the cancellation property.

We are now ready to explain the organization of our paper. First, in Section II, our main results are presented on three examples (the above one, the div-curl example and the square of divergence free vector-fields). We show, in these three examples, why the corresponding non-linear quantities lie in $\mathscr{H}^1$ by rather elementary arguments (using only Sobolev-Poincaré inequalities and the classical maximal theorem). And we will deduce from the proofs some results below L^1 ($\mathscr{H}^p$ for $p<1$). In section III, we present another approach to these results and show the relationship with classical results on commutators. We also indicate various variants and other examples.

Then, in section IV, we apply the $\mathscr{H}^1$ regularity to the convergence issues. In particular, we recall a recent result by P. Jones and J. L. Journé [29] showing the consistency between weak $*\,\mathscr{H}^1$ and a.e. convergences and we modify its proof to yield a similar result for the so-called "biting lemma" convergence due to J. K. Brooks and R. V. Chacon [6] (*see* also J. M. Ball and F. Murat [5], E. J. Balder [3]...).

Next, in section V, we develop the equivalences mentioned above for bilinear operators that commute with translations and dilations. Then, in section VI, we consider general quadratic expressions and raise the question of their $\mathscr{H}^1$ regularity when certain linear partial differential bounds are available.

The section VII deals with examples of situations where more cancellations are available (higher moments of the nonlinear quantities vanish), in which case we can lower the value of p and still obtain some $\mathscr{H}^p$ regularity. A typical example is given by the jacobian when $u=\nabla\Phi$ and $\Phi\in W^{2,Nq}(\mathbb{R}^N)$ with $q>N/(N+2)$: then $J(u)\in\mathscr{H}^q(\mathbb{R}^N)$.

Next, in section VIII, we consider again situations "below L^1" where the "compensated compactness" nonlinear quantities can be defined in the sense of distributions (and are shown to belong to some Hardy space $\mathscr{H}^p$ for some range of p). On the other hand, these expressions can also be defined pointwise. And we explain in this section the relationships between the distributional definition and the pointwise notion, recovering, in particular, another recent result of S. Müller [29].

Finally, section IX is devoted to a few applications of our results like, for instance, slights improvements of the known regularity for Leray solutions of three dimensional Navier-Stokes equations. We also mention an example of the recent and remarkable regularity result of F. Hélein ([27], [28]) for two-dimensional weak harmonic maps (*i.e.* weak solutions of the harmonic maps equation) which uses the $\mathscr{H}^1$ regularity of certain nonlinear expressions: in the case when the target manifold is a sphere – *see* F. Hélein [27] – the proof is a very simple and direct application of our $\mathscr{H}^1$ regularity.

Let us conclude this Introduction by mentioning that some of the results presented here were announced in [10].

II. Three basic examples

We first explain the setting of three examples even if we shall see that, after some simple algebraic manipulations, they can all be deduced from one of them namely the so-called "div-curl" expression.

We begin with the example of the Jacobian already mentioned in the Introduction: let u satisfy

$$u \in L^q_{loc}(\mathbb{R}^N)^N \quad \text{for all } q<\infty, \qquad \nabla u \in L^N(\mathbb{R}^N)^{N\times N}. \tag{2}$$

We consider the Jacobian $J(u)=\det(\nabla u)$ which clearly belongs to $L^1(\mathbb{R}^N)$. The second example deals with vector fields E, B on $\mathbb{R}^N$ satisfying

$$E \in L^p(\mathbb{R}^N)^N, \quad B \in L^{p'}(\mathbb{R}^N)^N \qquad \text{with} \quad 1<p<\infty, \quad \frac{1}{p}+\frac{1}{p'}=1 \tag{3}$$

and

$$\operatorname{div} E=0, \qquad \operatorname{curl} B=0 \quad \text{in } \mathscr{D}'(\mathbb{R}^N). \tag{4}$$

Then, we form the scalar product E.B which again clearly belongs to $L^1(\mathbb{R}^N)$. Finally, we consider a scalar function u and a vector field v on $\mathbb{R}^N$ for $N \geqq 2$ satisfying

$$\begin{cases} \nabla u \in L^2(\mathbb{R}^N)^N, \quad u \in L^{2N/(N-2)}(\mathbb{R}^N) \quad \text{if } N \geqq 3, \\ u \in L^q_{loc}(\mathbb{R}^N) \quad \text{for all } q<\infty \text{ if } N=2, \end{cases} \tag{5}$$

$$\begin{cases} \nabla v \in L^2(\mathbb{R}^N)^{N\times N}, \qquad \operatorname{div} v=0, \\ v \in L^{2N/(N-2)}(\mathbb{R}^N)^N \qquad \text{if} \quad N \geqq 3, \\ v \in L^q_{loc}(\mathbb{R}^N)^N \quad \text{for all } q<\infty \text{ if } N=2 \end{cases} \tag{6}$$

and we form $\nabla u.(\partial v/\partial x_i)$ for some fixed $i \in \{1, \ldots, N\}$.

The first two quantities above are standard and model examples in the compensated compactness theory while the third one, when we choose $u=v_j$ for each $j \in \{1, \ldots, N\}$ is a rewriting of the quantities identified by L. Mascarenhas [30].

Our main result is the

THEOREM II.1. — 1) *Let u satisfy* (2), *then* $J(u) \in \mathscr{H}^1(\mathbb{R}^N)$.

2) *Let* E, B *satisfy* (3)-(4), *then* $E.B \in \mathscr{H}^1(\mathbb{R}^N)$.

3) *Let u, v satisfy* (5)-(6), *then* $\nabla u.\partial v/\partial x_i \in \mathscr{H}^1(\mathbb{R}^N)$.

*Remark II.*1. — As we shall see below, many variants and extensions are possible. The one that we wish to mention at this stage is the possibility of localizing the above result. Indeed, it remains true if we replace all the global ($\mathbb{R}^N$) functions spaces in the above assumptions and results by their local versions.

Remark II.2. — Of course – it can be in fact deduced from functional analysis arguments – the above result not only provides an embedding but also a priori estimates in $\mathscr{H}^1(\mathbb{R}^N)$ in terms of $\|\nabla u\|_{L^N}^N$ [in the case of 1)], $\|E\|_{L^p}\|B\|_{L^{p'}}$ [in the case of 2)] and $\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}$ [in the case of 3)].

As we will see in this paper, several proofs of Theorem II.1 are possible. The one we are going to present now is both simple and elementary. But, before we go into more details, we explain how the cases 1) and 3) are in fact included in case 2). Indeed, in the case 3) of Theorem II.1, we observe that $\nabla u \in L^2(\mathbb{R}^N)^N$, $\partial v/\partial x_i \in L^2(\mathbb{R}^N)^N$ while

$$\operatorname{curl}(\nabla u)=0 \quad \text{and} \quad \operatorname{div}\left(\frac{\partial v}{\partial x_i}\right)=\frac{\partial}{\partial x_i}(\operatorname{div} v)=0 \quad \text{in } \mathscr{D}'(\mathbb{R}^N).$$

We thus only have to set $E=\partial v/\partial x_i$, $B=\nabla u$ to prove our claim (notice that $p=2$ when reducing 2) to 3)). The reason why 1) is also a reduction of 2) is quite classical in the theory of compensated compactness: indeed, we may write

$$J(u)=\det(\nabla u)=\nabla u^1.\sigma$$

with

$$\operatorname{div}\sigma=0 \quad \text{in } \mathscr{D}'(\mathbb{R}^N), \qquad |\sigma| \leqq \prod_{j=2}^{N} |\nabla u^j| \text{ a.e.} \tag{7}$$

Again, we are back in the situation of 2) with

$$B=\nabla u^1 \in L^N(\mathbb{R}^N)^N \quad \text{and} \quad E=\sigma \in L^{N/(N-1)}(\mathbb{R}^N)^N$$

in view of (7).

Therefore, we only have to prove the second assertion of Theorem II.1. The proof immediately follows from the following.

LEMMA II.1. – *Let* E, B *satisfy* (3)-(4). *For all* α, β *satisfying*

$$\frac{1}{\alpha}+\frac{1}{\beta}=1+\frac{1}{N}, \qquad 1\leqq\alpha\leqq p, \quad 1<\beta\leqq p', \tag{8}$$

there exists a constant C (*depending only on* h, α, β) *such that*

$$|\{h_t * (E.B)\}(x)| \leqq C\left(\fint_{B_t^x}|E|^\alpha\right)^{1/\alpha}\left(\fint_{B_t^x}|B|^\beta\right)^{1/\beta}$$
$$\text{for all } x\in\mathbb{R}^N,\ t>0.$$

Here and everywhere below, $B_t^x=B(x,t)=B_t(x)$ are various notations for the open ball centered at x, of radius t and $\fint_E$ denotes $1/\operatorname{meas}(E)\int_E$.

Admitting temporarily Lemma II.1. we conclude the proof of Theorem II.1: since $1<p<\infty$ and $(1/p)+(1/p')=1$, one can find α, β satisfying (8) and also $\alpha<p$, $\beta<p'$

$$\sup_{t>0}\left\{\left(\fint_{B_t^x}|E|^\alpha\right)^{1/\alpha}\left(\fint_{B_t^x}|B|^\beta\right)^{1/\beta}\right\}\leqq\left(\sup_{t>0}\fint_{B_t^x}|E|^\alpha\right)^{1/\alpha}\left(\sup_{t>0}\fint_{B_t^x}|B|^\beta\right)^{1/\beta},$$

we deduce from the maximal theorem that $\sup_{t>0} |h_t * (E.B)| \in L^1(\mathbb{R}^N)$ and more precisely that

$$\sup_{t>0} |h_t * (E.B)| \leqq CM(|E|^\alpha)^{1/\alpha} M(|B|^\beta)^{1/\beta} \tag{10}$$

where

$$M(f) = \sup_{t>0} \fint_{B_t^x} |f| \quad \text{and} \quad \fint_B |f|\,dx = \frac{1}{|B|}\int_B |f(u)|\,du.$$

We now turn to the

Proof of Lemma II.1. – We first introduce a scalar function π such that $\nabla\pi = B$, where $\pi \in L^{p^*}(\mathbb{R}^N)$ if $p' < N$, $\pi \in L^q_{\text{loc}}(\mathbb{R}^N)$ for all $q < \infty$ if $p' \geqq N$. Here and everywhere below, we denote by p^* the Sobolev exponent $p^* = Np'/(N-p')$. We next observe that

$$E.B = \operatorname{div}(E\pi) \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{11}$$

This identity, which is trivial formally since div $E = 0$, is easily justified when $E \in L^p$, $B \in L^{p'}$ by a straightforward mollification argument. Therefore, we may write for each fixed $x \in \mathbb{R}^N$, $t > 0$

$$\begin{aligned} h_t * (E.B)(x) &= \int \nabla h\left(\frac{x-y}{t}\right)\frac{1}{t^{N+1}}.E(y)\pi(y)\,dy \\ &= \int \nabla h\left(\frac{x-y}{t}\right)\frac{1}{t^{N+1}}.E(y)\left\{\pi - \fint_{B_t^x}\pi\right\}dy. \end{aligned}$$

Next, we use Hölder's inequality to deduce

$$|h_t * (E.B)| \leqq C\left(\fint_{B_t^x}|E|^\beta\right)^{1/\beta}\left(\fint_{B_t^x}\left\{|\pi - \fint_{B_t^x}\pi|\,t^{-1}\right\}^{\beta'}\right)^{1/\beta'}.$$

Then, we use the Sobolev-Poincaré inequality to bound

$$\left\{\fint_{B_t^x}\left\{|\pi - \fint_{B_t^x}\pi|\,t^{-1}\right\}^{\beta'}\right\}^{1/\beta'} \leqq C\left(\fint_{B_t^x}|\nabla\pi|^\alpha\right)^{1/\alpha}$$

since

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{1}{N} = 1 - \frac{1}{\beta} = \frac{1}{\beta'},$$

inequality which completes the proof of Lemma II.1.

Remark II.3. – Observe that part 1) of Theorem II.1 still holds if we assume $\nabla u^i \in L^{p_i}(\mathbb{R}^N)$ where $1 < p_i < \infty$ and $\sum_{i=1}^N 1/p_i = 1/N\ (N \geqq 2)$.

Remark II.4. – If we use (10) in the "reductions from 1) or 3) to 2)" we deduce in the case of 1) by choosing $\alpha=(N-1)\beta$

$$\sup_{t>0} |h_t * \det(\nabla u)| \leqq M(|\nabla u|^\alpha)^{N/\alpha} \tag{12}$$

where $\alpha = N^2/(N+1)$.

And in the case of 3) we choose $\alpha=\beta$ (by symmetry) and thus $\alpha=2N/(N+1)$. And we find

$$\sup_{t>0} \left| h_t * \left(\nabla u . \frac{\partial v}{\partial x_i}\right)\right| \leqq M(|\nabla u|^\alpha)^{1/\alpha} M\left(\left|\frac{\partial v}{\partial x_i}\right|^\alpha\right)^{1/\alpha}. \quad \square \tag{13}$$

At this stage, we want to recall one definition of $\mathcal{H}^p(\mathbb{R}^N)$ for $0<p<1$ namely

$$\mathcal{H}^p(\mathbb{R}^N) = \{ f \in \mathcal{S}'(\mathbb{R}^N) / \sup_{t>0} |h_t * f| \in L^p(\mathbb{R}^N) \}.$$

And we deduce immediately from (10), (12) or (13) a sharper result than Theorem II.1. We first detail the conditions we need in order to state it concisely:

$$\begin{cases} \nabla u \in L^p(\mathbb{R}^N)^N \quad \text{for some } N \geqq p > \dfrac{N^2}{N+1}, \\ u \in L^{p^*}(\mathbb{R}^N) \quad \text{if } p<N, \qquad u \in L^q_{\text{loc}}(\mathbb{R}^N) \quad \text{for all } q<\infty \text{ if } q=N, \end{cases} \tag{14}$$

$$\begin{cases} E \in L^p(\mathbb{R}^N)^N, \qquad B \in L^q(\mathbb{R}^N)^N, \\ \text{with} \\ 1<p<\infty, \qquad 1<q<\infty, \qquad \dfrac{1}{p}+\dfrac{1}{q}<1+\dfrac{1}{N}, \\ \operatorname{curl} E=0, \ \operatorname{div} B=0 \text{ in } \mathcal{D}'(\mathbb{R}^N), \end{cases} \tag{15}$$

$$\begin{cases} \nabla u \in L^p(\mathbb{R}^N)^N, \qquad u \in L^{p^*}(\mathbb{R}^N) \qquad \text{if} \quad p<N, \\ u \in L^r(\mathbb{R}^N) \quad \text{for all } r \text{ if } p \geqq N, \\ \nabla v \in L^q(\mathbb{R}^N)^{N\times N}, \qquad v \in L^{p^*}(\mathbb{R}^N)^N \quad \text{if } q<N, \\ v \in L^r(\mathbb{R}^N)^N \quad \text{for all } r \qquad \text{if} \quad q \geqq N, \qquad \operatorname{div} v=0, \\ 1<p<\infty, \qquad 1<q<\infty, \qquad \dfrac{1}{p}+\dfrac{1}{q}<1+\dfrac{1}{N}. \end{cases} \tag{16}$$

THEOREM II.2. – 1) *Let* u *satisfy* (14), *then* $J(u) \in \mathcal{H}^{p/N}(\mathbb{R}^N)$.

