MOSCOW MATHEMATICAL JOURNAL
Volume 3, Number 2, April–June 2003, Pages 711–737

ELEVEN GREAT PROBLEMS OF MATHEMATICAL HYDRODYNAMICS

V. I. YUDOVICH

Abstract. The key unsolved problems of mathematical fluid dynamics, their current state and outlook are discussed. These problems concern global existence and uniquness theorems for basic boundary and initial-boundary value problems in the theory of ideal and viscous incompressible fluids, the spectral problems in hydrodynamic stability theory for steady and time periodic flows, creation of secondary, tertiary, etc... flow regimes as a result of bifurcations and the asymptotics of vanishing viscosity. Several new problems are formulated.

2000 Math. Subj. Class. 37Nxx, 35Qxx.

Key words and phrases. Incompressible fluid, unsolved problems, existence, uniqueness, stability, asymptotics.

> The prince in person led his troops to attack eleven times.
>
> Alexander Dumas

> Eleven times the foolhardy battalion attacked the enemy.
>
> Nikolay Tikhonov

1. Introduction

Regardless of the fact that this paper was written on the other occasion, I respectfully dedicate it to Vladimir Arnold. An interviewer once asked me whether I had heroes in mathematics. I said that I had just heroes, not gods, and first mentioned Arnold. His unique ability to respond to all alive and new in mathematics and physics by unexpected and stimulating ideas, his impeccable mathematical taste, his extraordinary penetrating power, making us to recall classics, his remarkable gift to point out the research directions promising maximal results — all this made him one of the world leaders in modern mathematics.

This is a slightly extended version of a talk that was given at the Conference on mathematical hydrodynamics at Hull University, UK, on the 10th April 2001. This talk was also repeated at the Newton Institute, Cambridge, on the 23rd April 2001. The title of this paper was suggested by V. A. Vladimirov, who invited me

Received February 16, 2002.
Supported in part by the CDRF Award RMI-2084.

to participate at the conference in Hull. "Why exactly eleven?" I asked him. "For no particular reason," he replied. "Hilbert pointed out 20 problems, and Smale pointed out 19, but these referred to mathematics in general... Well, if you do not like it, we can change the title." However, I liked the number 11. It is not as common as, for instance, 12 or 7. Besides, I recalled that some of my favorite authors mentioned this number while talking about battles and attacks. During the talk, I was reminded that a football team consists of 11 players. More recently, I read in Kazantsev's article in *Shahmatnoe Obozrenie* (Chess Review) no. 2, 2001, that "11, according to investigations of V. I. Avinsky, a founder of alphametrics, is a module of the Universe involved in all dimensions of both micro- and macroworlds." So, this is an intriguing number, and let it stay in the title, although one shouldn't take it too seriously. As a matter of fact, there are actually a greater number of problems, and almost all of them split into even more new problems. While choosing the 11 problems for this list, I tried to use the following criteria:

(1) The solution of the problem should bring us to a new level of understanding of fluid dynamics, or at least help to explain a sufficiently wide range of hydrodynamical phenomena.
(2) In general, it is impossible to solve the problem with any known method. We need some new ideas and approaches which will probably give rise to new mathematical theories and promote the improving art of the description of natural phenomena.
(3) Again quoting S. Smale [1], "We believe that the questions, their solutions, partial results, or even attempts to solve them are likely to be of great importance for mathematics and its development in the next[1] century."
(4) I spent a good amount of time trying to solve the problem and know some things about it. At the very least, I now know of several approaches that do not lead to the desirable results.

As a result almost all of the problems in this list deal with incompressible homogeneous ideal and viscous fluids. However, some others were excluded from the list only for the sake of brevity. Those were various problems concerning compressible fluids, nonhomogeneous fluids, asymptotic models of convection, magnetohydrodynamics, multi-component and especially infinite-component media, analytical dynamics and differential geometry of continuous media, problems with unknown and particularly free boundaries, etc. Some of those omitted still satisfy all of the above four criteria, and I hope to return to them in future publications.

