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Solution "in the Large" of the Nonstationary Boundary Value Problem for the Navier-Stokes System with Two Space Variables

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1. A Priori Estimates

We consider the Navier-Stokes system

$$
\begin{aligned}
&v_t - \nu\Delta v + \sum_{k=1}^{2} v_k v_{x_k} = -\operatorname{grad} p + f(x, t),\\
&\operatorname{div} v = 0,
\end{aligned} \tag{1}
$$

for the functions $v = (v_1(x_1, x_2, t), v_2(x_1, x_2, t))$ and $p(x_1, x_2, t)$ in the region Ω of the Euclidean x-plane $x = (x_1, x_2)$ with boundary S. We assume the boundary and initial conditions

$$
v|_S = 0, \qquad v|_{t=0} = a(x), \qquad (\operatorname{div} a = 0, \quad a|_S = 0). \tag{2}
$$

It was proved in [1] that the problem (1)—(2) (in the case of two and three space variables) is uniquely solvable for all time $t \geqq 0$, if f has a potential and if the Reynolds number at the initial moment is small, and for a period of time which is short enough even if these conditions are not fulfilled. Moreover, the unique solvability "in the large" of the Cauchy problem for system (1) in the case of two space variables was proved by J. Leray [2] (and later by the author in a different way). As to the question of the unique solvability "in the large" of the boundary value problem (1)—(2), it seemed dubious even for two space variables (see the detailed investigations of J. Leray [3] on this question). Here we establish the following

THEOREM 1. *The problem* (1)—(2) *is uniquely solvable "in the large" (i.e. for all times $t \geqq 0$) for any value of the Reynolds number at the initial moment of time and for arbitrary forces* f, *if only the integrals*

$$
\int_\Omega v^2(x, 0)dx, \qquad \int_\Omega v_t^2(x, 0)dx, \qquad \int_0^t \int_\Omega (f^2 + f_t^2)dxdt
$$

are finite. The region Ω may be bounded or unbounded.[1]

[1]The solution will have the derivatives v_{x_k}, v_t, v_{tx_k} in $L_2(\Omega x[0, t])$ and $v_{x_k x_j}$ in $L_2(\Omega' x[0, t])$, $\Omega' \subset \Omega$.

In [1] the whole question of unique solvability "in the large" of problem (1)—(2) was reduced to obtaining *a priori* estimates for the integrals

$$\int_0^t \int_\Omega \mathbf{v}_t^2 \, dx dt, \qquad \int_\Omega \sum_{k=1}^2 v_k^4 \, dx, \tag{3}$$

or for max $|\mathbf{v}|$. Therefore we shall deal here only with the *a priori* estimates for the solutions of problem (1)—(2).

Let us introduce the following notation:

$$||\mathbf{v}|| = \left(\int_\Omega \mathbf{v}^2 \, dx\right)^{1/2}, \qquad \phi^2(t) = \int_\Omega \sum_{k=1}^2 \mathbf{v}_{x_k}^2(x, t) dx,$$

$$\psi^2(t) = \int_\Omega \mathbf{v}_t^2(x, t) dx, \qquad \mathscr{F}^2(t) = \int_\Omega \sum_{k=1}^2 \mathbf{v}_{tx_k}^2(x, t) dx.$$

It is known (see [1]) that the solutions of problem (1)—(2) satisfy the inequality

$$\begin{aligned} ||\mathbf{v}(x, t)||^2 + 2\nu \int_0^t \phi^2(t) dt &\leqq ||\mathbf{a}||^2 + 2||\mathbf{a}|| \int_0^t ||\mathbf{f}|| dt + 2\left(\int_0^t ||\mathbf{f}|| dt\right)^2 \\ &\leqq 2||\mathbf{a}||^2 + 3\left(\int_0^t ||\mathbf{f}|| dt\right)^2. \end{aligned} \tag{4}$$

Let us find one more estimate for $\mathbf{v}$. In order to do this we differentiate the system with respect to t, scalarly multiply the result by $\mathbf{v}_t$ and integrate over Ω. After simple transformations we obtain the identity

$$\frac{1}{2} \frac{d}{dt} \psi^2(t) + \nu \mathscr{F}^2(t) + \int_\Omega v_{kt} \mathbf{v}_{x_k} \mathbf{v}_t \, dx = \int_\Omega \mathbf{f}_t \mathbf{v}_t \, dx$$

from which

$$\frac{d}{dt} \psi^2(t) + 2\nu \mathscr{F}^2(t) \leqq 2\phi(t) \left(\int_\Omega \sum_{ik} v_{kt}^2 v_{it}^2 \, dx\right)^{1/2} + 2||\mathbf{f}_t|| \psi(t) \tag{5}$$

follows.