2) *Let* E, B *satisfy* (15), *then* $E.B \in \mathcal{H}^r(\mathbb{R}^N)$ *with* $1/r=(1/p)+(1/q)$.

3) *Let* u, v *satisfy* (16), *then* $\nabla u.(\partial v/\partial x_i) \in \mathcal{H}^r(\mathbb{R}^N)$ *with* $1/r=(1/p)+(1/q)$.

As it stands, the above result is a bit vague since the definition of $J(u)$, E.B or $\nabla u.(\partial v/\partial x_i)$ is not clear. To be specific, we may consider these expressions to be defined as limits in the sense of distributions (or in the corresponding Hardy spaces in view of the bounds implied by the proof of Theorem II.2, bounds that also show the existence of limits). An even more precise way consists in writing these expressions in conservative form: for instance, in the case of 2), we introduce as in the proof of Theorem II.1 $\pi \in L^{p^*}(\mathbb{R}^N)$ and we define $E.B = \operatorname{div}(\pi B)$, a meaningful expression since $\pi B \in L^1(\mathbb{R}^N)$. It is also easy to check that these two definitions coincide.

Remark II.5. – We define $\mathscr{H}_w^{N/(N+1)}$ by

$$\mathscr{H}_w^{N/(N+1)} = \{ f \in \mathscr{S}' / (\sup_{t>0} | f * h_t |)^{(N+1)/N} \in L_w^1(\mathbb{R}^N) \}$$

where $g \in L_w^1$ if meas $(|g| \geqq \lambda) \leqq C/\lambda$ for all $\lambda > 0$, for some $C \geqq 0$. Then, the conclusion of Theorem II.2 remains valid replacing $\mathscr{H}^{N/(N+1)}$ by $\mathscr{H}_w^{N/(N+1)}$ (and by even the closure of $C_0^\infty(\mathbb{R}^N)$ in that space), when $(1/p) + (1/q) = 1 + (1/N)$ in parts 2) and 3), when $p = N^2/(N+1)$ in part 1).

We now conclude this section with an even further extension of Theorem II.2 that we state only in the case of the "div-curl" situation (case 2) above) in order to simplify the presentation. We wish to allow now E, B to belong to Hardy spaces

$$\begin{gathered} E \in \mathscr{H}^p(\mathbb{R}^N), \qquad B \in \mathscr{H}^q(\mathbb{R}^N) \quad \text{with} \\ 0 < p < \infty, \qquad 0 < q < \infty, \qquad \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{N}, \\ \operatorname{curl} E = 0, \qquad \operatorname{div} B = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^N), \end{gathered} \tag{17}$$

– notice that necessarily p, $q > N/(N+1)$ and either p or q is strictly greater than 1. Of course, $\mathscr{H}^p(\mathbb{R}^N) = L^p(\mathbb{R}^N)$ if $p > 1$.

Then, we can form E.B either by a density argument (as a distribution) using the bound implied by the result below, or by writing it directly in conservative form essentially as we did above. For instance, if $q > 1$ (the other case requires a slightly different algebra...), we write $E = \nabla \pi$ where $\pi \in L^{p^*}(\mathbb{R}^N)$ (recall that Sobolev embeddings are valid for $\mathscr{H}^p$ spaces if $p > (N+1)/N$ – a direct consequence of the atomic decomposition, *see* R. Coifman [8], R. Coifman and G. Weiss [14] for instance). Then, E.B may be defined as above by div (πB) and this quantity makes sense since

$$(1/p^*) + (1/q) = (1/p) - (1/N) + (1/q) < 1$$

by (17). And we have the

THEOREM II.3. – *Let* E, B *satisfy* (17) *then* $E.B \in \mathscr{H}^r(\mathbb{R}^N)$ *with* $1/r = (1/p) + (1/q)$.

Proof. – To keep the ideas clear, we only consider as we just did above the case when $q > 1$. Then, we may follow mutatis mutandis the above arguments and we conclude provided we show the following

LEMMA II.2. – *Let* $p \in (N/(N+1), \infty)$ *let* $\alpha \in [1, p^*)$ *(recall that* $p^* = Np/(N-p)$*). Then, there exists a constant* $C \geqq 0$ *such that for all f satisfying* $\nabla f \in \mathscr{H}^p(\mathbb{R}^N)$ $(f \in L^{p^*}(\mathbb{R}^N))$ *we have*

$$\left\{ \int_{\mathbb{R}^N} \left[\sup_{t>0} \fint_{B_t^x} \left\{ \frac{1}{t} \left| f - \fint_{B_t^x} f \right| \right\}^\alpha dy \right]^{p/\alpha} dx \right\}^{1/p} \leqq C \| \nabla f \|_{\mathscr{H}^p}. \tag{18}$$

Remark II.6. – A similar argument to the proof of Lemma II.2 below shows that the Sobolev-Poincaré inequality holds if $\nabla f \in \mathcal{H}^p$ $(p > N/(N+1))$. In that case we may even allow the exponent $\alpha = p^*$. Of course, the above result is obvious for $p \geqq 1$.

Proof of Lemma II.2. – First of all, we consider $\Lambda = (-\Delta)^{1/2}$ and we observe that $\nabla f \in \mathcal{H}^p$ and $\Lambda f \in \mathcal{H}^p$ are equivalent (since the Riesz transforms are bounded on $\mathcal{H}^p$). Then, we introduce the operator

$$\mathrm{T} f(x) = \sup_{t>0} \left(\fint_{B_t^x} \left\{ \frac{1}{t} \left| g - \fint_{B_t^x} g \right| \right\}^\alpha dy \right)^{1/\alpha} \tag{19}$$

where $g = \Lambda^{-1} f$.

Since $\alpha \geqq 1$, this operator is obviously sublinear

$$[i.e.\ |\mathrm{T}(\lambda f + \mu g)| \leqq |\lambda|\,|\mathrm{T}(f)| + \mu\,|\mathrm{T}(g)|]$$

and Lemma II.2 amounts to the boundedness of the operator T from $\mathcal{H}^p$ into L^p. If $p<1$, it is enough to show that $\mathrm{T}(a)$ is uniformly bounded in L^p for all normalized p-atoms a *i.e.* compactly supported bounded functions a satisfying

$$\operatorname{Supp} a \subset Q, \qquad \int a\,dx = 0, \qquad \|a\|_{L^\infty} \leqq \frac{1}{\operatorname{meas}(Q)^{1/p}} \tag{20}$$

for some cube Q in $\mathbb{R}^N$ [recall that $p > N/(N+1)$].

By a simple translation and scaling argument, we readily see that we only have to prove this claim when Q is the unit cube centered at 0.

Having thus reduced the proof of the Lemma to this estimation, we proceed as follows and consider first points $|x| \leqq 10$. We then recall the elementary Poincaré inequality

$$\left(\fint_{B_t^x} \left| g - \fint_{B_t^x} g \right|^\alpha dy \right)^{1/\alpha} \leqq C\, t \left(\fint_{B_t^x} |\nabla g|^\alpha dy \right)^{1/\alpha}$$
$$\leqq C\, t\, M(|\nabla g|^\alpha)^{1/\alpha} \quad \text{for all } t \geqq 0,$$

where $g = \Lambda^{-1} a$. Therefore, we have

$$0 \leqq \mathrm{T}(a) \leqq M(|\nabla g|^\alpha)^{1/\alpha} \quad \text{a.e.}$$

On the other hand, if $\beta \in (\alpha, \infty)$

$$\|M(|\nabla g|^\alpha)^{1/\alpha}\|_{L^\beta} = \|M(|\nabla g|^\alpha)\|_{L^{\beta/\alpha}}^{1/\alpha}$$
$$\leqq C \|\,|\nabla g|^\alpha\|_{L^{\beta/\alpha}}^{1/\alpha} = C \|\nabla g\|_{L^\beta}.$$

And $\partial g / \partial x_i = R_i a$ (Riesz transform), thus $\|\nabla g\|_{L^\beta} \leqq C \|a\|_{L^\beta} \leqq C$. We thus deduce finally

$$\left(\int_{|x| \leqq 10} |\mathrm{T}(a)|^p dx \right)^{1/p} \leqq C \left(\int_{|x| \leqq 10} |\mathrm{T}(a)|^\beta dx \right)^{1/\beta} \tag{21}$$
$$\leqq C \|M(|\nabla g|^\alpha)^{1/\alpha}\|_{L^\beta} \leqq C.$$

We next treat the part ("away from the singularity") $|x| \geqq 10$. We first recall that $g = \Lambda^{-1} a = c_N a * 1/|x|^{N-1}$. Since $\int_{\mathbb{R}^N} a\,dx = 0$ and a is bounded with compact support, we deduce trivially:

$$|g(x)| \leqq C(1+|x|)^{-N}, \qquad |\nabla g(x)| \leqq C(1+|x|)^{-(N+1)}. \tag{22}$$

We then claim that by a brute force estimate we have

$$0 \leqq T(a) \leqq C(\log|x|)\, x^{-N-1} + C|x|^{-1-N/\alpha} \qquad \text{if} \quad |x| \geqq 10. \tag{23}$$

If this claim is shown, we conclude easily the proof of Lemma II.2. Indeed

$$\int_{|x| \geqq 10} |T(a)|^p\,dx \leqq C, \quad \text{since } p + \frac{Np}{\alpha} > N \quad \text{and} \quad p > \frac{N}{NH}$$

and we combine this inequality with (21).

There only remains to show (23). We first consider $0 < t \leqq |x|/2$. In that case, if $y \in B_t^x$, we have in view of (22)

$$\left| g(y) - \fint_{B_t^x} g \right| \leqq C \frac{t}{|x|^{N+1}}$$

and thus

$$\sup_{0 < t \leqq |x|/2} \left(\fint_{B_t^x} \left\{ \frac{1}{t} \left| g - \fint_{B_t^x} g \right| \right\}^\alpha \right)^{1/\alpha} \leqq \frac{C}{|x|^{N+1}} \qquad \text{for} \quad |x| \geqq 10. \tag{24}$$

Next, if $t > |x|/2$, we write

$$\begin{aligned} \left| g(y) - \fint_{B_t^x} g \right| &\leqq |g(y)| + \left| \fint_{B_t^x} g \right| \\ &\leqq |g(y)| + C t^{-N} \log t. \end{aligned}$$

Therefore, if $|x| \geqq 10$,

$$\begin{aligned} \sup_{t > |x|/2} \left(\fint_{B_t^x} \left\{ \frac{1}{t} \left| g - \fint_{B_t^x} g \right| \right\}^\alpha \right)^{1/\alpha} &\leqq C \sup_{t > |x|/2} \left(\frac{\log t}{t^{N+1}} + \frac{1}{t^{1+N/\alpha}} \right) \\ &\leqq C \left(\frac{\log|x|}{|x|^{N+1}} + \frac{1}{|x|^{1+N/\alpha}} \right). \end{aligned}$$

Combining this inequality with (24), the claim (23) is proven, concluding thus the proof of Lemma II.2.

III. Other approaches, more examples and variants

III.1. COMMUTATORS. – We want to show in this section how Theorem II.1 can be deduced from the well-known result of Coifman-Rochberg-Weiss [13] denoted here after as the CRW theorem: if R is any Riesz transform (*i.e.* $R=\partial/\partial x_j(-\Delta)^{-1/2}$ for some j) and if $b\in\mathrm{BMO}$, then the commutator $[b, R]$ is bounded from L^p into L^p for all $p\in(1,\infty)$.

Therefore, there exists a constant $C>0$ such that for all $b\in C_0^\infty(\mathbb{R}^N)$, $f\in L^p$, $g\in L^{p'}$ [for some $p\in(1,\infty)$], we have

$$\left|\int (b\,R(f)-R(bf))\,g\,dx\right|=\left|\int b\{(Rf)g+f(Rg)\}\,dx\right| \leq C\|b\|_{\mathrm{BMO}}\|f\|_{L^p}\|g\|_{L^{p'}}. \tag{25}$$

Therefore, by the classical duality $(\mathrm{VMO})^*=\mathscr{H}^1$, $(\mathscr{H}^1)^*=\mathrm{BMO}$ – where VMO is the closure of $C_0^\infty(\mathbb{R}^N)$ for the BMO norm "up to constants"... –, we see that an equivalent form of the CRW theorem is the

THEOREM III.1. – *Let* R *be any Riesz transform, let* $p\in(1,\infty)$ *and let* $f\in L^p(\mathbb{R}^N)$, $g\in L^{p'}(\mathbb{R}^N)$, *then* $(Rf)g+f(Rg)\in\mathscr{H}^1(\mathbb{R}^N)$.

Remark III.1. – If R, R′ are two Riesz transforms, then this result immediately implies that $(Rf)(R'g)-(R'f)(Rg)\in\mathscr{H}^1(\mathbb{R}^N)$ since we have

$$(Rf)(R'g)-(R'f)(Rg)=[(Rf)(R'g)+f(RR'g)]-[f(R'Rg)+(R'f)(Rg)].$$

Remark III.2. – It is quite clear that, in the above Theorem, R may be replaced by a general Calderón-Zygmund operator K but we will not pursue now in this direction since such extensions will be anyway consequences of section V. Of course, in that case, one forms $(Kf)\,g-f\,(K^t g)$ where K^t is the transposed on K.

Let us now explain how Theorem II.1 can be deduced from Theorem III.1. In order to avoid the rather messy algebra involved for the determinant, we only consider the case of 2, 3) (recalling anyway that 1) can be deduced from 2). In the case of 2), we introduce π such that $\nabla\pi=E$ or even better $f\in L^p(\mathbb{R}^N)$ such that $R_j f=E_j$ for all j. (The role of E and B are exchanged in the proof.) Then,

$$E.B=\sum_{j=1}^{N}(R_j f)B_j=\sum_{j=1}^{N}(R_j f)B_j+f(R_j B_j)$$

since

$$\sum_{j=1}^{N}R_j B_j=\operatorname{div}((-\Delta)^{-1/2}B)=(-\Delta)^{-1/2}(\operatorname{div}B)=0.$$

And we conclude since, in view of Theorem III.1, each of the functions

$$(R_j f)B_j+f(R_j B_j)\quad\text{belongs to }\mathscr{H}^1.$$

In the case of 3), it is extremely similar since we only need to introduce

$$f=(-\Delta)^{1/2}u\in L^2(\mathbb{R}^N),\qquad g_j=(-\Delta)^{1/2}v_j\in L^2(\mathbb{R}^N),\quad \text{for } 1\leqq j\leqq N.$$

Then,

$$\nabla u.\frac{\partial v}{\partial x_i}=\sum_{j=1}^{N}(R_j f)(R_i g_j)=\sum_{j=1}^{N}(R_j f)(R_i g_j)-(R_i f)(R_j g_j)$$

since

$$\sum_{j=1}^{N}R_j g_j=\operatorname{div} v=0.$$

And we conclude in view of Theorem III.1 (and Remark III.1).

