On the Stationary Solutions of the Navier-Stokes Equations in Two Dimensions

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Although in three dimensions it has been shown under various conditions that the time-independent Navier-Stokes equations[1]

$$\begin{aligned} \Delta w - w \cdot \nabla w - \nabla p &= 0 \\ \nabla \cdot w &= 0 \end{aligned} \tag{0.1}$$

admit solutions defined in a neighborhood $\mathscr{E}$ of infinity, which achieve prescribed continuous data w^* on $\partial\mathscr{E}$ and tend to a prescribed limit w_∞ at infinity [*1*, *2*, *3*, *4*, *5*, 6, *7*], the corresponding problem for solutions in two dimensions has remained open. To our knowledge, the only significant contribution of mathematical precision is that of LERAY [*1*][2], who proved the existence of a solution which achieves the data w^* and has finite Dirichlet integral. Unfortunately the method of LERAY yields little information on the important question of the sense in which the solution satisfies the condition at infinity. This question has a particular interest in view of the Stokes paradox, according to which the corresponding problem for the linearized equations

$$\begin{aligned} \Delta w - \nabla p &= 0 \\ \nabla \cdot w &= 0 \end{aligned} \tag{0.2}$$

has no solution in the case of greatest physical interest, $w^* \equiv 0$. Further doubt has been cast on the matter by the observations that in two dimensions the solutions are in general not unique [*4*, p. xi] and do not permit asymptotic developments at infinity in terms of elementary functions [*6*, p. 228][3].

In the present work we demonstrate the existence of strict solutions of the indicated problem for (0.1) when the data are small. We approach the problem from the point of view of singular perturbation theory, as developed in our preceding paper [*10*]. To fix the ideas, consider the physically important case, $w^* \equiv 0$, $w \to w_\infty \neq 0$ at infinity. The solution is sought as a (finite) perturbation of the

[1] We assume throughout that the viscosity coefficient $\nu = 1$. If not originally the case, this can always be arranged by a suitable coordinate change.

[2] Recently, some interesting computational methods have been proposed, which appear to converge at low Reynolds number to answers in qualitative agreement with experiment. See, *e.g.*, [*13*], also [*8*], Chapter VIII, for discussion and references. It should be remarked, however, that experimental (and computational) results on two dimensional flows may be open to some question, because of the tendency of a disturbance in a uniform flow field to spread large distances from its source. Demonstrations of LERAY's result which differ in some respects from his original one have been given by FUJITA [*5*] and by LADYZHENSKAIA, *cf.* [*4*, Chapter 5].

[3] The examples in both references are special cases of a general family of explicit solutions of (0.1), given in [*9*, p. 14].

solution $\boldsymbol{u}(x)$ of the equations linearized about the value $\boldsymbol{w}_\infty$ (OSEEN [11])

$$\begin{aligned} \Delta \boldsymbol{w} - \boldsymbol{w}_\infty \cdot \nabla \boldsymbol{w} - \nabla p &= 0 \\ \nabla \cdot \boldsymbol{w} &= 0 \end{aligned} \tag{0.3}$$

with the same data, the existence of which is shown in [10, Theorem 7.2]. Using the estimates of [12] in conjunction with methods developed in [7], it is possible to show the existence of a solution $\boldsymbol{w}(x)$ of (0.1), which tends to $\boldsymbol{w}_\infty$ at infinity, and which achieves on $\partial\mathscr{E}$ prescribed data which are to lie in a certain functional neighborhood $\mathscr{N}$ of $\boldsymbol{w}_\infty$.

The crux of the matter is to prove that for sufficiently small $|\boldsymbol{w}_\infty|$, the function $\boldsymbol{w}^* \equiv 0$ is interior to $\mathscr{N}$. This result is not evident, as $\mathscr{N}$ shrinks with $|\boldsymbol{w}_\infty|$ and degenerates to the single "point" $\boldsymbol{w} \equiv 0$ in the limit $\boldsymbol{w}_\infty = 0$. What occurs is a dispute between the value at infinity and the coefficient of the first order term in (0.3), which in the indicated problem equal each other. Letting the value at infinity tend to zero reduces the effect of the nonlinearity in (0.1) and facilitates the construction of a solution; on the other hand, as the coefficient $\boldsymbol{w}_\infty \to 0$ the equation (0.3) becomes singular in the sense that the value at infinity cannot be prescribed.

It is our good fortune that the dispute is won by the data at infinity; in this regard the basic lemma may be considered to be Theorem 8.3 of [10] which shows that if these data are unvaried, then the force $\mathscr{F}(\boldsymbol{w}_\infty)$ exerted on $\partial\mathscr{E}$ in the motion vanishes like $\big|\log|\boldsymbol{w}_\infty|\big|^{-1}$, as (the coefficient) $\boldsymbol{w}_\infty \to 0$. A general estimate follows on the asymptotic behavior in x, $\boldsymbol{w}_\infty$ of the solutions of (0.3). This result is contained in the material of [10], but we restate it here in a form convenient to our purpose as the First Estimate (Lemma 2.1).

The Second Estimate (Lemma 3.7) shows that the nonlinear contribution to any solution can be estimated by functions whose asymptotic properties are qualitatively as good or preferable to those of the solutions of the unperturbed equations (0.3). The quadratic character of the nonlinearity in (0.1) then permits us to prove in § 4 that for small data a solution of the indicated problem exists (Theorems 4.1 and 4.2; Corollary 4.2). Our method here is the classical one of successive approximations, based on properties of contraction mappings. It is in principle constructive.

It may be pointed out that the zero-outflux condition $\oint \boldsymbol{w}^* \cdot d\boldsymbol{\sigma} = 0$, which is necessary in a bounded domain, plays no role in the considerations of this paper.

We devote some effort to determining properties of the solutions. We note, for example (§ 6), that they are "physically reasonable" in the sense of [12] and hence satisfy the estimates of that paper. They are also unique (§ 7) among "small" solutions — a property which is perhaps significant in view of the fact, noted above, that uniqueness in general does not hold. They satisfy the same qualitative asymptotic estimates as do the solutions of (0.3) (Theorem 5.6).