Let us verify that for any continuously differentiable function $u(x_1, x_2)$ of compact support in the plane the inequality

$$\iint u^4(x_1, x_2) dx_1 dx_2 \leqq 2 \iint u^2(x_1, x_2) dx_1 dx_2 \iint (u_{x_1}^2 + u_{x_2}^2) dx_1 dx_2 \tag{6}$$

is true. (Here the integration is extended over the whole space.) It is clear that

$$u^2(x_1, x_2) = 2 \int_{-\infty}^{x_k} u u_{x_k} \, dx_k, \qquad k = 1, 2,$$

and hence

$$\max_{x_k} u^2(x_1, x_2) \leqq 2 \int_{-\infty}^{\infty} |u u_{x_k}| dx_k, \qquad k = 1, 2.$$

Therefore

$$\iint_{-\infty}^{\infty} u^4 dx_1 dx_2 \leqq \int_{-\infty}^{\infty} dx_2 \left(\max_{x_1} u^2 \cdot \int_{-\infty}^{\infty} u^2 dx_1\right)$$
$$\leqq 2\int_{-\infty}^{\infty} dx_2 \left(\int_{-\infty}^{\infty} |uu_{x_1}| dx_1 \cdot \max_{x_2} \int_{-\infty}^{\infty} u^2 dx_1\right)$$
$$\leqq 4\iint_{-\infty}^{\infty} |uu_{x_1}| dx_1 dx_2 \cdot \iint_{-\infty}^{\infty} |uu_{x_2}| dx_1 dx_2,$$

implying inequality (6).

Let us use (6) to estimate $\mathscr{I}(t) = (\int_\Omega \sum_{ik} v_{kt}^2 v_{it}^2 \, dx)^{1/2}$ in (5). Since the functions v_{kt} are equal to zero on the boundary of Ω, we have, because of (6), the estimate

$$\mathscr{I}(t) \leqq 2\psi(t)\mathscr{F}(t)$$

and therefore, from (5),

$$\tag{7} \begin{aligned} \frac{d}{dt}\psi^2(t) + 2\nu\mathscr{F}^2(t) &\leqq 2\|\boldsymbol{f}_t\|\psi(t) + 4\phi(t)\psi(t)\mathscr{F}(t) \\ &\leqq 2\|\boldsymbol{f}_t\|\psi(t) + \nu\mathscr{F}^2(t) + \frac{4}{\nu}\phi^2(t)\psi^2(t). \end{aligned}$$

This yields

$$\tag{8} \frac{d}{dt}\psi^2(t) \leqq 2\|\boldsymbol{f}_t\|\psi(t) + \frac{4}{\nu}\phi^2(t)\psi^2(t)$$

and

$$\tag{9} \psi(t) \leqq \psi(0)\exp\left\{\frac{2}{\nu}\int_0^t \phi^2(\tau)d\tau\right\} + \int_0^t \|\boldsymbol{f}_\xi(x,\xi)\| \exp\left\{\frac{2}{\nu}\int_\xi^t \phi^2(\tau)d\tau\right\} d\xi.$$

From (7) and (9) we deduce also

$$\tag{10} \nu\int_0^t \mathscr{F}^2(t)dt \leqq \psi^2(0) + 2\int_0^t \|\boldsymbol{f}_t\|\psi\, dt + \frac{4}{\nu}\int_0^t \phi^2\psi^2 dt.$$

These inequalities give us *a priori* estimates for the solutions which are even stronger than (3). From the proof given above it may be seen that neither the size of the region Ω nor the smoothness of the boundary influence the value of the constants in inequalities (9), (10).