Of course, Theorem III.1 is very much reminiscent of the famous fact on Hilbert transforms: let f, $g\in L^2(\mathbb{R})$, then $(Hf)g+f(Hg)\in\mathscr{H}^1(\mathbb{R})$. In that case, however, some form of converse is known (with precise estimates on (f, g)) showing in particular that the range of $(Hf)g+f(Hg)$ when f, g describe $L^2(\mathbb{R})$ is exactly $\mathscr{H}^1(\mathbb{R})$. This of course cannot be true if we naively replace H by arbitrary Calderón-Zygmund operators. However, it is very tempting to ask whether the map

$$u\in W^{1,2}(\mathbb{R}^2)^2\mapsto J(u)=\det(\nabla u)\in\mathscr{H}^1(\mathbb{R}^2)$$

is onto? We have been unable to answer this question which can be raised for almost all the nonlinear quantities arising in the theory of compensated compactness. However, we shall see in section III.3 that any element of $\mathscr{H}^1$ can be decomposed into a countable sum of "normalized jacobians": this will show in particular that $\mathscr{H}^1$ is the minimal vector space containing $J(u)$ for all $u\in W^{1,2}(\mathbb{R}^2)^2$. The argument is in fact quite general.

III.2. Variants and more examples. – We begin by mentioning briefly some nonhomogeneous situations. We only consider the "div-curl" example and E, B satisfying

$$E\in L^p_{loc},\qquad B\in L^{p'}_{loc},\qquad \operatorname{div} B\in W^{-1,r}_{loc},\qquad \operatorname{curl} E\in W^{-1,s}_{loc},$$

where $r>p'$, $s>p$. Then, $E.B\in\mathscr{H}^1_{loc}$.

This result is easily shown by a (Hodge) decomposition

$$E=E_0+E_1,\qquad E_0\in L^p_{loc},\quad E_1\in L^s_{loc},\quad \operatorname{curl} E_0=0,\quad \operatorname{div} E_1=0,$$
$$B=B_0+B_1,\qquad B_0\in L^{p'}_{loc},\quad B_1\in L^r_{loc},\quad \operatorname{div} B_0=0,\quad \operatorname{curl} B_1=0.$$

Then,

$$E.B=E_0.B_0+R,\qquad R\in L^t_{loc}\quad\text{for some } t>1.$$

And we apply Theorem II.1 (and Remark II.1 following it) to conclude.

We also wish to mention a special case of the div-curl expression namely linear or quasilinear elliptic equations

(26) $$\operatorname{div}(a(x).\nabla u)=0 \quad \text{in } \Omega, \qquad \nabla u \in L^2_{\text{loc}}$$

or

(27) $$\operatorname{div}(|\nabla u|^{p-2}\nabla u)=0 \quad \text{in } \Omega, \qquad \nabla u \in L^p_{\text{loc}},$$

where $p>1$, Ω is bounded open set in $\mathbb{R}^N$, a is a bounded function taking values in the set of nonnegative matrices. Then, we set $E=\nabla u$ so that $E \in L^2_{\text{loc}}$ (or L^p_{loc}) and $\operatorname{curl}(E)=0$. And, we consider $B=a(x).\nabla u$ or $B=|\nabla u|^{p-2}\nabla u$ so that $\operatorname{div} B=0$ and $B \in L^2_{\text{loc}}$ (or $B \in L^{p'}_{\text{loc}}$). Applying Theorem II.1, we find that $E.B \in \mathscr{H}^1_{\text{loc}}$. And we observe that $C=E.B$ is nothing but

$$C=\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \qquad \text{or} \qquad =|\nabla u|^p$$

which are nonnegative quantities. In particular, we immediately recover an improved integrability: $\int_K C|\log C|\,dx<\infty$ for all compact set $K \subset \Omega$. There is more to this improvement since if we now adapt (locally) the proof of Theorem II.1, we obtain an estimate of the (usual since $C \geqq 0$) maximal function of C in terms of respectively

$$M(|\nabla u|^\alpha)^{1/\alpha} M(|a.\nabla u|^\alpha)^{1/\alpha} \qquad \text{where} \quad \alpha=\frac{2N}{N+1}.$$

or

$$M(|\nabla u|^\alpha)^{1/\alpha} M(|\nabla u|^{\beta(p-1)})^{1/\beta}=M(|\nabla u|^\alpha)^{(N+1)/N}$$

where $\alpha=\beta(p-1)=Np/(N+1)$. In particular, if a is (uniformly in x) positive definite, we estimate essentially the maximal function of $|\nabla u|^2$ (resp. $|\nabla u|^p$) by

$$M(|\nabla u|^{2N/(N+1)})^{(N+1)/N} \text{ (resp. } M(|\nabla u|^{Np/(N+1)})^{(N+1)/N}).$$

And we recover basically the standard and celebrated reverse Hölder inequalities that are basic in the study of elliptic regularity: in fact, the scheme of proof of Theorem II.1 when explicitely translated in those cases is very much reminiscent to the standard proofs of those reverse Hölder inequalities...

Another remark relating our results to second-order elliptic equations is the following (that we detail only in $\mathbb{R}^N$ with the Laplace operator to keep the ideas clear): let $u \in L^{2N/(N-2)}$ such that $-\Delta u=f \in L^{2N/(N+2)}$ $(N \geqq 3)$ so that $\nabla u \in L^2$. We claim that $|\nabla u|^2-fu \in \mathscr{H}^1$. In order to show this claim, we follow the proof of Theorem II.1 and we find

$$\{h_t*(|\nabla u|^2-fu)\}(x)$$
$$=\int \nabla u(y)\frac{1}{t}\left[u(y)-\frac{1}{|B_t^x|}\int_{B_t^x} u\right]\nabla h\left(\frac{x-y}{t}\right)\frac{1}{t^N}dy-\frac{1}{|B_t^x|}\int_{B_t^x} u\int_{B_t^x} f(y)\,h\left(\frac{x-y}{t}\right)\frac{1}{t^N}dy.$$

Hence, we conclude as in the proof of Theorem II.1 provided we show that

$$\sup_{t>0}\left|\fint_{B_t^x} u\right|\left|\fint_{B_t^x} f(y)\,h\left(\frac{x-y}{t}\right)\frac{1}{t^N}dy\right|\in L^1.$$

And this is obvious since we can bound that quantity by $M(|u|)\,M(|f|)$ and

$$M(|u|)\in L^{2N/(N-2)}(\mathbb{R}^N),\qquad M(|f|)\in L^{2N/(N+2)}(\mathbb{R}^N).$$

In fact, the same result holds if $N=2$, $\nabla u\in L^2(\mathbb{R}^2)$, $\Delta u\in\mathscr{H}^1(\mathbb{R}^2)$ in which case one can show (*see* also section IX) that, up to a constant, $u\in C_0(\mathbb{R}^2)$. More generally, if $N\geqq 3$, one can still assert that $\Delta u\,u+|\nabla u|^2\in\mathscr{H}^1(\mathbb{R}^N)$ if $u\in L^p(\mathbb{R}^N)$ with $2N/(N-2)\leqq p\leqq\infty$ and $\Delta u\in L^{p'}(\mathbb{R}^N)$ with the substitution of $L^1(\mathbb{R}^N)$ by $\mathscr{H}^1(\mathbb{R}^N)$ when $p=+\infty$. The proof is exactly the same. Notice finally that this quantity is nothing but $(1/2)\,\Delta(|u|^2)$.

Also the arguments given at the end of section II allow to go below 1 but since "two moments vanish" it turns out that one can do even better and we shall come back on this example in section VII.

It is also worth remarking that analogous results are possible for the wave operator $\square=(\partial^2/\partial t^2)-\Delta$. Indeed, if

$$\square\,u=f\quad\text{in }\mathbb{R}_t\times\mathbb{R}_x^N \tag{28}$$

and $\partial u/\partial t$, $\nabla u\in L^2(\mathbb{R}^{1+N})$, $u\in L^p(\mathbb{R}^{1+N})$ with $2(N+1)/(N-1)\leqq p\leqq\infty$ and $f\in L^2(\mathbb{R}^{1+N})$ if $p<\infty$, $f\in\mathscr{H}^1(\mathbb{R}^{1+N})$ if $p=+\infty$ (and $N\geqq 2$), then

$$\frac{1}{2}\square\,(u^2)=fu+\left|\frac{\partial u}{\partial t}\right|^2-|\nabla u|^2\in\mathscr{H}^1(\mathbb{R}^{1+N}).$$

From these two examples, it is clear that only the fact that we are dealing with second-order (with constant coefficients) operators matters. In fact, similar results hold for higher-order (with an even order if we insist on local quantities) operators.

Much more examples are possible. These examples include all explicit examples of the theory of compensated compactness like: (i) minors of the Jacobian matrix of $u\in W^{1,p}(\mathbb{R}^N)^N$ where p is the order of the minor (ii) products of differential forms (like in R. C. Rogers and B. Temple [40]), (iii) specific quantities arising in Maxwell's equations... In fact, these examples can be ordered in the degree of generality but we detail only the last one.

Let E, B, D, $H\in L^2(\mathbb{R}_t\times\mathbb{R}_x^3)^3$ satisfy "Maxwell's equations":

$$\left\{\begin{aligned}&\frac{\partial B}{\partial t}+\operatorname{curl}E=0,\qquad \operatorname{div}B=0,\\&\frac{\partial D}{\partial t}-\operatorname{curl}H=0,\qquad \operatorname{div}D=0,\quad\text{in }\mathbb{R}_t\times\mathbb{R}_x^3.\end{aligned}\right. \tag{29}$$

Then, E.B, D.H and $E.D-B.H\in\mathscr{H}^1(\mathbb{R}^{1+3})$.

This can be shown exactly as in section II, introducing potential vectors (curl $\bar{B}=B$, div $\bar{B}=0$, curl $\bar{D}=D$, div $\bar{D}=0$...). We can also show this claim using the CRW theorem or equivalently Theorem III.1. Indeed, in the case of E.B (the proof is the same for D.H), since div $B=0$, we introduce A such that $R\times A=B$, $R.A=0$ where R is the "vector Riesz operator" given by $R_j=(\partial/\partial x_j)(-\Delta_{x,t})^{-1/2}$, $(\forall j=1, 2, 3)$. Then, $A\in L^2(\mathbb{R}^{1+2})^3$. And we have by Theorem III.1

$$\begin{aligned}(R\times A).E&=(R\times E).A+f_1, \qquad \text{where } f_1\in\mathcal{H}^1,\\ &=-(R_0 B).A+f_1\end{aligned}$$

in view of (29), where $R_0=(\partial/\partial t)(-\Delta_{x,t})^{-1/2}$. Next

$$(R_0 B).A=[R_0(R\times A)].A=[R\times(R_0 A)].A$$

and

$$\begin{aligned}[R\times(R_0 A)].A&=[R\times A].(R_0 A)+f_2, \qquad \text{where } f_2\in\mathcal{H}^1,\\ &=-[R_0(R\times A)].A+f_2+f_3, \qquad \text{where } f_3\in\mathcal{H}^1.\end{aligned}$$

In other words $(R_0 B).A=(1/2)(f_2+f_3)\in\mathcal{H}^1$ and we conclude.

Next, we consider $E.D-B.H$, introducing $C\in L^2(\mathbb{R}^{1+3})^3$ which satisfies $R.C=0$, $R\times C=D$. We then write

$$\begin{aligned}E.D-B.H&=E.(R\times C)-H.(R\times A)\\ &=(R\times E).C-(R\times H).A+f_1, \qquad \text{where } f_1\in\mathcal{H}^1,\\ &=-(R_0 B).C-(R_0 D).A+f_1 \quad \text{in view of (29).}\end{aligned}$$

And

$$\begin{aligned}(R_0 B).C&=[R_0(R\times A)].C=[R\times(R_0 A)].C\\ &=(R_0 A).(R\times C)+f_2, \qquad \text{where } f_2\in\mathcal{H}^1,\\ &=(R_0 A).D+f_2.\end{aligned}$$

We can now conclude since $(R_0 A).D+(R_0 D).A\in\mathcal{H}^1$.

We finally close this section with a few more examples. Let $u, v\in H^2(\mathbb{R}^2)$ $(=W^{2,2}(\mathbb{R}^2))$, then the quadratic expression arising in von Karman's equations namely

$$[u, v]=\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}-2\frac{\partial^2 u}{\partial x_1\partial x_2}\frac{\partial^2 v}{\partial x_1\partial x_2}$$

belongs to $\mathcal{H}^1(\mathbb{R}^2)$.

Let $u, v\in H^2(\mathbb{R}^N)$ $(N\geqq 2)$, then

$$|\Delta u|^2-\sum_{i,j=1}^{N}\left|\frac{\partial^2 u}{\partial x_i\partial x_j}\right|^2\in\mathcal{H}^1(\mathbb{R}^N).$$

It must be clear (by now) that the list is endless and we shall see in sections V, VI some abstract formulations covering most of these examples (and more) for quadratic expressions.

III.3. A DECOMPOSITION OF $\mathcal{H}^1(\mathbb{R}^N)$ INTO "div-curl" QUANTITIES. – We have mentioned several times above the problem of determining the exact range of the "compensated-compactness" quantities. We are going to answer partially this question here on the div-curl example – this type of answer applies also to other examples like the jacobian... We then denote by W the subset of $\mathcal{H}^1(\mathbb{R}^N)$ formed by the functions $w=\mathrm{E}.\mathrm{B}$ where $\mathrm{E}, \mathrm{B}\in \mathrm{L}^2(\mathbb{R}^N)^N$, $\|\mathrm{E}_j\|_{\mathrm{L}^2(\mathbb{R}^N)}, \|\mathrm{B}_j\|_{\mathrm{L}^2(\mathbb{R}^N)}\leqq 1$ and $\operatorname{div}\mathrm{E}=0$, $\operatorname{curl}\mathrm{B}=0$ in $\mathbb{R}^N$.

We then state the

THEOREM III.2. – *Any function* $f\in\mathcal{H}^1(\mathbb{R}^N)$ *can be written as* $f=\sum_{k=0}^{\infty}\lambda_k w_k$ *where* $w_k\in \mathrm{W}\ (\forall k\geqq 0)$, $\sum_{k=0}^{\infty}|\lambda_k|<\infty$.

This decomposition – somewhat reminiscent of the classical atomic decomposition – will be shown by an argument which relies on two simple functional analysis facts given by the following

LEMMA III.1. – *Let* V *be a bounded subset of a normed vector space* F. *We assume that* $\bar{\mathrm{V}}$ *(closure of* V *for the norm of* F) *contains the unit ball (centered at* 0) *of* F. *Then, any* x *in that ball can be written as* $x=\sum_{j=0}^{\infty}(1/2^j)\,y_j$ *where* $y_j\in\mathrm{V}$ *for all* $j\geqq 0$.