Finally we mention a property which has no counterpart in the three dimensional case (or, to our knowledge, in any other flow theory), and which seems to us of particular interest (Theorem 5.4). In the problem discussed above, there holds as $\boldsymbol{w}_\infty \to 0$,

$$\frac{1}{|\boldsymbol{w}_\infty|}\,\mathscr{F}(\boldsymbol{w}_\infty) = \left(4\pi\,\frac{\boldsymbol{w}_\infty}{|\boldsymbol{w}_\infty|} + o(1)\right)\big|\log|\boldsymbol{w}_\infty|\big|^{-1}.$$

Thus, *asymptotically, the force of reaction is determined entirely by* $\boldsymbol{w}_\infty$ *(which is the zeroth order term in an asymptotic expansion at infinity), and it is directed along the line of this vector. It is independent of the geometrical configuration.* These results are in fact a special case of a more general result for nonconstant boundary data, formulated as Theorem 5.5.

1. Formalities

Throughout this paper we lean heavily on the results of our preceding work [*10*], and also on some basic estimates due to SMITH [*12*], to which we shall refer freely. Notation is with minor exceptions the same; the section on notation in [*10*] is intended to serve also the present work. We shall use here, however, two new quantities $h_1(\xi)$, $h_2(\xi)$ defined as follows for $0<|\xi|<\infty$, ξ a point of two-dimensional Euclidean space,

$$
\begin{aligned}
&0<|\xi|\leqq 1: \qquad h_i(\xi)=\log\frac{2}{|\xi|}, \qquad i=1,2 \\
&|\xi|>1: \quad \begin{cases} h_1(\xi)=|\xi|^{-\frac{1}{2}} \\ h_2(\xi)=|\xi|^{-\frac{1}{2}-\varepsilon}, \qquad 0<\varepsilon<1. \end{cases}
\end{aligned}
\tag{1.1}
$$

We intend to majorize the solution of (0.3) in terms of the $h_i(|\boldsymbol{w}_\infty|x)$. The reason for introducing two such quantities, rather than a single majorant, is to take account of the differing asymptotic behavior of the two components w_1, w_2 of the solution vectors (*cf.* the remarks on pp. 346–7 of [*12*]). Our choice of functions $h_i(\xi)$ does not lead to best possible majorizations, but it has turned out to be convenient to the technical needs of the material. It permits us to place the solutions in the class PR [*12*], and qualitatively sharp majorizations (Theorem 5.6) then follow *a fortiori* by use of the methods and results of that reference.

The symbol $\boldsymbol{w}_\infty$ refers, in general, to a velocity vector prescribed at infinity. We suppose throughout that this vector has been normalized so that it is directed along the positive x_1-axis: $\boldsymbol{w}_\infty=(w_{\infty 1}, 0)$, $w_{\infty 1}>0$.

2. The First Estimate

Lemma 2.1. *Suppose* $0<\lambda\leqq\lambda_0<\infty$, *let* $\boldsymbol{w}^*$ *be prescribed data of class* $\mathscr{C}^{2+\alpha}$ *on* $\partial\mathscr{E}$. *Let* $|\boldsymbol{w}_\infty|=\lambda$, *let* $\boldsymbol{w}(x;\lambda)$ *be a solution of* (0.3) *in* $\mathscr{E}$, *such that* $\boldsymbol{w}\to\boldsymbol{w}^*$ *on* $\partial\mathscr{E}$ *and* $\boldsymbol{w}\to 0$ *at infinity. There exists a constant* C, *depending only on* λ_0, *on* $\boldsymbol{w}^*$ *and on the geometry, such that throughout* $\mathscr{E}$, $|w_i(x;\lambda)|<C h_i(\lambda x)|\log\lambda|^{-1}$.

Proof. Let Σ_1 be a circumference (we may suppose it of unit radius) surrounding $\partial\mathscr{E}$. Interior to Σ_1 we have $|w_i(x;\lambda)|<M<\infty$ by Theorem 6.52 of [*10*]. Exterior to Σ_1 we have the representation

$$
\boldsymbol{w}(x;\lambda)=-\oint_{\partial\mathscr{E}}\mathbf{E}\cdot T\boldsymbol{w}\cdot d\boldsymbol{\sigma}+\oint_{\partial\mathscr{E}}\boldsymbol{w}^*\cdot T\mathbf{E}\cdot d\boldsymbol{\sigma}+\lambda\oint_{\partial\mathscr{E}}\mathbf{E}\boldsymbol{\alpha}\cdot d\boldsymbol{\sigma}, \qquad \boldsymbol{\alpha}=\frac{\boldsymbol{w}_\infty}{|\boldsymbol{w}_\infty|}
$$

so that, letting $\mathscr{E}^0$ denote the convex closure of the complement of $\mathscr{E}$,

$$
\begin{aligned}
&|w_i(x;\lambda)-\mathscr{I}(\lambda)\cdot\mathbf{E}_i(x;\lambda)| \\
&\quad\leqq C\lambda\max_{y\in\partial\mathscr{E}}|\mathbf{E}_i(x-y;\lambda)|+C\max_{y\in\mathscr{E}^0}\{|\mathbf{e}_i(x-y)|+|\nabla\mathbf{E}_i(x-y;\lambda)|\}
\end{aligned}
\tag{2.1}
$$

where $\mathbf{E}_i$ is the vector $(\mathsf{E}_{i1}, \mathsf{E}_{i2})$ and C depends on the indicated parameters. The estimates (8) and (14) of [*12*] show that $|\mathbf{E}_i(x-y;\lambda)|=|\mathbf{E}_i(\lambda(x-y);1)|<Ch_i(\lambda x)$ when $y\in\mathscr{E}^0$ and x is exterior to Σ_1. The corresponding estimates for the derivatives of $\mathbf{E}$ and of $\mathbf{e}$ yield uniform bounds for these functions in the interval $1\leqq|x|\leqq|\lambda|^{-1}$, and show that they decay faster than $h_i(\lambda x)$ outside this interval. By Theorem 8.4 of [*10*], $|\mathscr{I}(\lambda)|<C|\log\lambda|^{-1}$. Thus, (2.1) and the bound near $\partial\mathscr{E}$ contain the assertion of the lemma.

3. The Second Estimate

For any tensor field $\mathbf{B}(x,y;\lambda)$ defined for $x,y\in\mathscr{E}$, $0<\lambda<1$, and subset $\hat{\mathscr{E}}$ of $\mathscr{E}$, we define the quantity

$$I_i(x;\lambda;\mathbf{B};\hat{\mathscr{E}})={}_\lambda\int_{\hat{\mathscr{E}}} h_j(\lambda y)\,h_k(\lambda y)\left|\frac{\partial \mathsf{B}_{ij}(x,y;\lambda)}{\partial y_k}\right|dy,\qquad i=1,2. \tag{3.1}$$

Let $\mathbf{G}(x,y;\lambda)$ be the GREEN's tensor for (0.3) in $\mathscr{E}$, with $\lambda=|w_\infty|$, introduced in § 6 of [*10*], $\mathbf{G}(x,y;\lambda)=\mathbf{E}(x-y;\lambda)-\mathbf{A}(x,y;\lambda)$. Let $\mathscr{E}_\rho$ be the part of $\mathscr{E}$ exterior to a circumference of radius ρ.