Of course, it would be desirable that a problem be formulated in a mathematically rigorous manner. However, unfortunately, this is not always possible. I suppose that a physical problem cannot be formulated completely until it is resolved. Only a beautiful solution will eventually confirm the correctness of the problem's initial statement.

Many of the problems discussed below are well-known, while some of them are rather new. In the current mathematical literature, one can find solutions for probably all known problems, especially for those where the hypothetical result is sufficiently clear and needs only to be rigorously justified. According to Francois

[1] twenty-first

Rabelais, "every respectable citizen should believe everything he is told and everything which is published." In mathematical hydrodynamics, this great principle should be applied carefully, since too many published proofs are erroneous.

The following is a list of problems with some comments. The first two problems concern the fundamentals of mathematical physics and are not part of the list of eleven great problems in fluid dynamics.

2. Mathematical Models of Hydrodynamics

Problem G1. *Construct mathematical models of continuous media including phase transitions* (*boiling water, ferroelectrics which can turn into dielectrics, liquid crystals, etc.*)*.*

This is mainly a question of the correct mathematical statement of the initial boundary-value problem under conditions when a continuous medium can undergo phase transitions at *a priori* unknown moments of time and in *a priori* unknown regions of the space occupied by it. For example, it is necessary to learn how to describe the flow of water under conditions when its temperature changes within the interval containing one or several points of phase transition (freezing–melting, boiling–condensation, or triple critical points of the equation of state nearby which all three phases can coexist). This problem belongs as much to physics as to mathematics, since the interpretation of the phase transitions is still an unsettled area of physics. Current physical journals regularly publish works on the fundamentals of this theory (see, for example, [2]).

The available phenomenological models of a liquid-gas mixture and boiling water are rather rough, while it would be interesting to obtain appropriate equations starting from the "first principles" of statistical thermodynamics. By the way, the possibility of transition of water into ice reminds us once again about the impossibility to establish partitions between the natural sciences once and for all, since such partitions do not exist in nature.

Interfaces arising at phase transitions very often turn out to be unstable, and waves appear on them. A number of interesting problems are connected with these phenomena; many of them do not even require the creation of new methods and are quite accessible to investigation. I recall the experiment conducted by Rostov physicists (Fridkin and Grekov) [3] as long ago as the 1970s. The edges of a rod made of ferroelectric material (such as barium titanate) were kept at constant temperatures. The temperature was lower than the Curie point on one edge and higher on the other. One could expect that the part of the rod close to the hot edge would be in the dielectric phase and the cold part would be in the ferroelectric phase. In general, the experiment confirmed these expectations, but the point of interface started to oscillate along the rod. As far as I know, an appropriate theory of this phenomenon was never constructed.

Another example: It is doubtless that a flat boundary between water and ice while freezing or melting is often unstable. It would be of interest to investigate the spontaneous waves appearing on such a boundary. It would also be of great interest to consider the parametrically excited waves generated by oscillations of the outer temperature and pressure. The results may become significant for the investigation of glacier motions, the formation and thawing of icebergs, and the freezing of water reservoirs. The obtained results can also be applied in the development of practical methods for the destruction of ice covers on rivers and lakes.

Problem G2. *Determine the dependence of kinetic coefficients* (*viscosity, thermoconductivity, diffusion, surface tension, permittivity, ...*) *on thermodynamic parameters* (*temperature T, pressure p, density ρ, impurity concentration c, ...*)*.*

It is very important to determine the restrictions on kinetic coefficients imposed by the requirement of global solvability in basic evolution initial boundary-value problems. I believe that it is possible to build up a general theory of globally solvable systems of ordinary and partial differential equations. Of course, the global existence of solutions to initial boundary value problems is a very special *physical* property of the system, since, as we know, there are some explosive continuous media which can exist only within a limited period of time. Speaking in mathematical language, the possibility of collapse is a generic property, while the global solvability is in a sense a degeneration. (Such is the eccentric mathematical language — the most interesting and beautiful systems are called *degenerate.*)