LEMMA III.2. – *Let* V *be a bounded symmetric* ($x\in\mathrm{V}\Rightarrow -x\in\mathrm{V}$) *subset of a normed vector space* F. *Then, the closed convex hull* $\tilde{\mathrm{V}}$ *of* V *(in* F) *contains a ball centered at* 0 *if and only if, for any* $l\in\mathrm{F}^*$, $\|l\|_{\mathrm{F}^*}$ *and* $\sup_{x\in\mathrm{V}}\langle l, x\rangle$ *are two equivalent norms.*

We shall give a proof of these facts later on and we first prove Theorem III.1 admitting temporarily those two lemmata. Clearly, in view of these results, it suffices to show that, for any $b\in\mathrm{BMO}(\mathbb{R}^N)$, $\|b\|_{\mathrm{BMO}}$ and $\sup_{w\in\mathrm{W}}\left\{\int_{\mathbb{R}^N} bw\,dx\right\}$ are two equivalent norms. In turn, this will be proven if we show the following claim: let $b\in\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^N)$ satisfy

$$\int_{\mathbb{R}^N} b\,\mathrm{E}.\mathrm{B}\,dx\leqq\|\mathrm{E}\|_{\mathrm{L}^2}\|b\|_{\mathrm{L}^2}$$

for all $\mathrm{E}, \mathrm{B}\in C_0^\infty(\mathbb{R}^N)$, with $\operatorname{div}\mathrm{E}=0$, $\operatorname{curl}\mathrm{B}=0$ in $\mathbb{R}^N$. Then, we claim that $b\in\mathrm{BMO}(\mathbb{R}^N)$ and $\|b\|_{\mathrm{BMO}}\leqq \mathrm{C}_N$ (for some constant C_N depending only on N). In fact this claim will be a consequence of a more precise estimate: let Q be an arbitrary cube in $\mathbb{R}^N$ centered at $x^0=(x_1^0,\ldots,x_N^0)$ and of sidelength $2d$. Let $\tilde{\mathrm{Q}}$ be the doubled cube (same center, sidelength $4d$). Then, we have

$$\left(⨍_Q \left|b-⨍_Q b\right|^2 dx\right)^{1/2} \leqq C_N \sup\left\{\int b\,E.B\,dx/E,\ B\in C_0^\infty(\tilde{Q}),\ \|E\|_{L^2}\leqq 1,\ \|B\|_{L^2}\leqq 1\right\}.$$

Indeed, let $\varphi_0 \in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi_0 \equiv 1$ on $[-1, +1]^N$, $\varphi_0 \equiv 0$ on $([-2, +2]^N)^c$. We then set $B=\gamma\,\text{meas}(Q)^{-1/2}\,\nabla((x_j - x_j^0)\,\varphi_Q(x))$ (for $1\leqq j\leqq N$ fixed) where $\varphi_Q=\varphi((x-x^0)/d)$ so that $B\in C_0^\infty(\tilde{Q})$, and where $\gamma>0$ is a normalization constant (independent of x^0 and d) such that $\|B\|_{L^2}=1$. Notice that $B=\gamma$ meas $(Q)^{-1/2} e_j$ on Q.

Next, we take $u\in C_0^\infty(Q)$ such that $\|\nabla u\|_{L^2}\leqq 1$ and we set

$$E=\left(-\frac{\partial u}{\partial x_j}, 0, \ldots, 0, \frac{\partial u}{\partial x_1}, 0, 0\ldots, 0\right),$$

so that $\partial u/\partial x_1$ is the j-th component of E and $\|E\|_{L^2}\leqq 1$, div $E=0$.

Then, we have

$$\int b\,E.B\,dx=\int_Q b\,\gamma\,\text{meas}(Q)^{-1/2}\frac{\partial u}{\partial x_1}\,dx.$$

Since u is arbitrary in $C_0^\infty(Q)$ with $\|\nabla u\|_{L^2}\leqq 1$, we deduce

$$\left\|\frac{\partial b}{\partial x_1}\right\|_{H^{-1}(Q)} \leqq C_N\,\text{meas}(Q)^{1/2}\sup\left\{\int b\,(E.B)\,dx/E,\ B\in C_0^\infty(\tilde{Q}),\ \|E\|_{L^2}\leqq 1,\ \|B\|_{L^2}\leqq 1\right\}.$$

We obtain in a similar way the same bound for $\|\partial b/\partial x_j\|_{H^{-1}(Q)}$ for all j. We then conclude easily in view of the classical inequality

$$\inf_{\lambda\in\mathbb{C}}\left(\int_Q |b-\lambda|^2\,dx\right)^{1/2} \leqq C\sum_{j=1}^N \left\|\frac{\partial b}{\partial x_j}\right\|_{H^{-1}(Q)}. \quad \square$$

Proof of Lemma III.1. — Clearly, $x\in\bar{V}$. Hence, there exists $y_0\in V$ such that $\|x-y_0\|<1/2$. Therefore, $2(x-y_0)\in\bar{V}$ and there exists $y_1\in V$ such that $\|2(x-y_0)-y_1\|<1/2$... Arguing by induction, we build a sequence $(y_k)_{k\geqq 0}$ in $\bar{V}$ such that $\left\|x-\sum_{j=0}^N (1/2^j)\,y_j\right\|<1/2^{N+1}$, concluding thus the proof.

Proof of Lemma III.2. — We first note that $\tilde{V}$ is also symmetric and that we have

$$\sup_{x\in V}\langle l, x\rangle=\sup_{x\in V}|\langle l, x\rangle|=\sup_{x\in\tilde{V}}\langle l, x\rangle,$$
$$=\sup_{x\in\tilde{V}}|\langle l, x\rangle|, \qquad \forall\, l\in F^*.$$

Therefore, if $\tilde{V}$ contains a ball centered at 0, these quantities define a norm which is clearly equivalent to $\|l\|_{F^*}$ (since $\tilde{V}$ is also bounded).

Conversely, if there exists $\alpha > 0$ such that we have for all $l \in F^*$

$$\sup_{x \in \tilde{V}} \langle l, x \rangle = \sup_{x \in V} \langle l, x \rangle \geqq \alpha \| l \|_{F^*},$$

we have to show that $\tilde{V}$ contains the (closed) ball centered at 0 of radius α. Indeed, arguing by contradiction, we assume there exists $\| x_0 \| < \alpha$ such that $x_0 \notin \tilde{V}$. Then, by Hahn-Banach theorem, there exists $l \in F^*$ with $\| l \|_{F^*} = 1$ such that

$$\langle l, x_0 \rangle \geqq \sup_{x \in \tilde{V}} \langle l, x \rangle.$$

And we easily reach a contradiction since $\langle l, x_0 \rangle \leqq \| l \|_{F^*} x_0 \| \leqq \| x_0 \|$.

IV. On weak convergence in $\mathscr{H}^1$

As we recalled in the Introduction, compensated compactness is primarily concerned with passages to the limit. Typically, in all examples stated above, compensated compactness deals with a bounded (the natural bounds corresponding to all the results stated) sequence of functions that we may assume without loss of generality to be weakly convergent to some limits (again for the natural corresponding weak topology). Then, the main statement is that the nonlinear expressions converge in the sense of distributions (or weakly in the sense of measures) to the same expression formed with the weak limits. To be specific, let us consider a model example namely the div-curl case. Let E^n, B^n be bounded respectively in $L^p(\mathbb{R}^N)$, $L^{p'}(\mathbb{R}^N)$ $(1 < p < \infty,\ N \geqq 2)$ and let us assume they satisfy for all $n \geqq 1$

$$\text{(30)} \qquad \operatorname{curl} E^n = 0, \qquad \operatorname{div} B^n = 0.$$

Assume in addition (and this is clearly the case up to the extraction of a subsequence) that $E^n . B^n$ converge weakly respectively in L^p, $L^{p'}$ to some E, B. Then, $E^n . B^n$ converges in the sense of distributions to E . B. If we want to relate this weak convergence to the CRW theorem (*see* section III . 1 above) we simply notice that $\mathscr{H}^1 = (\text{VMO})^*$ and that if $b \in \text{VMO}$, the CRW theorem immediately implies that $[b, R]$ is compact on L^p [in view of the definition of VMO and the fact that this statement is obvious if $b \in C_0^\infty(\mathbb{R}^N)$].

In fact, the slightly improved regularity we proved above show that, in such situations, the nonlinear expressions are thus bounded in $\mathscr{H}^1$. For instance, in the model case above, $E^n . B^n$ is bounded in $\mathscr{H}^1(\mathbb{R}^N)$. Since $\mathscr{H}^1$ is the dual of a separable Banach space namely VMO, $\mathscr{H}^1$ inherits of the usual weak-$*$ convergence. And we immediately deduce that the nonlinear quantities converge in fact in the weak-$*$ topology of $\mathscr{H}^1$. Again, in the model example, we deduce that $E^n . B^n$ converges to E . B in the weak-$*$ topology of $\mathscr{H}^1$ that we simply denote by ($\overset{*}{\rightharpoonup}$ in $\mathscr{H}^1$). It turns out that these elementary functional analysis considerations are useful! Or, in other words, the improvement from weak convergence in the sense of measures to weak convergence in $\mathscr{H}^1$ is useful. We shall present below an example illustrating this claim.

This will be a consequence of some properties of the weak convergence in $\mathscr{H}^1$, properties that are analogous of classical properties of the weak convergence in L^p for $p>1$. The first result in that direction is taken from P. Jones and J. L. Journé [29]—a striking result which in fact grew out of our work.

THEOREM [29]. – *Let f_n be bounded in $\mathscr{H}^1(\mathbb{R}^N)$. We assume that f_n converges a.e. to some $g\,(\in L^1(\mathbb{R}^N))$. Then, $g\in\mathscr{H}^1$ and $f_n \overset{*}{\rightharpoonup} g$ in $\mathscr{H}^1$.*

Since we will need to extend a bit this statement, we reproduce for the sake of completeness the proof of [29]. First of all, we may assume without loss of generality that $f_n \overset{*}{\rightharpoonup} f$ in $\mathscr{H}^1$ and we need to show that $f\equiv g$.

Let $\varphi\in C_0^\infty(\mathbb{R}^N)$, we wish to prove that $\int_{\mathbb{R}^N} f\varphi\,dx=\int_{\mathbb{R}^N} g\varphi\,dx$. Let $R>0$ be such that $\operatorname{Supp}\varphi\subset B(0,R)$. By Egorov theorem, for each $\varepsilon>0$, we can find a measurable set E such that meas $(E)<\varepsilon$, and f_n converges uniformly to g on E^c. The main (and only) difficulty is due to the fact that 1_E does not belong to BMO. This will be circumvented to the expense of "fattening a bit" 1_E. In order to do so, we consider $w_\lambda=(1+\lambda\log M(1_E))_+$. Notice that $1_E\leqq M(1_E)\leqq 1$ and $1_E\leqq w_\lambda\leqq 1$ a.e. On the other hand, by a result of R. Coifman and R. Rochberg [12], $\log M(1_E)\in\text{BMO}$ and $\|\log M(1_E)\|_{\text{BMO}}\leqq C_N$ (a constant that depends only on N). Since we have $\|b_+\|_{\text{BMO}}\leqq\|b\|_{\text{BMO}}$, we deduce that $w_\lambda\in\text{BMO}$ and that $\|w_\lambda\|_{\text{BMO}}\leqq C\lambda$ where C denotes here and below various constants independent of λ and ε.

Next, we need to make sure that we did not fatten 1_E too much. This can be seen by observing that

$$\{w_\lambda>0\}=\{M(1_E)>e^{-1/\lambda}\}.$$

Therefore, from the weak L^1 estimate on maximal functions, we deduce

$$\text{meas}(\{w_\lambda>0\})\leqq C\,\varepsilon\,e^{1/\lambda}. \tag{31}$$

Collecting these estimates on w_λ, it is now easy to conclude. Indeed, on one hand $\int_{\mathbb{R}^N}\varphi f_n\,dx$ goes to $\int_{\mathbb{R}^N}\varphi f\,dx$ as n goes to $+\infty$. On the other hand

$$\int_{\mathbb{R}^N}\varphi\,f_n\,dx-\int_{\mathbb{R}^N}\varphi\,g\,dx=\int_{\mathbb{R}^N}\varphi\,w_\lambda\,f_n\,dx+\int_{\mathbb{R}^N}\varphi\,(1-w_\lambda)(f_n-g)\,dx-\int_{\mathbb{R}^N}\varphi\,w_\lambda\,g\,dx.$$

For ε and λ fixed, the second term in the right-hand side goes to 0 as n goes to $+\infty$ since $1-w_\lambda$ vanishes on E. Hence, we deduce

$$\overline{\lim_n}\left|\int_{\mathbb{R}^N}\varphi\,f_n\,dx-\int_{\mathbb{R}^N}\varphi\,g\,dx\right|\leqq C\,\|\varphi\,w_\lambda\|_{\text{BMO}}+\int_{\{w_\lambda>0\}}|\varphi|\,|g|\,dx. \tag{32}$$

But we have for all cubes Q

$$\fint_Q |\varphi w_\lambda - \fint_Q \varphi w_\lambda \, dy| \, dx$$

$$\leqq \|\varphi\|_{L^\infty} \|w_\lambda\|_{BMO} + \fint_Q dx \fint_Q |\varphi(x) - \varphi(y)| \, 1_{(w_\lambda > 0)} \, dy$$

$$\leqq C\lambda + Ch \text{ if the size of Q is less than } h$$

$$\leqq C\lambda + \frac{C}{h^N} \operatorname{meas}(w_\lambda > 0) \leqq C\lambda + C \frac{\varepsilon e^{1/\lambda}}{h^N}$$

if the size of Q is more than h.

Similarly, we have in view of (31)

$$\int_{\{w_\lambda > 0\}} |\varphi| \, |g| \, dx \to 0 \quad \text{as } \varepsilon \to 0_+ \text{ for } \lambda > 0.$$

Therefore, we are able to conclude using (32) and letting first ε go to 0 and then λ to 0.

Our extension of the preceding result relies upon the notion of convergence in the sense of Chacon also called biting convergence. Let Ω be a measurable set of $\mathbb{R}^N$ (say) with finite measure, let f_n be bounded in $L^1(\Omega)$, we say that f_n converges in the sense of the biting lemma to some $f \in L^1(\Omega)$, for all $\varepsilon > 0$, there exists a measurable subset E of Ω such that meas (E) $< \varepsilon$ and $f_n \underset{n}{\rightharpoonup} f$ weakly in $L^1(\Omega \setminus E)$. We denote this convergence by $f_n \overset{b}{\underset{n}{\rightharpoonup}} f$. The interest of this notion is essentially due to the following result (the biting lemma) due to J. K. Brooks and R. V. Chacon [6] (*see* also E. J. Balder [3], J. Ball and F. Murat [5]...): for *any* bounded sequence f_n in $L^1(\Omega)$, there exist a subsequence n' and a function $f \in L^1(\Omega)$ such that $f_{n'} \overset{b}{\underset{n'}{\rightharpoonup}} f$.

In view of these facts, we may now just copy the proof of the above Theorem and we find the

COROLLARY IV.1. – *Let* $f_n \overset{*}{\underset{n}{\rightharpoonup}} f$ *in* $\mathscr{H}^1(\mathbb{R}^N)$. *For each* $R > 0$, *let* $g \in L^1(B(0, R))$ *be such that there exists a subsequence* n' *for which* $f_{n'} \overset{b}{\underset{n'}{\rightharpoonup}} g$ *in* $B(0, R)$, *then* $g = f$ a.e. *on* $B(0, R)$.

Remark IV.1. – Of course, all the results stated or mentioned above have local analogues.

Remark IV.2. – It was shown by K. Zhang [48] (*see* also S. Müller [33]) that compensated compactness quantities (like jacobians for example) converge to their weak limits in the sense of the biting lemma.