Lemma 3.1. $I_i(x;\lambda;\mathbf{E};\mathscr{E}_{\lambda^{-1}})<Ch_i(\lambda x)$, *where C depends only on w^* and on the geometry.*

Proof. From $\mathbf{E}(x-y;\lambda)=\mathbf{E}(\lambda(x-y);1)$ follows

$$\int_{\mathscr{E}_{\lambda^{-1}}} h_j(\lambda y)\,h_k(\lambda y)\left|\frac{\partial \mathsf{E}_{ij}(x-y;\lambda)}{\partial y_k}\right|dy=\lambda^{-1}\int_{\mathscr{E}_1} h_j(\eta)\,h_k(\eta)\left|\frac{\partial \mathsf{E}_{ij}(\xi-\eta;1)}{\partial \eta_k}\right|d\eta. \tag{3.2}$$

In $\mathscr{E}_1$ there holds $h_1(\eta)<|\eta|^{-\frac{1}{2}}$, $h_2(\eta)<|\eta|^{-\frac{1}{2}-\varepsilon}$, so that if $x\in\mathscr{E}_{\lambda^{-1}}$, the result is obtained as a particular case of the Estimates 1 and 2 (pp. 352, 358) of [*12*]. If $x\notin\mathscr{E}_{\lambda^{-1}}$, then $|\xi|=|\lambda x|<1$ and the integral on the right in (3.2) is bounded.

Lemma 3.2. $I_i(x;\lambda;\mathbf{E};\mathscr{E}-\mathscr{E}_{\lambda^{-1}})<Ch_i(\lambda x)$, *where C depends only on w^* and on the geometry.*

Proof. Let $\mathscr{E}^\lambda$ denote the image of $\mathscr{E}$ under the transformation $\xi=\lambda x$. The right side of (3.2) is now to be replaced by an integration over $\mathscr{E}^\lambda-\mathscr{E}_1$. If $\lambda x\in\mathscr{E}^\lambda-\mathscr{E}_2$, the integral is bounded, while if $\lambda x\in\mathscr{E}_2$, the pointwise estimate $|\nabla E(x;1)|<C|x|^{-1}$ (*cf.* [*12*, § 2]) yields the result.

Lemma 3.3. *Let $x\in\mathscr{E}_2$. Then $I_i(x;\lambda;\mathbf{A};\mathscr{E}_1)<Ch_i(\lambda x)$, where C depends only on w^* and on the geometry.*

Proof. Lemma 6.61 of [*10*] implies that if $x\in\mathscr{E}_2$, then

$$|\mathbf{A}_i(x,y;\lambda)|\leqq C\,h_i(\lambda x)\max_{z\in\partial\mathscr{E}}|\mathbf{E}(z-y;\lambda)|,$$

a relation which may be differentiated formally in y. The estimate (14-iii) of [*12*] thus implies, for all $x\in\mathscr{E}_2$,

$$\begin{aligned} I_i(x;\lambda;\mathbf{A}_i;\mathscr{E}_1)&<C\,h_i(\lambda x)\int_{\mathscr{E}^\lambda} h_j(\eta)\,h_k(\eta)\,|\nabla\mathbf{E}(-\eta;1)|\,d\eta\\ &<C\,h_i(\lambda x). \end{aligned}$$

Lemma 3.4. *Let* $x \in \mathscr{E}_2$. *Then* $I_i(x;\,\lambda;\,\mathbf{A};\,\mathscr{E}-\mathscr{E}_1) < C\lambda\,h_i(\lambda x)\log^2\lambda$.

Proof. By Lemma 6.51 of [10] and the estimates (14) of [12], $x\in\mathscr{E}_2$, $y\in\mathscr{E}-\mathscr{E}_1$ imply $|\nabla_y A_i(x,y;\,\lambda)| \leqq C Q_i(x;\,\lambda)$ where $Q_i(x;\,\lambda)$ denotes a bound, for $z\in\partial\mathscr{E}$, of the magnitudes of $\mathbf{E}(x-z;\,\lambda)$ and its derivatives up to second order. Since $\mathbf{E}(x-z;\,\lambda) = \mathbf{E}(\lambda(x-z);\,1)$, the estimates (14) of [12] yield $Q_i < C h_i(\lambda x)$. Hence $x\in\mathscr{E}_2$ implies

$$
\begin{aligned}
I_i(x;\lambda;\mathbf{A}_i;\mathscr{E}-\mathscr{E}_1) &< C\lambda\,h_i(\lambda x)\int_{\mathscr{E}-\mathscr{E}_1} h_j(\lambda y)\,h_k(\lambda y)\,dy\\
&< C\lambda^{-1}h_i(\lambda x)\int_0^{\lambda} t\log^2 t\,dt\\
&< C\lambda\,h_i(\lambda x)\log^2\lambda .
\end{aligned}
$$

Lemma 3.5. *Let* $x\in\mathscr{E}-\mathscr{E}_2$. *Then* $I_i(x;\,\lambda;\,\mathbf{A};\,\mathscr{E}-\mathscr{E}_4) < C\lambda\log^2\lambda$.

Proof. By Lemma 6.51 of [10], if $x\in\mathscr{E}-\mathscr{E}_2$ and $y\in\mathscr{E}-\mathscr{E}_4$, then

$$|\nabla_y\mathbf{A}(x,y;\,\lambda)| < C|x-y|^{-1}.$$

Thus

$$
\begin{aligned}
I(x;\lambda;\mathbf{A};\mathscr{E}-\mathscr{E}_4) &< C\lambda\int_{\mathscr{E}-\mathscr{E}_4} h_i(\lambda y)\,h_k(\lambda y)\,|x-y|^{-1}\,dy\\
&< C\int_0^{4\lambda}\log^2 t\,dt < C\lambda\log^2\lambda .
\end{aligned}
$$

Lemma 3.6. *Let* $x\in\mathscr{E}-\mathscr{E}_2$. *Then* $I_i(x;\,\lambda;\,\mathbf{G};\,\mathscr{E}_4) < C$.