2. Stability of Solutions of Problem (1)—(2)

It is known that the vector space $L_2(\Omega)$ can be decomposed into two orthogonal subspaces: $\mathring{J}(\Omega)$ and $G(\Omega)$. The subspace $G(\Omega)$ consists of gradients of simple valued functions; our solution $\boldsymbol{v}$ and $\boldsymbol{v}_t$ belong to the subspace $\mathring{J}(\Omega)$. Corresponding to this decomposition of $L_2(\Omega)$ we decompose $\boldsymbol{f}$ into two components and add the gradient part to grad p. For the remaining part we shall retain the old notation $\boldsymbol{f}$. It is this part of the force that is included into (4)—(10). For example, if the forces have a potential, then $\boldsymbol{f}$ in (4)—(10) is equal to zero.

Here as well as above we assume that the initial disturbances $\boldsymbol{a}$ are such that $\|\boldsymbol{a}\|$ and $\|\boldsymbol{v}_t(x, 0)\|$ are finite; it is easy to see that these conditions are really fulfilled if $\boldsymbol{a} \in W_2^2(\Omega)$ and $\operatorname{div} \boldsymbol{a} = 0$, $\boldsymbol{a}|_S = 0$.

From inequalities (4)—(10) it follows that for arbitrary initial disturbances the estimate

$$\|\boldsymbol{v}(x, t)\|^2 + \int_0^\infty \phi^2(t)dt \leqq \text{const.}$$

holds if $\int_0^\infty \|\boldsymbol{f}\|dt \leqq$ const., and that the estimate

$$\psi^2(t) + \int_0^\infty \mathscr{F}^2(t)dt \leqq \text{const.}$$

holds if in addition $\int_0^\infty \|\boldsymbol{f}_t\|dt \leqq$ const.. These estimates imply

THEOREM 2. *The solution* $\boldsymbol{v}$ *of problem* (1)—(2) *for which* $\|\boldsymbol{v}(x, 0)\|$ *and* $\|\boldsymbol{v}_t(x, 0)\|$ *are finite and the forces* $\boldsymbol{f}$ *satisfy the condition* $\int_0^\infty (\|\boldsymbol{f}\|+\|\boldsymbol{f}_t\|)dt \leqq$ const. *tends to zero when* $t \to \infty$; *this means that for* $\boldsymbol{v}$ *the integrals* $\int_\Omega \sum_{k=1}^2 \boldsymbol{v}_{x_k}^2(x, t)dx$ *and* $\int_{\Omega'} \boldsymbol{v}^2(x, t)dx$ (Ω' *is any finite part of* Ω) *tend to zero when* $t \to \infty$.

To prove this it is sufficient to use the finiteness of the integrals $\int_0^\infty \phi^2(t)dt$ and $\int_0^\infty \mathscr{F}^2(t)dt$ and to take into consideration that

$$\frac{d\phi^2}{dt} \leqq 2\phi\mathscr{F}, \qquad \int_{\Omega'} \boldsymbol{v}^2 dx \leqq c_{\Omega'} \phi^2.$$

The solution of problem (1)—(2) is stable with respect to changes in the initial conditions and in the external forces. Indeed, the following theorem is true

THEOREM 3. *Let* $\boldsymbol{v}'$ *and* $\boldsymbol{v}''$ *be the solutions of problem* (1)—(2) *corresponding to* $\boldsymbol{a}'(x)$ *and* $\boldsymbol{a}''(x)$ *and to the forces* $\boldsymbol{f}'(x, t)$ *and* $\boldsymbol{f}''(x, t)$. *For the difference* $\boldsymbol{u}(x, t)$ *of these solutions the estimate*

$$\begin{aligned} \|\boldsymbol{u}(x, t)\| \leqq \|\boldsymbol{a}'-\boldsymbol{a}''\| \exp\left\{\frac{2}{\nu}\int_0^t \tilde{\phi}^2(\tau)d\tau\right\} \\ + \int_0^t \|\boldsymbol{f}'_\xi(x, \xi)-\boldsymbol{f}''_\xi(x, \xi)\| \exp\left\{\frac{2}{\nu}\int_\xi^t \tilde{\phi}^2(\tau)d\tau\right\} d\xi \end{aligned} \tag{11}$$

holds. Here $\tilde{\phi}^2(t) = \int_\Omega \sum_{k=1}^2 (\boldsymbol{v}''_{x_k}(x, t))^2 dx$.