We now conclude this section with an application of these facts. It was shown by E. Acerbi and N. Fusco [1] that the functional $E(u)=\int_{\mathbb{R}^N} a(x)\,|\det(\nabla u)|\,dx$ defined on $W^{1,N}(\mathbb{R}^N)^N$ is weakly sequentially lower semicontinuous in that space as soon as $a\in L^\infty(\mathbb{R}^N)$ and $a\geqq 0$ a.e. Indeed, let $u_n \underset{n}{\rightharpoonup} u$ weakly in $W^{1,N}(\mathbb{R}^N)^N$. Without loss of generality we may assume that $E(u_n)$ converges to some $\bar{E}$. From the remarks made above, we know that $\det(\nabla u_n) \overset{*}{\underset{n}{\rightharpoonup}} \det(\nabla u)$ in $\mathcal{H}^1(\mathbb{R}^N)$. Then, let $R>0$. By the above Corollary and the biting lemma, there exists a subsequence n' such that $\det(\nabla u_{n'}) \overset{b}{\underset{n'}{\rightharpoonup}} \det(\nabla u)$. We then use the definition of that convergence to deduce

$$\int_{\mathbb{R}^N} a(x)\,|\det(\nabla u_{n'})|\,dx \geqq \int_{B(0,R)} a(x)\,|\det(\nabla u_{n'})|\,dx$$

$$\geqq \int_{E^c} a(x)\,|\det(\nabla u_{n'})|\,dx.$$

For each $\varepsilon>0$, $\det(\nabla u_{n'}) \underset{n'}{\rightharpoonup} \det(\nabla u)$ weakly in $L^1(E^c)$, therefore

$$\bar{E} \geqq \underline{\lim}_{n'} \int_{E^c} a\,|\det(\nabla u_{n'})|\,dx \geqq \int_{E^c} a\,|\det(\nabla u)|\,dx.$$

We then let ε go to 0 and we find $\bar{E}\geqq \int_{B(0,R)} a\,|\det(\nabla u)|\,dx$. And we deduce the desired inequality $\bar{E}\geqq \int_{\mathbb{R}^N} a(x)\,|\det(\nabla u)|\,dx$ letting R go to $+\infty$.

V. Relations with Coifman-Meyer analysis of bilinear operators

In the previous sections, we have seen that the nonlinear expressions arising in the theory of compensated compactness belong in fact to $\mathcal{H}^1$ (under natural conditions). And we recall that these expressions were considered for the weak continuity properties recalled in section IV. Roughly speaking, we have on one hand "weakly continuous nonlinear quantities" and on the other hand "nonlinear quantities" that belong to $\mathcal{H}^1$. A natural – but vague – question is then to determine whether these two classes coincide. However, it is not clear how one should formulate precisely this (too) general question.

We now present one possible formulation where we shall show that the two classes coincide. This formulation will involve only bilinear operators even if extensions to general multilinear operators are clearly possible – one such partial extension can be found in L. Grafakos [23]. The result we are going to present illustrates two more facts:

(i) the close connections that exist between this work and Coifman-Meyer "multilinear analysis" [11], (ii) the importance of cancellations for the theory of compensated compactness. We strongly believe that the phenomenon of cancellation is the real origin of compensated compactness.

We may now introduce our formulation. Let B be a bilinear continuous operator from $C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ into $\mathscr{D}'(\mathbb{R}^N)$. We assume B commutes with translations and dilations *i.e.*

$$(33) \quad \begin{cases} B(\varphi(.+h), \psi(.+h)) = B(\varphi, \psi)(.+h), \\ \forall h \in \mathbb{R}^N, \quad \forall \varphi, \psi \in C_0^\infty(\mathbb{R}^N), \end{cases}$$

$$(34) \quad \begin{cases} B(\varphi(\lambda .), \psi(\lambda .)) = B(\varphi, \psi)(\lambda .), \\ \forall \lambda > 0, \quad \forall \varphi, \psi \in C_0^\infty(\mathbb{R}^N). \end{cases}$$

Then, by standard results, there exists $m \in \mathscr{D}'(\mathbb{R}^N \times \mathbb{R}^N)$ such that

$$(35) \quad B(e^{i\xi\cdot x}, e^{i\eta\cdot x}) = m(\xi, \eta)\, e^{i(\xi+\eta)\cdot x}$$

$$(36) \quad m(\lambda\xi, \lambda\eta) = m(\xi, \eta) \quad \text{on } \mathbb{R}^N \times \mathbb{R}^N, \text{ for all } \lambda > 0.$$

We shall assume that m is bounded and *smooth* for $(\xi, \eta) \neq (0, 0)$ – and we will not bother to estimate the precise degree of smoothness required. Then, by the results of [11], we deduce that B maps $L^2 \times L^2$ into $L^1(\mathbb{R}^N)$. Let us give a few example

Example V.1 (The ordinary product). – $B(f, g) = fg$. Then, $m \equiv 1$.

Example V.2 (The pseudo-product of S. Dobyinsky [20]):

$$B(f, g) = fg - 2\int_0^\infty Q_t f Q_t g \frac{dt}{t}, \qquad \text{where} \quad Q_t = -t\frac{\partial}{\partial t}(e^{-t\Delta}).$$

Then, $m = (|\xi|^2 - |\eta|^2)^2/(|\xi|^2 + |\eta|^2)^2$.

We may now state our main result.

THEOREM V.1. – *With the above notations and assumptions, the following assertions are equivalent*:

(i) $\forall f, g \in C_0^\infty(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} B(f, g)\, dx = 0$,

(ii) $\forall f, g \in L^2(\mathbb{R}^N)$, $B(f, g) \in \mathscr{H}^1(\mathbb{R}^N)$,

(iii) $\forall f \in L^p(\mathbb{R}^N)$ $(1 < p < \infty)$, $\forall g \in L^{p'}(\mathbb{R}^N)$, $B(f, g) \in \mathscr{H}^1(\mathbb{R}^N)$,

(iv) *If* $f_n \overset{*}{\rightharpoonup} f$, $g_n \overset{*}{\rightharpoonup} g$ *weakly in* $L^2(\mathbb{R}^N)$, $B(f_n, g_n) \overset{*}{\rightharpoonup} B(f, g)$ *in* $\mathscr{D}'(\mathbb{R}^N)$.

(v) $m(\xi, -\xi) = 0$ *for all* $\xi \neq 0$.

Remark V.1. – If these conditions hold, one may then prove that if

$$f \in \mathscr{H}^p(\mathbb{R}^N), \quad g \in \mathscr{H}^q(\mathbb{R}^N), \qquad \left(p, q > \frac{N}{N+1}\right)$$

and

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}<1+\frac{1}{N},$$

then $B(f, g)\in \mathcal{H}^r(\mathbb{R}^N)$.

Remark V.2. – Clearly, the above conditions are satisfied in the Example V.2. The Example V.2 is systematically studied in S. Dobyinsky [20] where many interesting properties of this pseudo-product are investigated, together with its use to understand the structure of the nonlinear expressions we are considering in this work.

Remark V.3. – It is worth observing that if we choose $B(f, g)=f(Rg)+(Rf)g$ where R is a Riesz transform, then, $m=i(\xi_j/|\xi|+\eta_j/|\eta|)$ and m satisfies all the conditions of Theorem V.1 except for the smoothness requirement. Thus, Theorem III.1 is not really a consequence of Theorem V.1 (even if it should be...).

Proof of Theorem V.1. – Clearly, (iii) ⇒ (ii). And (ii) ⇒ (i) since $\int_{\mathbb{R}^N}\varphi\,dx=0$ for all $\varphi\in\mathcal{H}^1(\mathbb{R}^N)$. The implication (i) ⇒ (ii) is shown in R. Coifman and Y. Meyer [11] (or follows easily by duality from the results of [11]). Next, (i) and (v) are easily shown to be equivalent since for all $f, g\in C_0^\infty(\mathbb{R}^N)$

$$B(f, g)=(2\pi)^{-2N}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\hat f(\xi)\hat g(\eta)\,m(\xi,\eta)\,e^{-i(\xi+\eta)\cdot x}\,d\xi\,d\eta, \tag{37}$$

where $\hat f$, $\hat g$ are the Fourier transforms of f, g respectively. Therefore, we have also

$$\int_{\mathbb{R}^N}B(f, g)\,dx=(2\pi)^{-N}\int_{\mathbb{R}^N}\hat f(\xi)\hat g(-\xi)\,m(\xi,-\xi)\,d\xi.$$

And the equivalence between (i) and (v) is then clear.

We next show that (iv) implies (v). To this hand, we fix $\xi_0\neq 0$ and let

$$f_n(x)=e^{inx\cdot\xi_0}\varphi(x),\ g_n(x)=e^{-inx\cdot\xi_0}\varphi(x)$$

where $\varphi=e^{-|x|^2/2}$ (for instance). Clearly, $f_n, g_n \overset{*}{\rightharpoonup} 0$ weakly in $L^2(\mathbb{R}^N)$. Therefore, if (iv) holds, we have

$$\lim_n B(f_n, g_n)=0\quad\text{in } \mathcal{D}'(\mathbb{R}^N). \tag{38}$$

On the other hand, in view of (37), we find

$$\begin{aligned}B(f_n, g_n)&=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\hat\varphi(\xi-n\xi_0)\hat g(\eta+n\xi_0)\,m(\xi,\eta)\,e^{-i(\xi+\eta)\cdot x}\,d\xi\,d\eta\\&=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\hat\varphi(\xi)\hat g(\eta)\,e^{-i(\xi+\eta)\cdot x}\,m(\xi+n\xi_0,\eta-n\xi_0)\,d\xi\,d\eta.\end{aligned}$$

Then, (36) implies that $m(\xi+n\xi_0, \eta-n\xi_0)=m(\xi_0+\xi/n, -\xi_0+(\xi/n))$. And we deduce from the dominated convergence theorem that

$$B(f_n, g_n) \underset{n}{\rightarrow} m(\xi_0, -\xi_0) \iint_{\mathbb{R}^N \times \mathbb{R}^N} \hat{\varphi}(\xi)\hat{\varphi}(\eta) e^{-i(\xi+\eta)\cdot x} d\xi\, d\eta,$$

say uniformly on bounded sets. And we conclude easily from (38) that $m(\xi_0, -\xi_0)=0$ since

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \hat{\varphi}(\xi)\hat{\varphi}(\eta) e^{-i(\xi+\eta)\cdot x} d\xi\, d\eta = \varphi(x)^2 = e^{-|x|^2} > 0.$$

There only remains to show that (v) implies (iv). We thus assume (v). In view of (37), it is enough to show that if h_n converges weakly to h in $L^2(\mathbb{R}^N)$ and if $\varphi \in C_0^\infty(\mathbb{R}^N)$ then

$$(39) \qquad \int_{\mathbb{R}^N} h_n(\eta)\, m(\xi, \eta)\, \varphi(\xi+\eta)\, d\eta \underset{n}{\rightarrow} \int_{\mathbb{R}^N} h(\eta)\, m(\xi, \eta)\, \varphi(\xi+\eta)\, d\eta \quad \text{in } L^2(\mathbb{R}^N_\xi).$$

Clearly, this quantity converges pointwise (for $\xi \neq 0$) and is uniformly bounded. Therefore, in order to prove (39) we only have to show that

$$(40) \qquad \lim_{R\to\infty} \sup_n \int_{|\xi|\geqq R} |H_n(\xi)|^2\, d\xi = 0$$

where

$$H_n(\xi) = \int_{\mathbb{R}^N} h_n(\eta)\, m(\xi, \eta)\, \varphi(\xi+\eta)\, d\eta.$$

Let $R_0>0$ be such that Supp $\varphi \subset B(0, R_0)$. Then, we have

$$H_n(\xi) = \int_{|\xi+\eta|\leqq R_0} h_n(\eta)\, m(\xi, \eta)\, \varphi(\xi+\eta)\, d\eta$$

and thus because of (36)

$$H_n(\xi) = \int_{|\xi+\eta|\leqq R_0} h_n(\eta)\, m\left(\frac{\xi}{|\xi|}, \frac{\eta}{|\xi|}\right) \varphi(\xi+\eta)\, d\eta.$$

But, for $R \geqq 2R_0$ and $|\xi| \geqq R$, we have $|\eta| \geqq R/2$ and

$$\left|\frac{\eta}{|\xi|} - \left(-\frac{\xi}{|\xi|}\right)\right| \leqq \frac{R_0}{R}.$$

Therefore, from the regularity of m and the assumption (v), we deduce that for $R \geqq 2R_0$, $|\xi| \geqq R$, $|\xi+\eta| \leqq R_0$ we have

$$\left| m\left(\frac{\xi}{|\xi|}, \frac{\eta}{|\xi|}\right)\right| \leqq \frac{C}{R} \quad \text{for some } C \geqq 0 \text{ independent of } R.$$

Hence, we can estimate $H_n(\xi)$ as follows

$$|H_n(\xi)| \leqq \frac{C}{R} \int_{\mathbb{R}^N} |h_n(\eta)|\, |\varphi(\xi+\eta)|\, d\eta, \qquad \text{for} \quad |\xi| \geqq R \quad \text{and} \quad R \geqq 2R_0.$$

And (40) is proven since we obtain for $R \geqq 2R_0$

$$\begin{aligned}\int_{|\xi| \geqq R} |H_n(\xi)|^2\, d\xi &\leqq \frac{C^2}{R^2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} |h_n(\eta)|^2\, |\varphi(\xi+\eta)|\, d\eta\, d\xi \,.\, \|\varphi\|_{L^1} \\ &\leqq \frac{C^2}{R^2} \|\varphi\|_{L^1}^2 \|h_n\|_{L^2}^2.\end{aligned}$$

Remark V.3. − The implication (v) $\Rightarrow$ (iv) uses only the boundedness of m and the continuity of m for $|\xi| \neq 0$ and $|\eta| \neq 0$.

VI. General quadratic expressions

In this section, we want to work in the context of the general algebraic frameworks of compensated compactness studied in F. Murat ([34], [35], [36]), L. Tartar ([44], [45]) − related works include R. C. Rogers and B. Temple [40], B. Dacorogna ([16], [17]), B. Hanouzet [25], B. Hanouzet and J. L. Joly [26], A. Bachelot [2], P. Pedregal [37]... We will restrict our attention to quadratic nonlinearities even if our arguments can be adapted to general multilinear ones. As we shall see even for quadratic expressions, we seem to need a certain rank condition which is quite classical in the theory of compensated compactness and is needed there too at least for general multilinear quantities − even if for quadratic quantities it can be eliminated by a tricky argument due to L. Tartar. Since it is not clear how we can avoid this constant rank assumption*, we stick for clarity and brevity to the quadratic case and we shall only make later on a few remarks on non constant rank situations where we can work out some specific examples. These technical remarks being made, we now explain the setting.

Let q be a quadratic form on $\mathbb{R}^p$ $(p \geqq 1)$. Let $B: \mathbb{R}^p \times \mathbb{R}^N \to \mathbb{R}^m$ $(N, m \geqq 1)$ be bilinear and let us assume that q vanishes on $\Lambda = \{x \in \mathbb{R}^p / B(x, \xi) = 0 \text{ for some } \xi \in \mathbb{R}^N,\ \xi \neq 0\}$ − the critical condition in the theory of compensated compactness. We write

$$B(x, \xi)_i = \sum_{j=1}^{p} \sum_{k=1}^{N} B_{ijk}\, x_j\, \xi_k \qquad \text{for} \quad 1 \leqq i \leqq m.$$

* *Added in proofs:* this constant rank assumption has been removed by A. McIntosh, Macquarie University, NSW 2109 (Australia).