Proof. Lemma 6.51 of [10] shows that on Σ_3 there holds $|\mathbf{G}|+|T\mathbf{G}|<C$. The representation

$$\mathbf{G}(x,y;\,\lambda) = \oint_{\Sigma_3}\{\mathbf{G}\cdot T\,\mathbf{E}^* - \mathbf{E}^*\cdot T\,\mathbf{G} + \lambda\,\mathbf{G}\cdot\mathbf{E}^*\alpha\}\cdot d\boldsymbol{\sigma}$$

shows that if $y\in\mathscr{E}_4$, then

$$|\mathbf{G}(x,y;\,\lambda)| < C\max_{z\in\Sigma_3}\{|\mathbf{E}^*(y-z;\,\lambda)| + |T_z\,\mathbf{E}^*(y-z;\,\lambda)|\}$$

and this relation can be differentiated formally in y. Using again the estimates (14) of [12], we obtain

$$
\begin{aligned}
I_i(x;\lambda;\mathbf{G};\mathscr{E}_4) &= \lambda\int_{\mathscr{E}_4} h_j(\lambda y)\,h_k(\lambda y)\left|\frac{\partial \mathrm{G}_{ij}}{\partial y_k}\right| dy\\
&< C\int_{\mathscr{E}_{4\lambda}} h_j(\eta)\,h_k(\eta)\max_{\zeta\in\Sigma_{3\lambda}}\left|\frac{\partial \mathrm{E}^*_{ij}(\eta-\zeta;\,1)}{\partial\eta_k}\right| d\eta < C .
\end{aligned}
$$

Corollary 3.6. *Let* $x\in\mathscr{E}-\mathscr{E}_2$. *Then* $I_i(x;\,\lambda;\,\mathbf{A};\,\mathscr{E}_4) < C$.

Proof. Apply Lemmas 3.1, 3.2, 3.6.

From Lemmas 3.1 to 3.6 we conclude:

Lemma 3.7. *There is a constant* H, *depending only on* w^* *and on the geometry, such that* $I_i(x;\,\lambda;\,\mathbf{G}(x,y;\,\lambda);\,\mathscr{E}) < H h_i(\lambda x)$.

4. Existence

If $w(x; w_\infty)$ is a solution in $\mathscr{E}$ of the Navier-Stokes equations

$$(4.1)\qquad \begin{gathered} \Delta w - w \cdot \nabla w - \nabla p = 0 \\ \nabla \cdot w = 0 \end{gathered}$$

such that $w(x; w_\infty) \to w_\infty$ at infinity, then for any positive τ, the vector field

$$u(x; w_\infty) = \tau^{-1} \frac{w - w_\infty}{|w_\infty|}$$

defines a solution of the system

$$(4.2)\qquad \begin{gathered} \Delta u - w_\infty \cdot \nabla u - \nabla p = \tau\, |w_\infty|\, u \cdot \nabla u \\ \nabla \cdot u = 0 \end{gathered}$$

and $u(x; w_\infty) \to 0$ at infinity. Conversely, every solution of (4.2) defines a corresponding solution of (4.1).

We seek a solution of (4.1) which achieves prescribed data, w^* on $\partial\mathscr{E}$ and w_∞ at infinity. We begin by finding a solution of (4.2) which achieves the data

$$u^* = \frac{w^* - w_\infty}{|w_\infty|}$$

and which vanishes at infinity. Setting

$$\alpha = \frac{w_\infty}{|w_\infty|}, \qquad \lambda = |w_\infty|$$

and introducing the GREEN's tensor $\mathbf{G}(x, y; \lambda)$ for the linearized system (0.3), we seek $u(x; \lambda)$ as solution of the integral equation

$$(4.3)\qquad u(x; \lambda) = u^{(0)}(x; \lambda) - \tau \lambda \int_{\mathscr{E}} u \cdot u \cdot \nabla_y \mathbf{G}(x, y; \lambda)\, dy.$$

Here $u^{(0)}(x; \lambda)$ is the unique solution of (0.3) corresponding to the given data (*cf.* [*10*], Theorem 7.2). We shall obtain a solution as a formal expansion

$$(4.4)\qquad u(x; \lambda) = u^{(0)}(x; \lambda) + \sum_1^\infty u^{(n)}(x; \lambda)\, \tau^n$$

corresponding to vector functions $u^{(n)}(x; \lambda)$ which vanish on $\partial\mathscr{E}$ and at infinity. It is thus required to determine the $\{u^n(x; \lambda)\}$, and to prove the convergence of (4.4) in $\mathscr{E}$.

Inserting (4.4) into (4.3) and equating coefficients of equal power of τ yields

$$(4.5)\qquad u_i^{(n+1)}(x; \lambda) = -\lambda \int_{\mathscr{E}} \sum_{\nu=0}^{n} (u_j^{(\nu)} u_k^{(n-\nu)}) \frac{\partial \mathbf{G}_{ij}}{\partial y_k}\, dy.$$

Let us write $u_i^{(k)}(x; \lambda) = v^{(k)}(x; \lambda)\, h_i(\lambda x)$, $i = 1$ or 2. Then (4.4) becomes

$$(4.6)\qquad v(x; \lambda) = v^{(0)}(x; \lambda) + \sum_1^\infty v^{(n)}(x; \lambda)\, \tau^n$$

and (4.5) becomes

$$(4.7) \qquad v^{(n+1)}(x;\lambda) = -\lambda\, h_i^{-1}(\lambda x) \int_{\mathscr{E}} \sum_{\nu=0}^{n} (v^{(\nu)} v^{(n-\nu)})\, h_j\, h_k \frac{\partial \mathsf{G}_{ij}}{\partial y_k}\, dy.$$