To prove this we form the system for $\boldsymbol{u} = \boldsymbol{v}'-\boldsymbol{v}''$, $p = p'-p''$:

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + v'_k \boldsymbol{u}_{x_k} + u_k \boldsymbol{v}''_{x_k} = -\operatorname{grad} p + \boldsymbol{f}, \qquad \operatorname{div} \boldsymbol{u} = 0,$$

where $\boldsymbol{f} = \boldsymbol{f}'-\boldsymbol{f}''$. Let us multiply it scalarly by $\boldsymbol{u}$ and integrate the result over Ω. After some simple transformations we get

$$\frac{1}{2}\frac{d}{dt}||\boldsymbol{u}||^2+\nu\phi^2(t)+\int_\Omega u_k \boldsymbol{v}''_{x_k}\boldsymbol{u}\,dx=\int_\Omega \boldsymbol{f}\boldsymbol{u}\,dx,$$

where $\phi^2(t)=\int_\Omega \sum_{k=1}^2 \boldsymbol{u}^2_{x_k}(x,t)dx$. From this equality we have

$$\frac{d}{dt}||\boldsymbol{u}||^2+2\nu\phi^2(t)\leqq 4\tilde{\phi}||\boldsymbol{u}||\phi+2||\boldsymbol{f}||\cdot||\boldsymbol{u}||, \tag{12}$$

and hence estimate (11) and also the estimate

$$\nu\int_0^t \phi^2(t)dt\leqq ||\boldsymbol{u}(x,0)||^2+2\int_0^t||\boldsymbol{f}||\cdot||\boldsymbol{u}||dt+\frac{4}{\nu}\int_0^t\tilde{\phi}^2||\boldsymbol{u}||^2dt \tag{13}$$

follow. The theorem is proved.

Let us suppose that the force $\boldsymbol{f}''$ does not depend on t and let $\boldsymbol{v}''(x)$ be the corresponding solution of the stationary problem. Let us show that if the "generalized Reynolds number" $2\tilde{\phi}c_\Omega/\nu$ corresponding to $\boldsymbol{v}''(x)$ is less than one, then the solutions $\boldsymbol{v}'(x,t)$ of the non-stationary problem corresponding to the same $\boldsymbol{f}''(x)$ and any $\boldsymbol{a}(x)$ tend, in a certain sense, to $\boldsymbol{v}''(x)$ when $t\to\infty$. Here c_Ω is a constant depending only on the region Ω; c^2_Ω is the supremum of

$$\frac{\int_\Omega \boldsymbol{b}^2\,dx}{\int_\Omega\sum_{k=1}^2 b^2_{x_k}\,dx}$$

taken for all continuously differentiable functions $\boldsymbol{b}(x)$ which vanish on the boundary of Ω.

THEOREM 4. *If for the solution $\boldsymbol{v}''(x)$ of the stationary problem, corresponding to the forces $\boldsymbol{f}''(x)$, "the generalized Reynolds number" $2\tilde{\phi}c_\Omega/\nu$ is less than one, then for all arbitrary solutions of problem* (1)—(2), *corresponding to the same $\boldsymbol{f}''(x)$, the integral*

$$\int_0^\infty\int_\Omega\sum_{k=1}^2[\boldsymbol{v}'_{x_k}(x,t)-\boldsymbol{v}''_{x_k}(x)]^2\,dxdt$$

is finite.

Indeed, from (12), which is true for $\boldsymbol{u}=\boldsymbol{v}'-\boldsymbol{v}''$, and from the inequality $||\boldsymbol{u}(x,t)||\leqq c_\Omega\phi(t)$, we have

$$\frac{d}{dt}||\boldsymbol{u}||^2+2\nu\phi^2(t)\leqq 4c_\Omega\tilde{\phi}\phi^2(t),$$

which implies the statement of our theorem.