Finally, let $u \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$ satisfy

$$(41)\qquad \sum_{j=1}^{p}\sum_{k=1}^{N} B_{ijk}\frac{\partial u_j}{\partial x_k} \in W^{-1,r}_{\mathrm{loc}} \text{ for some } r>2, \text{ for } 1 \leqq i \leqq m.$$

Theorem VI.1. – *With the above notations and conditions and if, in addition, we assume the rank of* $B(.,\xi)$ *(as a linear map from* $\mathbb{R}^p$ *into* $\mathbb{R}^m$*) to be constant for* $\xi \neq 0$, *then* $q(u) \in \mathscr{H}^1_{\mathrm{loc}}$.

Of course, examples (and illustrations...) of such a setting can be found in the references we recalled above. Let us only briefly explain how the div-curl example of section II fits in this setting: we set $p=2N$, $u=(E, B)$, $m=(N(N-1)/2)$ (curl)$+1$ (div), $B(x, \xi)=(x_1 \wedge \xi, x_2 . \xi)$ where $x=(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$. Then, we find

$$\Lambda = \bigcup_{\xi \neq 0} (\mathbb{R}\xi) \times (\mathbb{R}\xi)^{\perp}.$$

Therefore, $q(x)=x_1 . x_2$ vanishes on Λ. Finally, rank $B(.,\xi)=N$ for all $\xi \neq 0$.

Proof of Theorem VI.1. – By a simple multiplication of y by a smooth cut-off function, we immediately deduce that, without loss of generality, we can assume that u is compactly supported (hence $u \in L^2$) and thus $W^{-1,r}_{\mathrm{loc}}$ can be replaced by $W^{-1,r}$ in (41) (in fact all these distributions are also compactly supported).

In order to prove that $q(u) \in \mathscr{H}^1_{\mathrm{loc}}$, we introduce $\varphi \in \mathscr{S}$ such that $\hat{\varphi}(\xi)=1$ if $|\xi| \leqq 1$, $\hat{\varphi}(\xi)=0$ if $|\xi| \geqq 2$ ($\hat{\varphi}$ denotes the Fourier transform of φ). Let $\varphi_t=(1/t^n)\varphi(./t)$ and define the operator P_t by $P_t f=\varphi_t * f$. Set $Q_t=t(d/dt)P_t$ so that $Q_t f=\psi_t * f$ where $\hat{\psi}(\xi)=0$ unless $1 \leqq |\xi| \leqq 2$.

We then write $q(u)=A(u, u)$ where A is a real symmetric bilinear form on $\mathbb{R}^p$. Thus, we can write

$$q(u)=P_1 q(P_1 u)-\int_0^1 t\frac{\partial}{\partial t}\{P_t A(P_t u, P_t u)\}\frac{dt}{t}$$

and thus

$$(42)\qquad q(u)=P_1 q(P_1 u)-\int_0^1 \{Q_t A(P_t u, P_t u)+2P_t A(Q_t u, P_t u)\}\frac{dt}{t}.$$

The first term is clearly smooth and we are then left to prove that the term defined by the integral belongs to $\mathscr{H}^1_{\mathrm{loc}}$.

We next rewrite that integral in the following way. We pick $\tilde{\varphi} \in \mathscr{S}$ so that $\hat{\tilde{\varphi}}(\xi)=1$ if $|\xi| \leqq 1/100$, $\hat{\tilde{\varphi}}(\xi)=0$ if $|\xi| \geqq 1/50$. We then define $\tilde{\varphi}_1$ and $\tilde{P}_t$ as we did before and we finally set $\tilde{Q}_t=P_t-\tilde{P}_t$. We next replace P_t by $\tilde{P}_t+\tilde{Q}_t$ in the above integral and expand the quadratic terms. All the terms involving twice $\tilde{P}_t$ in these expansions vanish in view of the restrictions on the supports of $\hat{\psi}_t$ and $\hat{\tilde{\varphi}}(\xi)$. And we are left with

$$(43)\quad \int_0^1 \{ Q_t A(\tilde{Q}_t u, \tilde{Q}_t u) + 2 Q_t A(\tilde{Q}_t u, \tilde{P}_t u) + 2 \tilde{Q}_t A(Q_t u, \tilde{Q}_t u) + 2 \tilde{Q}_t A(Q_t u, \tilde{P}_t u) + 2 \tilde{P}_t A(Q_t u, \tilde{Q}_t u) \} \frac{dt}{t}.$$

The next step consists in showing that all these terms but the last one belong to $\mathscr{H}^1$. In fact, this claim will be proven using only the fact that $u \in L^2$. In order to do so, we multiply all these terms by $b \in VMO$ (or BMO) and we integrate on $\mathbb{R}^N$. Since Q_t is self-adjoint, we only have to show that

$$(44)\quad \left\{ \begin{array}{l} \displaystyle\int_0^1 \int_{\mathbb{R}^N} |Q_t b| \{ |\tilde{Q}_t u|^2 + |\tilde{Q}_t u| \, |\tilde{P}_t u| \} \\ \qquad + |\tilde{Q}_t b| \{ |Q_t u| \, |\tilde{Q}_t u| + |Q_t u| \, |\tilde{P}_t u| \} \, dx \frac{dt}{t} \leqq C \|b\|_{BMO} \|u\|_{L^2}^2, \\ \text{for some } C \geqq 0 \text{ independent of } b \text{ and } u. \end{array} \right.$$

Then, we remark that we have by Plancherel equality

$$\int_0^1 \int_{\mathbb{R}^N} |\tilde{Q}_t u|^2 \, dx \frac{dt}{t} \leqq \tilde{C} \|u\|_{L^2}^2,$$

$$\int_0^1 \int_{\mathbb{R}^N} |Q_t u|^2 \, dx \frac{dt}{t} \leqq C \|u\|_{L^2}^2,$$

for some constants $\tilde{C}$, C which are given by respectively (up to some irrelevant constants depending only on N)

$$\tilde{C} = \left\| \int_0^\infty |\hat{\tilde{\psi}}(t\,.)|^2 \frac{dt}{t} \right\|_{L^\infty}, \qquad C = \left\| \int_0^\infty |\hat{\psi}(t\,.)|^2 \frac{dt}{t} \right\|_{L^\infty}.$$

Therefore, in order to prove the claim (44), using Cauchy-Schwarz inequality, we only have to show that terms of the form $\int_{\mathbb{R}^N} \int_0^1 |Q_t b|^2 |\tilde{P}_t u|^2 \, dx \, (dt/t)$ can be bounded by $C \|b\|_{BMO}^2 \|u\|_{L^2}^2$. And this follows easily from Carleson's inequality and the fact that $|Q_t b|^2 \, dx \, (dt/t)$ is a Carleson measure if $b \in BMO$.

Therefore, it only remains to prove that the last term of (43) belongs to $\mathscr{H}^1_{loc}$ namely that

$$(45)\quad \int_0^1 \tilde{P}_t A(Q_t u, \tilde{Q}_t u) \frac{dt}{t} \in \mathscr{H}^1_{loc}.$$

This is at this point that we shall really use the compensated compactness setting although we deduce in fact from the proof presented below that this quantity lies in a smaller space (L^p_{loc} + a smaller Besov space). In other words, $q(u) = q_1(u) + q_2(u)$ where $q_1(u) \in \mathscr{H}^1_{loc}$ only because $u \in L^2_{loc}$ and $q_2(u)$ belongs to a smaller space than $\mathscr{H}^1_{loc}$. This phenomenon is explained for certain nonlinear quantities (like the div-curl example) in S. Dobyinski [20].

For each $\xi \in \mathbb{R}^N$, let π_ξ denote the orthogonal projection onto $\{y \in \mathbb{R}^p / B(y, \xi) = 0\}$. Thus, π_ξ is homogeneous of degree 0 in ξ and it depends smoothly on ξ for $\xi \neq 0$ because of the constant rank condition. Let $\pi^\perp_\xi = I - \pi_\xi$ and define the following decomposition of u

$$\hat{u}^1(\xi) = \pi_\xi(\hat{u}(\xi)), \quad \hat{u}^2(\xi) = \pi^\perp_\xi(\hat{u}(\xi)), \tag{46}$$

so that $u = u^1 + u^2$. Of course, $u^1 \in L^2$.

We then want to use (41) to deduce that $u^2 \in L^r$. One way to prove this claim is to observe that by the definition of π_ξ, one can build a linear map (for each $\xi \neq 0$) $T_\xi : \mathbb{R}^m \to \mathbb{R}^p$ homogeneous of degree 0 in ξ and smooth in ξ for $|\xi| \neq 0$ such that $\pi^\perp_\xi(y) = |\xi|^{-1} T_\xi(B(y, \xi))$ for all $y \in \mathbb{R}^p$. By the classical multipliers theory we deduce that $u^2 \in L^r$. We then expand the term given by (45) using $u = u^1 + u^2$ and we find four terms, three of which can be analysed in a straightforward manner. More precisely, from standard maximal estimates, we see that

$$\int_0^1 \tilde{P}_t A(Q_t u^2, \tilde{Q}_t u^2) \frac{dt}{t} \in L^{r/2},$$

$$\int_0^1 \tilde{P}_t A(Q_t u^2, \tilde{Q}_t u^1) \frac{dt}{t}, \int_0^1 \tilde{P}_t A(Q_t u^1, \tilde{Q}_t u^2) \frac{dt}{t} \in L^s,$$

where $s > 1$ is defined by $1/s = (1/2) + (1/r)$.

Hence, there only remains to show that

$$\int_0^1 \tilde{P}_t A(Q_t u^1, \tilde{Q}_t u^1) \frac{dt}{t} \in \mathscr{H}^1_{loc}. \tag{47}$$

Notice of course that u^1 satisfies: $u^1 \in L^2$ and

$$\sum_{j=1}^{p} \sum_{k=1}^{N} B_{ijk} \frac{\partial u^1_j}{\partial x_k} = 0 \quad \text{for} \quad 1 \leqq i \leqq m. \tag{48}$$

We then compute the Fourier transform of $A(Q_t u^1, \tilde{Q}_t u^1)$. We can of course assume that ψ and $\tilde{\psi}$ are radial and real so that the same is true of their Fourier transforms. We find

$$\mathscr{F}(A(Q_t u^1, \tilde{Q}_t u^1))(\xi) = \int_{\mathbb{R}^N} A(\hat{\psi}(t(\xi - \eta)) \hat{u}^1(\xi - \eta), \hat{\tilde{\psi}}(t\eta) \hat{u}_1(\eta)) \, d\eta.$$

Since we have on one hand $\pi_\eta = \pi_{-\eta}$, $\pi_{\xi-\eta}(\hat{u}^1(\xi-\eta)) = \hat{u}^1(\xi-\eta)$, $\pi_\eta(\hat{u}^1(\eta)) = \hat{u}^1(\eta)$ and on the other hand $A(\pi_\eta(x), \pi_\eta(y)) = 0$ ($\forall x, y \in \mathbb{R}^p$) for q vanishes on Λ, we can write

$$\mathscr{F}(A(Q_t u^1, \tilde{Q}_t u^1))(\xi) = \int_{\mathbb{R}^N} A(\hat{\psi}(t(\xi-\eta))\pi_{\xi-\eta}(\hat{u}^1(\xi-\eta)), \hat{\tilde{\psi}}(t\eta)\hat{u}_1(\eta))\,d\eta - \int_{\mathbb{R}^N} A(\hat{\psi}(t(\xi-\eta))\pi_\eta(\hat{u}^1(\xi-\eta)), \hat{\tilde{\psi}}(t\eta)\hat{u}_1(\eta))\,d\eta$$

hence, finally

$$\mathscr{F}(A(Q_t u^1, \tilde{Q}_t u^1))(\xi) = \int_{\mathbb{R}^N} A(\pi_{\xi-\eta} - \pi_{-\eta})(\hat{\psi}(t(\xi-\eta))\hat{u}^1(\xi-\eta)), \hat{\tilde{\psi}}(t\eta)\hat{u}^1(\eta)\,d\eta. \tag{49}$$

In view of the properties satisfied by π_ξ, we can write

$$\pi_{\xi-\eta} - \pi_{-\eta} = \sum_{i=1}^{N} \xi_i m_i(\xi, \eta), \tag{50}$$

where each m_i is a smooth matrix-valued function defined for $|\xi| \leqq 1/20$, $1/2 \leqq |\eta| \leqq 1$. We can of course extend m_i to a C^∞ function on $\mathbb{R}^N \times \mathbb{R}^N$ with compact support, even if (50) will then only hold on the afore-mentioned range. In addition, since m_i is smooth for each i, we can represent m_i as

$$m_i(\xi, \eta) = \sum_\alpha \mu_{i\alpha} f_{i\alpha}(\xi-\eta) g_{i\alpha}(\eta), \tag{51}$$

where $\|f_{i\alpha}\|_{L^\infty} \leqq 1$ ($f_{i\alpha}$ is matrix-valued), $\|g_{i\alpha}\|_{L^\infty} \leqq 1$ ($g_{i\alpha}$ is scalar) and $\sum_\alpha |\mu_{i\alpha}| \leqq C$ for all $i \in \{1, \ldots, N\}$. Thus, when $|\xi| \leqq 1/(20t)$, $1/(2t) \leqq |\eta| \leqq (1/t)$ [$t \in (0, 1)$], we have

$$\pi_{\xi-\eta} - \pi_{-\eta} = \pi_{t\xi-t\eta} - \pi_{-t\eta} = \sum_{i,\alpha} t\xi_i \mu_{i\alpha} f_{i\alpha}(t(\xi-\eta)) g_{i\alpha}(t\eta),$$

where we used (50), (51), and the homogeneity of π_ξ with respect to ξ.

We may now go back to (49) and we obtain

$$\mathscr{F}(A(Q_t u^1, \tilde{Q}_t u^1))(\xi) = \int_{\mathbb{R}^N} \sum_{i,\alpha} t\xi_i \mu_{i\alpha} A(f_{i\alpha}(t(\xi-\eta))\hat{\psi}(t(\xi-\eta))\hat{u}^1(\xi-\eta), g_{i\alpha}(t\eta)\hat{\tilde{\psi}}(t\eta)\hat{u}^1(\eta))\,d\eta.$$

Therefore, we have

$$A(Q_t u^1, \tilde{Q}_t u^1) = \sum_{i,\alpha} t \frac{\partial}{\partial x_i}[\mu_{i\alpha} A(F_t^{i\alpha} Q_t u^1, G_t^{i\alpha} \tilde{Q}_t u^1)],$$

where $F_t^{i\alpha}$, $Q_t^{i\alpha}$ are defined by

$$\mathscr{F}(F_t^{i\alpha}(\varphi))(\xi)=f_{i\alpha}(t\xi)\,\hat{\varphi}(\xi),$$
$$\mathscr{F}(G_t^{i\alpha}(\varphi))(\xi)=g_{i\alpha}(t\xi)\,\hat{\varphi}(\xi).$$

Then, if we introduce $\hat{Q}_t^i=t(\partial/\partial x_i)\tilde{P}_t$, (41) might be rewritten as

$$\sum_\alpha \mu_{i\alpha}\int_0^1 \hat{Q}_t^i\{A(F_t^{i\alpha}Q_t(u^1),\, G_t^{i\alpha}\tilde{Q}_t(u^1))\}\frac{dt}{t}.$$

And one easily deduces from standard square function estimates the fact that this quantity belongs to $\mathscr{H}^1$.