By Lemma 3.7

$$\lambda\, h_i^{-1}(\lambda x) \int_{\mathscr{E}} h_j(\lambda y)\, h_k(\lambda y) \left| \frac{\partial \mathsf{G}_{ij}(x,y;\lambda)}{\partial y_k} \right| dy < H$$

uniformly in $0<\lambda<1$. It follows that the series with constant coefficients

$$(4.8) \qquad V = V^{(0)} + \sum_{1}^{\infty} V^{(n)} \tau^n$$

will be a dominant series for (4.6), provided

$$(4.9) \qquad |v^{(0)}(x;\lambda)| \leqq V^{(0)}$$

in $\mathscr{E}$, and

$$(4.10) \qquad V^{(n+1)} = H \sum_{\nu=0}^{n} V^{(\nu)} V^{(n-\nu)}.$$

The convergence of (4.8, 4.10) implies

$$(4.11) \qquad V = V^{(0)} + \tau H V^2.$$

The solution of this equation has a branch which is analytic in τ interior to a circle whose radius is determined by the vanishing of the discriminant. We conclude that the series (4.8), and hence the series (4.4), converges throughout $\mathscr{E}$, whenever $\tau < (4HV^{(0)})^{-1}$. We shall show that this function is a solution of (4.2) with the indicated data, but it will in general not solve the original problem, as the corresponding solution $w(x;\lambda) = w_\infty + \tau\,|w_\infty|\, u(x;\lambda)$ assumes the data $w_\infty + \tau(w^* - w_\infty)$ on $\partial\mathscr{E}$. The original problem will however be solved as soon as the convergence of (4.6) can be established with $\tau=1$. Evidently this will be the case if $V^{(0)} < (4H)^{-1}$. We are thus required to choose the data in such a way that each of the two functions $|v^{(0)}(x;\lambda)| = |u_i^{(0)}(x;\lambda)|\, h_i^{-1}(\lambda x)$, $i=1$ or 2 satisfies

$$|v^{(0)}(x;\lambda)| < (4H)^{-1}$$

throughout $\mathscr{E}$. By Lemma 2.1, $|u_i^{(0)}(x;\lambda)| < M h_i(\lambda x)\,|\log\lambda|^{-1}$, so that

$$|v^{(0)}| < M\,|\log\lambda|^{-1}.$$

For fixed $\lambda = |w_\infty|$, $M\to 0$ with the boundary data and its first two derivatives. Thus for sufficiently small data there will be a solution of the integral equation (4.3) in the form of an expansion (4.4) with $\tau=1$. Using (4.3), one then verifies successively the Hölder continuity of $u(x;\lambda)$ and its differentiability in $\mathscr{E}$. We have proved:

Theorem 4.1. *Let $w_\infty \neq 0$ be given and let $w^* = |w_\infty|\, u^* + w_\infty$ be prescribed data on $\partial\mathscr{E}$. Then if u^* and its first two derivatives in arc are sufficiently small in magnitude, there exists a solution $w(x)$ of* (4.1) *in $\mathscr{E}$ such that $w(x)\to w^*$ on $\partial\mathscr{E}$ and $w(x)\to w_\infty$ at infinity.*

A case of somewhat deeper interest is that in which $\lambda = |w_\infty|$ is small, M being suitably restricted. We find:

Theorem 4.2. *Let $w^*(\lambda)$ be data on $\partial\mathscr{E}$ and let M be a bound in magnitude for*

$$u^* = \frac{w^* - w_\infty}{|w_\infty|}$$

and its first two derivatives in arc. If as

$$\lambda = |w_\infty| \to 0, \qquad M = o\left(\log\frac{1}{\lambda}\right),$$

then for all sufficiently small λ there will exist a solution $w(x;\lambda)$ of (4.1) *in $\mathscr{E}$ such that $w(x;\lambda)\to w^*$ on $\partial\mathscr{E}$ and $w(x;\lambda)\to w_\infty$ at infinity.*

The proof is the same as for Theorem 4.1. We need only note that under the hypotheses, $|v^{(0)}(x;\lambda)|\to 0$ uniformly in $\mathscr{E}$ as $\lambda\to 0$.

The physical problem, $w^*\equiv 0$, has a particular interest. We then have:

Corollary 4.2. *If $|w_\infty| = \lambda$ is sufficiently small, depending only on the geometry, there is a solution $w(x;\lambda)$ of* (4.1) *in $\mathscr{E}$ such that $w(x;\lambda)=0$ on $\partial\mathscr{E}$ and $w(x;\lambda)\to w_\infty$ at infinity.*

5. Asymptotic Properties

We show first that the strict solutions constructed in the preceding section have asymptotic properties, in x and in λ, similar to those of the linearized equations (0.3) described in Lemma 2.1.

Lemma 5.1. *Let $w^*(\lambda)$ be data satisfying the hypotheses of Theorem* 4.2 *with $M<M_0<\infty$, all λ. Then the solutions $w(x;\lambda)$ satisfy*

$$|u_i(x;\lambda)| = \frac{|w_i(x;\lambda) - w_{\infty i}|}{|w_\infty|} < M_1\, h_i(\lambda x)\, |\log\lambda|^{-1}$$

uniformly in x and λ, as $\lambda\to 0$.

Proof. Since $|v_i^{(0)}(x;\lambda)| = |u_i^{(0)}(x;\lambda)\, h_i^{-1}(\lambda x)| < C|\log\lambda|^{-1}$, $i=1,2$, it follows that there is a λ_0 such that $|v^{(0)}(x;\lambda_0)| < C|\log\lambda_0|^{-1} < (4H)^{-1}$. Choose $V^{(0)}$ such that $C|\log\lambda_0|^{-1} < V^{(0)} < (4H)^{-1}$. Suppose $\lambda\leqq\lambda_0$, set

$$\eta(\lambda;\lambda_0) = |\log\lambda|\,|\log\lambda_0|^{-1}.$$

Then $|v^{(0)}(x;\lambda)| < C|\log\lambda|^{-1} = C|\log\lambda_0|^{-1}\,\eta^{-1}$ so that

$$|v^{(0)}(x;\lambda)|\,\eta(\lambda;\lambda_0) < V^{(0)}.$$

Then (4.9) holds, and the series (4.6) with coefficients (4.7) will converge. But since $\eta\geqq 1$, the method of formation of the $v^{(n)}(x;\lambda)$ shows that

$$|v^{(n)}(x;\lambda)|\,\eta(\lambda;\lambda_0) \leqq |v^{(n)}(x;\lambda)|\,\eta^{n+1} < V^{(n)}.$$

Thus the series (4.8) dominates, for $i=1$ or 2, the series

$$\begin{aligned} u_i(x;\lambda)\, h_i^{-1}(\lambda x)\,\eta(\lambda;\lambda_0) &= \sum u_i^{(n)}(x;\lambda)\, h_i^{-1}(\lambda x)\,\eta(\lambda;\lambda_0) \\ &= \sum v^{(n)}(x;\lambda)\,\eta(\lambda;\lambda_0) \end{aligned}$$

so that $u_i(x;\lambda) < V h_i(\lambda x)\,\eta^{-1}(\lambda;\lambda_0) < C h_i(\lambda x)|\log\lambda|^{-1}$, which was to be proved.