Appendix

1. Let us say a few words about the case when Ω is an unbounded region and when for the function $\boldsymbol{a}(x)$ the integral $\int_\Omega \boldsymbol{a}^2(x)dx$ is infinite. Let $\boldsymbol{a}(x)$ be represented at infinity as

$$\boldsymbol{a}(x) = \boldsymbol{a}^{(0)} + \frac{\boldsymbol{a}^{(1)}}{|x|} + O\left(\frac{1}{|x|^2}\right), \qquad \boldsymbol{a}^{(0)} = \text{const.}, \qquad \boldsymbol{a}^{(1)} = \text{const.}.$$

More precisely, it is sufficient to know that $\operatorname{div} \boldsymbol{a} = 0$, $\max_{|x|\gg 1} |\boldsymbol{a}, \boldsymbol{a}_{x_k}| \leqq$ const., and $\boldsymbol{a}_{x_k}, \boldsymbol{a}_{x_k x_j} \in L_2(\Omega)$. Let us take some twice differentiable solenoidal function $\boldsymbol{u}'(x)$ which is equal to zero on S and for which $\max |\boldsymbol{u}', \boldsymbol{u}'_{x_k}| \leqq$ const., $\boldsymbol{a}(x) - \boldsymbol{u}'(x)$, $\boldsymbol{u}'_{x_k}$, $\Delta \boldsymbol{u}' \in L_2(\Omega)$. We shall seek the solution $\boldsymbol{v}(x, t)$ of problem (1)—(2) corresponding to $\boldsymbol{a}(x)$ in the form $\boldsymbol{v}(x, t) = \boldsymbol{u}'(x) + \boldsymbol{u}(x, t)$. To determine $\boldsymbol{u}(x, t)$ we have the system

$$\boldsymbol{u}_t - \nu\Delta \boldsymbol{u} + (u'_k + u_k)(\boldsymbol{u}'_{x_k} + \boldsymbol{u}_{x_k}) = -\operatorname{grad} p + \boldsymbol{f} + \nu\Delta \boldsymbol{u}', \qquad \operatorname{div} \boldsymbol{u} = 0,$$

and the conditions

$$\boldsymbol{u}|_S = 0, \qquad \boldsymbol{u}(x, 0) = \boldsymbol{a}(x) - \boldsymbol{u}'(x) \in L_2(\Omega).$$

It is easy to verify that for $\boldsymbol{u}$ a theorem of the type of Theorem 1 holds. The case of non-homogeneous boundary conditions is treated similarly.

2. Estimates (3), (9), (10) and the results of [1] imply the existence of a "generalized solution" $\boldsymbol{v}$ of problem (1)—(2), which has only the derivatives $\boldsymbol{v}_t$, $\boldsymbol{v}_{x_k}$, $\boldsymbol{v}_{tx_k}$. This solution $\boldsymbol{v}$ satisfies a certain integral identity (see [1]). Proceeding from this identity we proved that $\boldsymbol{v}$ has also the derivatives $\boldsymbol{v}_{x_i x_j}$. Based on J. Leray's article [3], K. Golovkin investigated when "the generalized solution" possesses continuous derivatives.

3. The method of estimates given above permits one to prove the unique solvability "in the large" of problem (1)—(2) for three space variables $x = (x_1, x_2, x_3)$ in case of axial symmetry, provided that the domain Ω has no points in common with the axis of symmetry.

4. While obtaining the *a priori* estimates for the solutions of the non-stationary problem (1)—(2) we encountered the question of existence of inequalities of the type (6). For functions of two variables also the inequality

$$\iint_\Omega u^3 dx_1 dx_2 \leqq \tfrac{9}{8} \iint_\Omega u\, dx_1 dx_2 \iint_\Omega (u_{x_1}^2 + u_{x_2}^2) dx_1 dx_2, \quad \text{if } u|_S = 0, \quad u \geqq 0, \tag{16}$$

is true. The proof of inequality (6) given above is analogous to the proof of (16) which was given by A. O. Gelfond. For functions of three and more independent variables inequalities (6) and (16) do not hold. Instead the following relation, for instance,

$$\iiint_\Omega u^4 dx_1 dx_2 dx_3 \leqq 4 \left(\iiint_\Omega u^2 dx_1 dx_2 dx_3\right)^{1/2} \left(\iiint_\Omega \sum_{k=1}^{3} u_{x_k}^2 dx_1 dx_2 dx_3\right)^{3/2},$$

$$\text{if } u|_S = 0$$

is true for functions of three variables.

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