Remark VI.1. – By a careful inspection of the above argument, we obtain the following

COROLLARY VI.1. – *Under the same conditions as in Theorem* VI.1 *with* $u\in L^2_{\text{loc}}$ *replaced by* $u\in L^q_{\text{loc}}$ *where* $2N/(N+1)<q<2$ *and* $r>q$ *in* (41) *replaced by* $r\geqq q'$, *then* $q(u)\in\mathscr{H}^{q/2}_{\text{loc}}$.

Remark VI.2. – It is plausible that one only needs $r>q$.

Remark VI.3. – If $q=2N/(N+1)$, working a bit more, one can show the same result as above replacing $\mathscr{H}^{q/2}_{\text{loc}}$ by the closure of $C_0^\infty(\mathbb{R}^N)$ in the "weak $\mathscr{H}^{q/2}_{\text{loc}}$" space.

We now conclude this section with a typical example where the constant rank assumption is not satisfied but weak continuity results are known (*see* F. Murat [36]). We are going to show that the $\mathscr{H}^1$ regularity is still true strongly indicating that the constant rank assumption is not optimal. An extension of this setting can be found in P. Pedregal [37] and our analysis also extends to the same setting.

Let $N\geqq 2$, let $u\in L^N_{\text{loc}}(\mathbb{R}^N)^N$ satisfy

$$\frac{\partial u_i}{\partial x_j}\in W^{-1,r}_{\text{loc}}\quad\text{for some } r>N,\quad \text{for all } 1\leqq i\neq j\leqq N. \tag{52}$$

And we consider $P=\prod_{i=1}^N u_i(x)$. We claim that $P\in\mathscr{H}^1_{\text{loc}}$.

Let us first check, in the case $N=2$ for instance, that the constant rank assumption is not satisfied: we take $p=N=2$, $m=2$, $B(x,\xi)=(x_1\xi_2, x_2\xi_1)$ so that

$$\Lambda=(\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R})\qquad\text{and}\qquad q(x)=x_1x_2$$

clearly vanishes on Λ. Now, rank $(B(.,\xi))=2$ if $\xi_1\neq 0$ and $\xi_2\neq 0$, rank $(B(.,\xi))=1$ if ξ_1 or $\xi_2=0$ (and $\xi\neq 0$).

We now want to show that $P\in\mathscr{H}^1_{\text{loc}}$. To this end, we first localize with a smooth cut-off function: thus, we may assume without loss of generality that $u\in L^N(\mathbb{R}^N)^N$ is compactly supported. Next, we write $u_i=-\sum_{\alpha=1}^N R_\alpha^2 u_i$ and we observe that (52) yields:

$R_\alpha^2 u_i \in L_{loc}^r$ for some $r > N$, for all $1 \leqq i \neq \alpha \leqq N$. Therefore,

$$P = (-1)^N \sum_{j_1=1}^{N} \dots \sum_{j_N=1}^{N} \prod_{k=1}^{N} R_{j_k}^2 u_k$$

and in view of the preceding remark, all these terms but one belong to L_{loc}^q for some $q > 1$ $(1/q = (1/r) + ((N-1)/N))$. Hence, we only have to show that $\prod_{i=1}^{N} R_i^2 u_i \in \mathscr{H}_{loc}^1$. We then denote by $v_i = R_i u_i$ and we observe that we still have $R_\alpha v_i \in L_{loc}^r$ for some $r > N$, for all $1 \leqq i \neq \alpha \leqq N$. Then, we need to show that $\prod_{i=1}^{N} R_i v_j \in \mathscr{H}_{loc}^1$ or, equivalently, by the same argument as above, that $\det(R_i v_j) \in \mathscr{H}_{loc}^1$. But, this is precisely the $\mathscr{H}_{loc}^1$ regularity for the Jacobian: set $f_j = (-\Delta)^{-1/2} v_j$, $f \in W^{1,N}(\mathbb{R}^N)^N$, then $\det(R_i v_j) = \det(\nabla f)$!

Remark VI.4. – It is possible to make another (and more general) proof of the above claim, using a methodology quite similar to the proof of Theorem VI.1.

VII. Examples with two cancellations

We have seen in section V how much the results presented in this work (and compensated compactness phenomena) depend upon some cancellation. This cancellation also allowed (*see* sections II, III, V, VI) to define these "cancelling" nonlinear expressions "below L^1" and to verify that they belong to some Hardy spaces. We want to show in this section on a few examples taken from PDE's theory that this can be pushed further if more cancellations are present – *i.e.* if higher moments vanish. In order to keep the ideas clear – and in an unsuccessful attempt to limit the length of this paper –, we shall restrict our attention to four examples where two moments vanish (two cancellations). This rather vague terminology will become clear in the course of discussing these examples. Abstract formulations covering these four examples are certainly possible if not necessarily interesting-one possible direction is to extend the analysis made in section V and it is investigated in R. Coifman and L. Grafakos [9].

We next present our model examples.

Example VII.1. – Let u, v satisfy

$$\left\{\begin{array}{ll} \nabla u \in \mathscr{H}^p(\mathbb{R}^N), & \nabla v \in \mathscr{H}^q(\mathbb{R}^N), \\ \operatorname{div} u = \operatorname{div} v = 0 & \text{in } \mathscr{D}'(\mathbb{R}^N), \end{array}\right. \tag{53}$$

where $N/(N+1) < p$, $q < \infty$ – so that $u.v \in L_{loc}^1$... It is in fact possible to take only p, $q > N/(N+2)$ in the analysis below but this extension would create some unpleasant technicalities and we prefer to skip it.

We wish to consider $\sum_{i,j=1}^{N} (\partial/\partial x_j)(u_i)(\partial/\partial x_i)(v_j)$ which of course, as such, is not really meaningful. In order to define this quantity in a proper way, we observe that when

$p=q=2$ then it can be rewritten as

$$\sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j}(u_i v_j).$$

And we have the following result whose proof we postpone until we present all the examples.

THEOREM VII.1. – *Assume* (53) *and* $1/r=(1/p)+(1/q)<1+(2/N)$, *then*

$$\sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j}(u_i v_j) \in \mathscr{H}^r(\mathbb{R}^N).$$

Remark VII.1. – Observe that $u_i v_j \in L^1_{loc}$, $u \in L^{p^*}$, $v \in L^{q^*}$ and

$$\frac{1}{p^*}=\frac{1}{p}-\frac{1}{N}, \qquad \frac{1}{q^*}=\frac{1}{q}-\frac{1}{N},$$

so that

$$\frac{1}{p^*}+\frac{1}{q^*}=\frac{1}{p}+\frac{1}{q}-\frac{2}{N}<1,$$

(at least if $p, q<N$, otherwise the claim is even simpler to prove...).

Remark VII.2. – As usual, the case $1/r=1+(2/N)$ can be treated as well and we obtain in that case that the above quantity lies in the closure of $C_0^\infty(\mathbb{R}^N)$ in the space "weak $\mathscr{H}^{N/(N+2)}$".

Example VII.2. – Let $N \geqq 2$, $u \in W^{2,p}(\mathbb{R}^N)$ where $p>N^2/(N+2)$. We want to consider $\det(D^2 u)$ and we first need to explain how to define it. To simplify the algebra and keep the ideas clear, we do so only for $N=2$. Then, if $u \in H^2_{loc}(\mathbb{R}^N)$, we observe that we have

$$\det(D^2 u)=\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^1}-\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2=\frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\right) -\frac{1}{2}\frac{\partial^2}{\partial x_1^2}\left(\left(\frac{\partial u}{\partial x_2}\right)^2\right)-\frac{1}{2}\frac{\partial^2}{\partial x_2^2}\left(\left(\frac{\partial u}{\partial x_1}\right)^2\right).$$

And this last expression makes sense as soon as $\nabla u \in L^2_{loc}(\mathbb{R}^2)^2$ which is the case as soon as $D^2 u \in L^1_{loc}$ (or even is a measure). Notice that in this case $N^2/(N+2)=1$.

THEOREM VII.2. – *Let* $N \geqq 2$, $u \in W^{2,p}(\mathbb{R}^N)$ *with* $p>N^2/(N+2)$. *Then, the expression*

$$\frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\right)-\frac{1}{2}\frac{\partial^2}{\partial x_1^2}\left(\left(\frac{\partial u}{\partial x_2}\right)^2\right)-\frac{1}{2}\frac{\partial^2}{\partial x_2^2}\left(\left(\frac{\partial u}{\partial x_1}\right)^2\right)$$

belongs to $\mathscr{H}^r(\mathbb{R}^N)$ *with* $r=p/N$.

Remark VII.3. — If $N \geqq 3$ and $p/N = N/(N+2)$ or if $N=2$ and $D^2 u$ is a bounded measure, then the above result still holds with $\mathcal{H}^r$ replaced, as usual, by the closure of $C_0^\infty(\mathbb{R}^N)$ in the "weak $\mathcal{H}^{N/(N+2)}$".

Example VII.3. — Let $N \geqq 1$, let $u \in L^p(\mathbb{R}^N)$ with $2 < p \leqq \infty$ satisfy $\nabla u \in \mathcal{H}^r(\mathbb{R}^N)$ with $r > 2N/(N+2)$ and $\Delta u \in \mathcal{H}^q(\mathbb{R}^N)$ with $q > N/(N+2)$. We assume that $(1/p)+(1/q) = 2/r$. We consider the quantity $(\Delta u) u + |\nabla u|^2$ that we define to be $(1/2)\Delta(|u|^2)$.

THEOREM VII.3. — *Under the above conditions,* $(1/2)\Delta(|u|^2) \in \mathcal{H}^s(\mathbb{R}^N)$ *with* $s = r/2$.

Example VII.4. — Let $N \geqq 2$, let $u \in W^{2,p}(\mathbb{R}^N)$ with $p > 2N/(N+2)$ $(N \geqq 2)$. We consider the quantity $|\Delta u|^2 - \sum_{i,j=1}^{N} |\partial^2 u/\partial x_i \partial x_j|^2$ that we define to be

$$\sum_{i \neq j} \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right) - \frac{1}{2}\frac{\partial^2}{\partial x_i^2}\left(\left(\frac{\partial u}{\partial x_j}\right)^2\right) - \frac{1}{2}\frac{\partial^2}{\partial x_j^2}\left(\left(\frac{\partial u}{\partial x_i}\right)^2\right).$$

Notice that this last expression makes sense in view of Sobolev's embeddings.

THEOREM VII.4. — *Under the above conditions, the above quantity belongs to* $\mathcal{H}^{p/2}(\mathbb{R}^N)$.

Remark VII.4. — Again, the limit case $p = 2N/(N+2)$ can be treated as well (see the above remarks)...

Remark VII.5. — If $N=2$, the examples 1, 2 and 4 coincide.

Remark VII.6. — It is possible to combine the examples 1 and 4 by considering all minors of the Hessian matrix $D^2 u$. We can then prove that they belong to $\mathcal{H}^{p/k}$ if the minor is of order k and $u \in W^{2,p}(\mathbb{R}^N)$ for $p > k\, N/(N+2)$.

Before briefly explaining the proofs, we want to make a "fundamental" observation on all the quantities, denoted generically by C, we introduced in the above examples. This observation will explain what we mean by cancellation and is really the heart of the matter. Indeed, since all these quantities may be written as second derivatives of some functions in $L^p(\mathbb{R}^N)$ for some $p > 1$, therefore, at least formally, we expect

$$\int_{\mathbb{R}^N} C\, dx = 0, \qquad \int_{\mathbb{R}^N} C\, x_j\, dx = 0, \quad (\forall\, 1 \leqq j \leqq N). \tag{54}$$

We now prove Theorem VII.1. Since the proofs of Theorem VII.2 and VII.4 are very much similar, we shall skip them. We have to estimate

$$h_t * \left(\sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j}(u_i v_j)\right)(x)$$
$$= \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} h_t(x-y) \frac{\partial^2}{\partial y_i \partial y_j}\left[\left(u_i - ⨍_{B_t^x} u_i\right)\left(v_i - ⨍_{B_t^x} v_i\right)\right] dy$$

$$= \sum_{i,j=1}^{N} \int_{B_t^x} \frac{1}{t^N} \left(\frac{\partial^2 h}{\partial x_i \partial x_j} \right) \left(\frac{x-y}{t} \right) \times \left\{ \frac{1}{t} \left(u_i - \fint_{B_t^x} u_i \right) \right\} \left\{ \frac{1}{t} (v_j - \fint_{B_t^x} v_j) \right\} dy.$$

Therefore,

$$\left| h_t * \left(\sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} (u_i v_j) \right) \right| \leqq C \sum_{i,j=1}^{N} \int_{B_t^x} \left| \frac{1}{t} \left(u_i - \fint_{B_t^x} u_i \right) \right| \left| \frac{1}{t} \left(v_j - \fint_{B_t^x} v_j \right) \right| dy.$$

And we deduce from Hölder's inequality

$$\sup_{t>0} \left| h_t * \left(\sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} (u_i v_j) \right) \right| \leqq C \left[\sup_{t>0} \fint_{B_t^x} \left| \frac{1}{t} \left(u - \fint_{B_t^x} u \right) \right|^{\alpha} \right]^{1/\alpha} \cdot \left[\sup_{t>0} \fint_{B_t^x} \left| \frac{1}{t} \left(v - \fint_{B_t^x} v \right) \right|^{\beta} \right]^{1/\beta}.$$

where α, β satisfy: $(1/\alpha) + (1/\beta) = 1$, $1 < \alpha < p^* = Np/(N-p)$, $1 < \beta < q^* = Nq/(N-q)$. This is clearly possible since

$$\frac{1}{p^*} + \frac{1}{q^*} = \frac{1}{p} + \frac{1}{q} - \frac{2}{N} < 1$$

by assumption. Using Lemma II.2, we deduce that

$$\left[\sup_{t>0} \fint_{B_t^x} \left| \frac{1}{t} \left(u - \fint_{B_t^x} u \right) \right|^{\alpha} \right]^{1/\alpha}, \quad \left[\sup_{t>0} \fint_{B_t^x} \left| \frac{1}{t} \left(v - \fint_{B_t^x} v \right) \right|^{\beta} \right]^{1/\beta},$$

belong respectively to $L^p(\mathbb{R}^N)$, $L^q(\mathbb{R}^N)$. Therefore, $\sup_{t>0} \left| h_t * \left(\sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} (u_i v_j) \right) \right|$ belongs to $L^r(\mathbb{R}^N)$ with $1/r = (1/p) + (1/q)$ and Theorem VII.1 is proven.