Lemma 5.2. *Corresponding to prescribed data w^* on $\partial\mathscr{E}$, the solutions $w(x;\lambda)$ of Theorem* 4.2 *satisfy*

$$\int_{\mathscr{E}} |\nabla w(x;\lambda)|^2\, dx < M\lambda^2,$$

M depending only on a bound for $w^* - w_\infty/|w_\infty|$ and its derivatives of first two orders and on the geometry.

Proof. We suppose the origin of coordinates to be not in $\mathscr{E}$, and choose the real quantity γ_0 such that the field $\gamma = \gamma_0 \nabla \log r$ satisfies

$$\oint_{\partial\mathscr{E}} (\lambda^{-1} w^* - \gamma)\cdot d\boldsymbol{\sigma} = 0.$$

We introduce a solenoidal extension $\zeta(x)$ into $\mathscr{E}$ of the field $(w^* - w_\infty/|w_\infty|) - \gamma$, such that the estimates of Lemma 6.31 in [*10*] will hold. The field $W(x;\lambda) = \lambda^{-1}[w(x;\lambda) - w_\infty] - \gamma - \zeta(x)$ then satisfies

$$\begin{aligned} \Delta W - w_\infty\cdot\nabla W - \nabla p = {} & -\Delta\zeta + w_\infty\cdot\nabla\zeta + \lambda W\cdot\nabla W + \\ & + \lambda\zeta\cdot\nabla W + \lambda W\cdot\nabla\zeta + \lambda\zeta\cdot\nabla\zeta + \\ & + w_\infty\cdot\nabla\gamma + \lambda\gamma\cdot\nabla W + \lambda W\cdot\nabla\gamma + \lambda\gamma\cdot\nabla\gamma \end{aligned} \tag{5.1}$$

in $\mathscr{E}$, and $W\to 0$ on $\partial\mathscr{E}$. Let $\hat{\mathscr{E}}$ be the support of ζ. We multiply (5.1) by W and integrate over the region between $\partial\mathscr{E}$ and a circumference Σ_R of (large) radius R. After certain integrations by parts, noting that

$$\frac{\partial\gamma_i}{\partial x_j} = \frac{\partial\gamma_j}{\partial x_i},$$

and neglecting terms which integrate to zero, we may write the result in the form

$$\begin{aligned} \int_{\mathscr{E}-\mathscr{E}_R} |\nabla W|^2\, dx = {} & \tfrac12 \oint_{\Sigma_R} W^2 w_\infty\cdot d\boldsymbol{\sigma} + \oint_{\Sigma_R} p\, W\cdot d\boldsymbol{\sigma} + \tfrac12 \oint_{\Sigma_R} (\nabla W^2)\cdot d\boldsymbol{\sigma} + \\ & + \oint_{\Sigma_R} \gamma\cdot w_\infty\, W\cdot d\boldsymbol{\sigma} + \lambda \oint_{\Sigma_R} (\gamma\cdot W)(\gamma + W)\cdot d\boldsymbol{\sigma} + \\ & + \frac{\lambda}{2}\oint_{\Sigma_R} W^2(\gamma+W)\cdot d\boldsymbol{\sigma} + \int_{\hat{\mathscr{E}}} \nabla W\cdot\nabla\zeta\, dx + \int_{\hat{\mathscr{E}}} W\cdot w_\infty\cdot\nabla\zeta\, dx + \\ & + \lambda\int_{\hat{\mathscr{E}}} W\cdot\zeta\cdot\nabla\zeta\, dx. \end{aligned} \tag{5.2}$$

Lemma 5.1 shows that the boundary integrals which do not involve p or ∇W vanish in the limit as $R\to\infty$. The same lemma shows that $w(x;\lambda)$ is "physically reasonable" in the sense of [*12*]. The estimates of that paper now imply that all boundary integrals approach zero. We may therefore use (5.2) in conjunction with Lemma 1.1 of [*10*] to obtain an inequality

$$\int_{\mathscr{E}} |\nabla W|^2\, dx \le \tfrac12 \int_{\hat{\mathscr{E}}} |\nabla W|^2\, dx + C$$

where C depends only on ζ (and hence only on $\lambda^{-1}(w^* - w_\infty)$ and on $\mathscr{E}$) as $\lambda\to 0$. Since $\hat{\mathscr{E}}\subset\mathscr{E}$, this completes the proof of the lemma.

Theorem 5.3. *Let $w^* = w^*(\lambda)$ be data on $\partial\mathscr{E}$ such that*

$$u^*(\lambda) = \frac{w^* - w_\infty}{|w_\infty|}$$

and its first two derivatives in arc converge uniformly as $\lambda = |w_\infty| \to 0$, and let u_0^* be the limit function. We suppose $u^*(\lambda)$ sufficiently small to yield

$$|v^{(0)}(x;\lambda)| < V^{(0)} < (4H)^{-1},$$

all λ, and we denote by $w(x;\lambda)$ the corresponding solutions of (4.1). *Then the functions*

$$u(x;\lambda) = \frac{w(x;\lambda) - w_\infty}{|w_\infty|}$$

converge uniformly in every compact subset of $\mathscr{E} + \partial\mathscr{E}$, to a solution $u_0(x)$ of the Stokes equations (0.2), *which assumes the data u^* on $\partial\mathscr{E}$ and has finite Dirichlet integral. The force associated with this solution is zero, and $u_0(x)$ is unique among all solutions of* (0.2) *which achieve the same data and have finite Dirichlet integral.*

Proof. A procedure parallel to the proof of Lemma 6.52 in [*10*], using Lemma 5.2, shows the equicontinuity of the $\{u(x;\lambda)\}$ in compact subsets up to the boundary. Thus there is a sequence $\lambda_i \to 0$ for which the solutions converge in $\mathscr{E} + \partial\mathscr{E}$, uniformly on compact subsets, to a continuous vector field $u_0(x)$. The representation (3.32) of [*10*] yields, however, for points removed from $\partial\mathscr{E}$,

$$u(x;\lambda) = \lambda \int_{\mathscr{D}} \mathbf{E}^\gamma(x-y;\lambda)\cdot u \cdot \nabla u\, dy + \int_{\mathscr{D}} \mathbf{H}^\gamma \cdot u\, dy,$$

$\mathscr{D}$ being a disk of radius γ centered at x. This relation may be applied using the uniform bound on Dirichlet integral (Lemma 5.2), to verify successively the Hölder continuity of the $u(x;\lambda_i)$ and their derivatives of all orders, and we conclude that these functions converge with their derivatives in $\mathscr{E}$, so that $u_0(x)$ is a solution of (0.2) with data u_0^* on $\partial\mathscr{E}$, and

$$\int_{\mathscr{E}} |\nabla u_0(x)|^2\, dx < \infty\,.$$

The uniqueness of $u_0(x)$ and the vanishing of the associated force follow as in the proof of Theorem 8.3 in [*10*], and from the uniqueness follows that $u(x;\lambda) \to u_0(x)$ for every sequence $\lambda_i \to 0$.

In case $w^* = 0$ the limiting solution is

$$u_0(x) \equiv \lim_{|w_\infty|\to 0} \frac{-w_\infty}{|w_\infty|} = \alpha \equiv \text{const.}$$

As a consequence we obtain:

Theorem 5.4. *Let $\{w(x;\lambda)\}$ denote the solutions of Corollary* 4.2, *and let $\mathscr{F}(\lambda)$ be the force of reaction against $\partial\mathscr{E}$ in the motion. Then*

$$u(x;\lambda) = \frac{w(x;\lambda) - w_\infty}{|w_\infty|}$$

admits the representation

$$u(x;\lambda)=\oint_{\partial\mathscr{E}} \mathbf{E}(x-y;\lambda)\cdot T u\cdot d\sigma_y+\lambda\oint_{\partial\mathscr{E}}(\mathbf{E}\cdot u)\,u\cdot d\sigma_y-\lambda\int_{\mathscr{E}} u\cdot u\cdot \nabla\mathbf{E}\,dy \tag{5.3}$$

in $\mathscr{E}$. *As* $|w_\infty|=\lambda\to 0$, *there holds*

$$\lim_{\lambda\to 0} u(x;\lambda)=\alpha$$

uniformly on every compact subset of $\mathscr{E}+\partial\mathscr{E}$, *and*

$$\frac{1}{\lambda}\mathscr{F}(\lambda)=\left(4\pi\alpha+o(1)\right)|\log\lambda|^{-1} \tag{5.4}$$

where $o(1)$ *denotes a vector quantity tending to zero with* λ. *There holds also*

$$u_i(x;\lambda)=4\pi\,\mathsf{E}_{ij}(x;\lambda)\,\alpha_j\,|\log\lambda|^{-1}+Q_i(x;\lambda) \tag{5.5}$$

where, denoting by $\mathscr{E}^0$ *the convex closure of the complement of* $\mathscr{E}$,

$$|Q_i(x;\lambda)|<\varepsilon(\lambda)\{|\mathsf{E}_{ij}(x;\lambda)|\;|\log\lambda|^{-1}+\max_{\substack{y\in\mathscr{E}^0\\ j=1,2}}|\nabla_y\mathsf{E}_{ij}(x-y;\lambda)|\}+$$
$$+C\lambda|\log\lambda|+C\,h_i(\lambda x)\,|\log\lambda|^{-2}$$

and $\varepsilon(\lambda)$ *is a quantity tending to zero with* λ.

N.B. The remarks following the corresponding Theorem 8.3 in [*10*] apply equally well in this case.

Integrating by parts over the *interior* region bounded by $\partial\mathscr{E}$, we obtain

$$\oint_{\partial\mathscr{E}} T\mathbf{E}\cdot d\sigma+\lambda\oint_{\partial\mathscr{E}}\mathbf{E}\,\alpha\cdot d\sigma=0$$

so that (5.24) of [*10*] implies

$$u(x;\lambda)=\oint_{\partial\mathscr{E}}\mathbf{E}(x-y;\lambda)\cdot Tu\cdot d\sigma_y+\lambda\int_{\mathscr{E}}\mathbf{E}\cdot u\cdot\nabla u\,dy.$$

The last term may be integrated by parts in $\mathscr{E}$. The outer boundary integrals tend to zero, and we obtain (5.3). Using it, we find

$$\begin{aligned}|u_i(x;\lambda)-\mathsf{E}_{ij}(x;\lambda)\cdot\mathscr{F}_j(\lambda)|<C\max_{\substack{y\in\mathscr{E}^0\\ i=1,2}}|\nabla_y\mathbf{E}_i(x-y;\lambda)|\max_{y\in\partial\mathscr{E}}|Tu|+\\ +C\lambda|\log\lambda|+C\,h_i(\lambda x)\,|\log\lambda|^{-2}\end{aligned} \tag{5.6}$$

by Lemmas 3.1, 3.2 and 5.1. The remainder of the proof parallels that of Theorem 8.3 in [*10*], as all terms on the right side of (5.6) are of negligible order as $\lambda\to 0$.

We have also an analogue of Theorem 8.4 in [*10*]:

Theorem 5.5. *Under the hypotheses of Theorem* 5.3, *there exists*

$$\lim_{\lambda\to 0}\frac{1}{4\pi\lambda}\mathscr{F}(\lambda)\log\frac{1}{\lambda}=u_\infty. \tag{5.7}$$

For the limit solution $\boldsymbol{u}_0(x)$, *there holds*

$$\lim_{x\to\infty} \boldsymbol{u}_0(x)=\boldsymbol{u}_\infty .$$

Proof. Analogous to (8.9) in [*10*] we may write

$$\begin{aligned}\boldsymbol{u}(x;\lambda)=&\frac{1}{4\pi\lambda}\,\mathscr{F}(\lambda)\log\frac{1}{\lambda}-\oint_{\partial\mathscr{E}}\mathbf{E}_0\cdot T\boldsymbol{u}\cdot d\boldsymbol{\sigma}_y+\\ &+\oint_{\partial\mathscr{E}}\boldsymbol{u}^*\cdot T\mathbf{E}_0\cdot d\boldsymbol{\sigma}_y-\lambda\int_{\mathscr{E}}\boldsymbol{u}\cdot\boldsymbol{u}\cdot\nabla\mathbf{E}\,dy+o(1)\end{aligned}\tag{5.8}$$

as $\lambda\to 0$. The last integral on the right is also $o(1)$, by Lemmas 3.1, 3.2 and 5.1. We have

$$\lim_{\lambda\to 0}\frac{1}{4\pi\lambda}\,\mathscr{F}(\lambda)\log\frac{1}{\lambda}=\boldsymbol{u}_\infty$$

exists, since this is the case for all other terms in (5.8). Passing to the limit, we obtain

$$\oint_{\partial\mathscr{E}} T\boldsymbol{u}_0(y)\cdot d\boldsymbol{\sigma}_y=0,$$

and

$$\boldsymbol{u}_0(x)=\boldsymbol{u}_\infty-\oint_{\partial\mathscr{E}}\mathbf{E}_0\cdot T\boldsymbol{u}_0\cdot d\boldsymbol{\sigma}+\oint_{\partial\mathscr{E}}\boldsymbol{u}^*\cdot T\mathbf{E}_0\cdot d\boldsymbol{\sigma}.$$

Since both boundary integrals vanish as $x\to\infty$, we find

$$\lim_{x\to\infty}\boldsymbol{u}_0(x)=\boldsymbol{u}_\infty ,$$

which completes the proof.