We conclude this section by proving Theorem VII.3. We write

$$h_t * \left(\frac{1}{2} \Delta |u|^2 \right)(x) = \int_{\mathbb{R}^N} h_t(x-y) \frac{1}{2} \Delta \left| u - \fint_{B_t^x} u \right|^2 dy + \fint_{B_t^x} u \,.\, (\Delta u * h_t)$$

$$= \fint_{B_t^x} c_N \Delta h \left(\frac{x-y}{t} \right) \frac{1}{2} \left| \frac{1}{t} \left(u - \fint_{B_t^x} u \right) \right|^2 dy + \fint_{B_t^x} u \,.\, (\Delta u * h_t).$$

Therefore, we have

$$\sup_{t>0} \left| h_t * \left(\frac{1}{2} \Delta |u|^2 \right) \right| \leqq C \sup_{t>0} \fint_{B_t^x} \left| \frac{1}{t} \left(u - \fint_{B_t^x} u \right) \right|^2 dy + M(u) \sup_{t>0} |\Delta u * h_t|.$$

We may then conclude in view of Lemma II.2 since

$$\sup_{t>0} \fint_{B_t^x} \left|(1/t)\left(u - \fint_{B_t^x} u\right)\right|^2 dy \in L^{r/2}(\mathbb{R}^N),\ M(u) \in L^p(\mathbb{R}^N),\ \sup_{t>0} |\Delta u * h_t| \in L^q(\mathbb{R}^N),$$

with $(1/p)+(1/q)=2/r$.

VIII. Pointwise definition of these nonlinear quantities

We have seen in the preceding sections that it is possible to define the "compensated compactness" nonlinear expressions as distributions "below L^1" and to prove they belong to some $\mathscr{H}^p$. On the other hand, in most of the situations where we did so, it is also possible to define these expressions pointwise, obtaining thus measurable functions that lie into L^p. We want to explain in this section the relationships between these definitions. Of course, we do not want to go through the full list of examples we treated in the preceding sections and we shall explain what can be shown in general on only one example namely the div-curl example.

Hence, let us take $E \in L^p(\mathbb{R}^N)^N$, $B \in L^q(\mathbb{R}^N)^N$ where $1<p<\infty$, $1<q<\infty$ and $(1/p)+(1/q)<1+(1/N)$ and let us assume

$$\text{div } B=0, \qquad \text{curl } E=0, \quad \text{in } \mathscr{D}'(\mathbb{R}^N). \tag{55}$$

We have seen (for instance in section II) that $E.B$ may be defined as a distribution and then lies into $\mathscr{H}^r(\mathbb{R}^N)$ with $1/r=(1/p)+(1/q)$. Recall that $E.B$ is defined by: $\text{div}(B\pi)$ where $\nabla\pi=E$. Of course, this result also shows that if we smoothe E and B say $E_t=E*h_t$, $B_t=B*h_t$ then $E_t.B_t$ converges in $\mathscr{H}^r$, as t goes to 0_+, to $\text{div}(B\pi)$.

On the other hand, the product $E.B$ makes sense pointwise and yields a measurable function which belongs to $L^r(\mathbb{R}^N)$. The relationships between those quantities is not clear when $r<1$, in particular if the pointwise product does not belong to L^1. But, even if the pointwise product belongs to L^1, the two quantities are in general different. Let us give one example of such a phenomenon (many more interesting examples are possible but we shall not pursue this matter here). Take $\pi=x_1/r$ so that

$$E=\left(\frac{1}{r}-\frac{x_1^2}{r^3},\ -\frac{x_1 x_2}{r^3}\right) \in L^p_{\text{loc}}(\mathbb{R}^2) \quad \text{for all } p<2\ (E \in L^{2,\infty})$$

and take

$$B=\left(\frac{1}{r}-\frac{x_2^2}{r^3},\ \frac{x_1 x_2}{r^3}\right) \in L^p_{\text{loc}}(\mathbb{R}^2) \quad \text{for all } p<2\ (B \in L^{2,\infty});$$

then, $E.B$ computed pointwise vanishes identically while $\text{div}(B\pi)=2\pi\delta_0$.

To clearly distinguish between these two definitions, we write $(E.B)_d=\text{div}(B\pi)$ and $(E.B)_{ae}$ the pointwise defined measurable function.

In order to state our main result connecting these two quantities, we have to recall a more or less classical fact on Hardy spaces $\mathscr{H}^r(\mathbb{R}^N)$ when $r\in(0,1)$ – it is a simple consequence of the maximal function characterization of $\mathscr{H}^p$. There exists a linear, continuous map P from $\mathscr{H}^p$ into L^p such that $P(f)=f$ if $f\in\mathscr{H}^p\cap L^1_{loc}$ and $f*h_t$ converges a.e. to $P(f)$ (and in L^p) as t goes to 0_+ for every $f\in\mathscr{H}^p$. In other words, one can define the "pointwise part" of elements of $\mathscr{H}^p$. For instance, if f is a bounded measure, $P(f)$ is its regular part.

Then, we have the

COROLLARY VIII.1. – *Let* $E\in L^p(\mathbb{R}^N)^N$, $B\in L^q(\mathbb{R}^N)^N$ *where* $1<p<\infty$, $1<q<\infty$ *and* $(1/p)+(1/q)<1+(1/N)$. *Assume that* (55) *holds. Then,* $(E.B)_d\in\mathscr{H}^r$ *with* $1/r=(1/p)+(1/q)$ *and* $P((E.B)_d)=(E.B)_{ae}$.

Remark VIII.1. – The same result holds locally.

Remark VIII.2. – We can also treat the borderline case $(1/p)+(1/q)=1+(1/N)$.

Remark VIII.3. – It is even possible to take $E=\mathscr{H}^p(\mathbb{R}^N)^N$, $B\in\mathscr{H}^q(\mathbb{R}^N)$ where $0<p$, $q<\infty$ with $(1/p)+(1/q)<1+(1/N)$. Then $P((E.B)_d)=(P(E).P(B))_{ae}$.

The above result is indeed a consequence of the arguments of section II since it suffices to show that $\{(E.B)_d\}*h_t-(E*h_t).(B*h_t)$ converges a.e. to 0 as t goes to 0_+. We recall from section II that we have

$$\{(E.B)_d\}*h_t=\int\left[\pi(y)-\fint_{B_t^x}\pi\right]B(y).\left[\frac{1}{t^{N+1}}\nabla h\left(\frac{x-y}{t}\right)\right]dy$$

$$=E_t.B_t+\int\left[\pi(y)-\fint_{B_t^x}\pi\right][B-B_t(x)].\left[\frac{1}{t^{N+1}}\nabla h\left(\frac{x-y}{t}\right)\right]dy,$$

where we denote by $E_t=E*h_t$, $B_t=B*h_t$.

Therefore, exactly as in section II, we deduce

$$|\{(E.B)_d\}*h_t-E_t.B_t|\leq C\left(\fint_{B_t^x}|E|^\alpha dy\right)^{1/\alpha}\left(\fint_{B_t^x}|B-B_t(x)|^\beta dy\right)^{1/\beta}$$

for some α, β satisfying: $1<\alpha<p$, $1<\beta<q$, $(1/\alpha)+(1/\beta)=1+(1/N)$.

This allows us to conclude since the right-hand side goes to 0 a.e. in x by classical measure theory results.

Let us finally conclude this section by mentioning that the above result contains a recent result by S. Müller [32] showing under the same conditions that if $(E.B)_d\in L^1_{loc}$, then $(E.B)_d=(E.B)_{ae}$. This is clearly the case since $P((E.B)_d)=(E.B)_d$ in that case by the very definition of P.

IX. Applications

We begin by showing how the $\mathscr{H}^1$ regularity of the Jacobian yields various known results. First of all, since $W^{1,N}(\mathbb{R}^N) \subsetneq \text{VMO}$, by duality $\mathscr{H}^1 \subsetneq W^{-1,N}(\mathbb{R}^N)$. Therefore, if $u \in W^{1,N}(\mathbb{R}^N)^N$, $\det(\nabla u) \in W^{-1,N}(\mathbb{R}^N)$ – a fact shown by H. Wente [47], L. Tartar [46], H. Brezis and J. M. Coron [7] when $N=2$, note that the proof in [46] can be adapted to the case $N \geqq 3$. In addition, if we solve

$$(56) \qquad -\Delta\varphi = \det(\nabla u) \quad \text{in } \mathbb{R}^2,$$

(a unique solution vanishing at infinity exists...), then it was shown that, if $u \in W^{1,2}(\mathbb{R}^2)^2$, then $\varphi \in C_0(\mathbb{R}^2)$ (*see* [47], [7]) and even $\varphi \in \mathscr{F} L^1(\mathbb{R}^2)$ (*see* [46]). This last result can be extended a bit since, for all $1 \leqq i, j \leqq 2$, $\partial^2 \varphi/\partial x_i \partial x_j = R_i R_j(\det(\nabla u)) \in \mathscr{H}^1(\mathbb{R}^2)$. And if $\Delta\varphi \in \mathscr{H}^1(\mathbb{R}^2)$ then $\hat{\varphi} = (1/|\xi|^2)\widehat{(\Delta\varphi)} \in L^1(\mathbb{R}^2)$. Indeed, recall that if $f \in \mathscr{H}^1(\mathbb{R}^N)$ then $(1/|\xi|^N)\hat{f} \in L^1(\mathbb{R}^N)$.

Another result that can be deduced from the $\mathscr{H}^1$ regularity of the jacobian is of course the result by S. Müller [31]: indeed, if $u \in W^{1,N}_{\text{loc}}(\mathbb{R}^N)^N$ and $\det(\nabla u) \geqq 0$ a.e., then $(\det \nabla u)\log(\det(\nabla u)) \in L^1_{\text{loc}}$ just because ([41], [42]) $\varphi \geqq 0$ belongs to $\mathscr{H}^1_{\text{loc}}$ if and only if $\varphi \log \varphi \in L^1_{\text{loc}}$.

The next fact we want to mention is the crucial role played by the improved $\mathscr{H}^1$ regularity in the results of F. Hélein ([27], [28]) about the regularity of weak harmonic maps from two dimensional open manifolds into arbitrary manifolds – see also L. C. Evans [21]. We do not want of course to repeat the delicate arguments in [27], [28] but it is possible to repeat them in one simple case namely for a weak harmonic map from an open set Ω in $\mathbb{R}^2$ into S^N ($N \geqq 2$). We thus consider $u \in H^1(\Omega)^{N+1}$ such that $|u|=1$ a.e. in Ω and

$$(57) \qquad -\Delta u = u|\nabla u|^2 \quad \text{in } \mathscr{D}'(\Omega).$$

By standard elliptic theory, it is easy to deduce that $u \in C^\infty(\Omega)^{N+1}$ if we show that $u \in C(\Omega)^{N+1}$. And by the arguments shown above, it is enough to show that $\Delta u \in \mathscr{H}^1_{\text{loc}}(\Omega)^{N+1}$. To this end, let us observe first that (57) implies

$$(58) \qquad \operatorname{div}(u_i \nabla u_j - u_j \nabla u_i) = 0 \quad \text{in } \mathscr{D}'(\Omega), \quad \text{for all } 1 \leqq i,\ j \leqq N+1.$$

And since $|u|=1$ a.e., we also find

$$(59) \qquad \sum_{j=1}^{N+1} u_j \frac{\partial u_j}{\partial x_i} = 0 \quad \text{a.e. in } \Omega, \quad \text{for all } 1 \leqq i \leqq 2.$$

Combining (57) and (59), we may write for all $1 \leqq j \leqq N+1$

$$-\Delta u_j = u_j|\nabla u|^2 = \sum_{k=1}^{N+1} (u_j \nabla u_k - u_k \nabla u_j).\nabla u_k \quad \text{in } \mathscr{D}'(\Omega).$$

Next, we see that $u_j \nabla u_k - u_k \nabla u_j \in L^2(\Omega)^q$, $\nabla u_k \in L^2(\Omega)^2$ and in view of (58) we deduce that for all $1 \leqq j, k \leqq N+1$

$$(u_j \nabla u_k - u_k \nabla u_j) \cdot \nabla u_k \in \mathscr{H}^1_{loc}(\Omega).$$

And we conclude the proof of the regularity of u.

We now conclude this section with a few remarks on weak solutions of incompressible Navier-Stokes equations in 3 dimensions: we thus consider

$$u \in L^2(0, \infty; H^1(\mathbb{R}^3))^3 \cap L^\infty(0, \infty; L^2(\mathbb{R}^3))^3$$

satisfying

$$\left\{ \begin{aligned} &\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u + \nabla p = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^3, \\ &\operatorname{div} u = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^3, \end{aligned} \right. \tag{60}$$

for some $p \in L^1(0, T; L^1_{loc}(\mathbb{R}^3))$ $(\forall T < \infty)$. We assume that $\nu > 0$ and we normalize the pressure by assuming it vanishes at infinity in a rather weak sense like, for example, $\{|p| \geqq \delta\}$ has finite measure in $((0, T) \times \mathbb{R}^3)$ for all $\delta > 0$ (for all $T < \infty$). Then we have the

THEOREM IX.1. – *With the above notations and conditions,*

$$(u \cdot \nabla) u, \ \nabla p \in L^2(0, \infty; \mathscr{H}^1(\mathbb{R}^3))^3, \qquad \partial^2 p / \partial x_i \partial x_j \, (1 \leqq i, j \leqq 3) \in L^1(0, \infty; \mathscr{H}^1(\mathbb{R}^3))$$

and thus

$$\nabla p \in L^1(0, \infty; L^{3/2, 1}(\mathbb{R}^3))^3, \qquad p \in L^1(0, \infty; L^{3, 1}(\mathbb{R}^3))^3.$$

In addition, $u \in L^1(0, T; C_0)^3$ $(\forall T < \infty)$ *and if* curl u *is a bounded measure on* $\mathbb{R}^3$ *then* $\nabla u \in L^\infty(0, \infty; L^1(\mathbb{R}^3))$.

Remark IX.1. – Those last two facts are essentially known: the first one was shown in [24] while the second one is a small extension of a result by P. Constantin [15].

Proof. – By the results of section II, we see that $(u \cdot \nabla) u \in \mathscr{H}^1(\mathbb{R}^3)$ a.e. $t \in (0, \infty)$ and $\|(u \cdot \nabla) u\|_{\mathscr{H}^1} \leqq C \|u\|_{L^2} \|\nabla u\|_{L^2}$ and thus $(u \cdot \nabla) u \in L^2(0, \infty; \mathscr{H}^1)^3$. Next, we take the divergence of (60) and we find

$$-\Delta p = \operatorname{div}[(u \cdot \nabla) u] \quad \text{in } \mathscr{D}'(\mathbb{R}^3). \tag{61}$$

Therefore, $\nabla p = \nabla(-\Delta)^{-1} \operatorname{div}[(u \cdot \nabla) u] \in L^2(0, \infty; \mathscr{H}^1)^3$. In addition, by the results of section II,

$$\operatorname{div}((u \cdot \nabla) u) = \sum_{i, j=1}^{3} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \in L^1(0, \infty; \mathscr{H}^1(\mathbb{R}^3))$$

and we deduce easily the results claimed on p.

These results (*see* P. Constantin [15]) imply the fact that $\nabla u \in L^\infty(0, \infty; L^1)$: one just has to write by a simple differentiation of (60)

$$\frac{\partial}{\partial t}(\nabla u) - \nu \Delta (\nabla u) = -\nabla((u.\nabla) u + \nabla p) = -(u.\nabla)(\nabla u) + f,$$

where $f \in L^1(0, \infty; L^1)$ (in fact $L^1(0, \infty; \mathcal{H}^1)$). And this is enough to conclude.

Similarly, the regularity of u follows from simple considerations on linear parabolic equations with divergence free first order terms since we have

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u.\nabla) u = -\nabla p \in L^1(0, \infty; L^{3/2, 1}(\mathbb{R}^3)).$$

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