As in the linearized case, we may note that the limiting (Stokes) equations control the solution in the region $|x|=o(\lambda^{-1})$, the non-uniformity in the perturbation $\lambda\to 0$ appearing outside this region.

Qualitatively, the behavior of the solutions at infinity is similar to that of the solutions of the Oseen linearized equations (0.3), the effect of the nonlinear terms being of smaller order asymptotically. We may summarize this behavior in the following way:

Theorem 5.6. *For the solutions constructed in Theorems* 4.1 *and* 4.2, *the functions*

$$\boldsymbol{u}(x;\lambda)\equiv\frac{\boldsymbol{w}(x;\lambda)-\boldsymbol{w}_\infty}{|\boldsymbol{w}_\infty|}$$

satisfy the results of Theorem 8.5 *in* [*10*].

The demonstration follows in general outline that of Lemmas 3.1 to 3.6. It requires a somewhat painstaking estimation in the spirit of these results and of our earlier papers [*7*, *12*]. We omit details.

6. General—Properties

The solutions constructed in the preceding section are "physically reasonable" in the sense of [*12*]. Hence we may apply immediately the results of that paper to obtain:

Theorem 6.1. *Setting*

$$(\operatorname{def} w)_{ij}=\frac{1}{2}\left(\frac{\partial w_i}{\partial x_j}+\frac{\partial w_j}{\partial x_i}\right),$$

there holds

$$\mathscr{F}\cdot w_\infty=2\int_{\mathscr{E}}(\operatorname{def} w)^2\,dx.$$

Corollary 6.1. $\mathscr{F}\cdot w_\infty>0$, *that is, the solution exhibits a "drag force" in the direction* w_∞.[4]

Theorem 6.2. *Suppose the solution of Theorem* 4.1 *or* 4.2 *satisfies*

$$|w_1(x)-w_{\infty 1}|=o(|x|^{-\frac{1}{2}})$$

as $x\to\infty$. *Then* $w(x)\equiv w_\infty$ *in* $\mathscr{E}$.

Thus the estimate (1.1), considered as a uniform estimate in $\mathscr{E}$ for fixed λ, cannot be improved. We remark, however, that the second component $w_2(x)$ does decay more rapidly at infinity; in fact, $|w_2(x)|=O(|x|^{-1}\log|x|)$ as $x\to\infty$. (We assume $w_\infty=(w_{\infty 1},0)$.)

7. Uniqueness

According to Lemma 5.1, the solutions of Theorem 4.2 satisfy

$$|w_i(x;\lambda)-w_{\infty i}|<M\lambda h_i(\lambda x)|\log\lambda|^{-1}.$$

Thus, for any prescribed $\varepsilon>0$ there will hold

$$\frac{|w_i(x;\lambda)-w_{\infty i}|}{|w_\infty|}<\varepsilon h_i(\lambda x)$$

throughout $\mathscr{E}$ for all sufficiently small λ. We now show that if ε is small, the solution is unique in a corresponding class.

Theorem 7.1. *Let* H *be chosen as in Lemma* 3.7, *let* $0<\varepsilon<(2H)^{-1}$. *Then there is at most one solution* $w(x;\lambda)$ *of* (0.1) *in* $\mathscr{E}$, *such that* $w(x;\lambda)\to w^*$ *on* $\partial\mathscr{E}$, $w(x;\lambda)\to w_\infty$ *at infinity, and*

$$\frac{|w_i(x;\lambda)-w_{\infty i}|}{|w_\infty|}<\varepsilon h_i(\lambda x).$$

Proof. Let $w^{(a)}$, $w^{(b)}$ be two such solutions,

$$u^{(j)}(x;\lambda)=\frac{w^{(j)}-w_\infty}{|w_\infty|},\qquad j=a,b.$$

Then[5]

$$u^{(j)}(x;\lambda)=\lambda\int_{\mathscr{E}}u^{(j)}\cdot u^{(j)}\cdot\nabla\mathbf{G}(x,y;\lambda)\,dy+\int_{\partial\mathscr{E}}u^{*(j)}\cdot T\mathbf{G}\,d\sigma,$$

$$(u^{(a)}-u^{(b)})=\lambda\int_{\mathscr{E}}u^{(a)}\cdot(u^{(a)}-u^{(b)})\cdot\nabla\mathbf{G}\,dy+\lambda\int_{\mathscr{E}}(u^{(a)}-u^{(b)})\cdot u^{(b)}\cdot\nabla\mathbf{G}\,dy.$$

---

[4] For small $|w_\infty|$ this follows alternatively from Theorem 5.4.

[5] The asymptotic estimates of [12] and Theorem 6.61 of [10] show that the outer boundary integrals vanish in the limit.

Let

$$\mu_i = \underset{\mathscr{E}}{\mathrm{l.u.b.}} \frac{|u_i^{(a)} - u_i^{(b)}|}{h_i(\lambda x)}, \qquad i=1,2, \qquad \mu = \max(\mu_1, \mu_2)$$

By Lemma 3.7,

$$|u_i^{(a)} - u_i^{(b)}| \leqq 2\varepsilon\lambda\mu \int_{\mathscr{E}} h_j(\lambda y)\, h_k(\lambda y) \left| \frac{\partial G_{ij}}{\partial y_k} \right| dy$$
$$\leqq 2\varepsilon\mu H\, h_i(\lambda x)$$

so that $\mu \leqq 2\varepsilon\mu H$, a contradiction.

We remark that for solutions in three dimensions, the uniqueness of a small solution has been proved in the class of all solutions which exhibit the same qualitative asymptotic properties at infinity [7]. We have been unable to obtain a comparable result in the present case.

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