3. Uniqueness, global existence, and nonexistence of a solution

Problem 1. *The global solvability of the basic boundary-value problems for the* $3D$ *Euler and Navier–Stokes equations in the case of homogeneous incompressible fluid and regularity of the solutions.*

There is no point in giving a detailed description of these well-known problems, even more so that I recently wrote an extensive article about them [4]. However, it is worth noting that similar problems arise in many areas of nonlinear mathematical physics. The situation in the $2D$ Euler and Navier–Stokes equations is good enough, since there are global theorems on the existence of generalized and smooth solutions, as well as rather strong uniqueness theorems. That is why it is widely believed that only $3D$ problems are difficult, while $2D$ problems are not so hard to solve. As a matter of fact, “$2D$ or not $2D$, that is not the question”. The issue here is not so much two-dimensionality but the specific properties of the Euler and Navier–Stokes equations, which make it possible to obtain strong *a priori* estimates of the solutions. In the case of Euler equations, this is the existence of the vortex integrals. In the case of Navier–Stokes equations, this is the specific embedding theorems for the function spaces that play the decisive role. The kinetic energy in the $3D$ case is still a quadratic functional; however in order to follow in a fashion similar to that for plane flows, one needs the velocity norm in the L_3 space (in the n-dimensional case, we need L_n).

If we consider the generalized solutions of Euler equations with initial velocity fields which possess only finite kinetic energy with no additional assumptions on smoothness, then the advantage of the $2D$ case fails immediately. The question of the existence and uniqueness of a global solution to the basic initial boundary-value problem in the $2D$ case turns out to be as complicated as for the similar $3D$ problem.

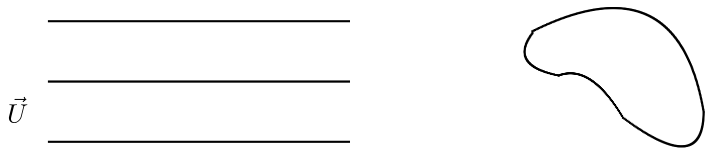

FIGURE 1.

Let us consider also the equations of ideal convection

$$\frac{d\vec{v}}{dt} = -\vec{\nabla} p + \theta \hat{k} \quad (\text{where } \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}), \tag{1}$$

$$\vec{\nabla} \cdot \vec{v} = 0, \tag{2}$$

$$\frac{d\theta}{dt} = 0, \tag{3}$$

$$v_n\big|_{\partial D} = 0, \quad \vec{v}\big|_{t=0} = \vec{v}_0, \quad \theta\big|_{t=0} = \theta_0, \tag{4}$$

where $\vec{v}$ is the velocity field, p is the pressure, θ is temperature, $\hat{k}$ is a unit vector directed upwards, and D is a bounded domain in $\mathbb{R}^3$ or $\mathbb{R}^2$; $\vec{v}_0 \equiv \vec{v}_0(x)$ and $\theta_0 \equiv \theta_0(x)$, where $x \in D$, are the initial velocity and initial temperature fields.

If $\theta_0 = \text{const}$, then $\theta(x, t)$ is constant for all x and t, and we face the problem for Euler equations. The proof of the global existence theorem in the class of smooth solutions is quite inaccessible for non-isothermal flows even in the $2D$ case. The question of whether or not blow up is possible arises once again. Two-dimensionality is of no use in this case, since the conservation law for the vorticity in a fluid particle is no longer valid.

In the right way, all these problems are stated informally: find a proper definition of a (generalized) solution, so that both the global existence theorem and the uniqueness theorem could be proved.

Problem 2. *Global existence theorems for stationary and periodic flows.*

After the classic works by J. Leray [7] and his successors (see, for instance, [5]), the following two problems still resist the efforts of researchers.

Problem 2a. *A global existence theorem for a solution to the $2D$ problem on a viscous fluid flow past a rigid body.*

The velocity at infinity is assumed to be given and equal to a prescribed constant vector $\vec{U}$ (see Fig. 1).

This problem goes back to the Stokes paradox. Stokes established that, in the linear case with the term $(v, \nabla)v$ neglected, the solution doesn't exist. This is in a sharp contrast with the existence of a $3D$ flow past a bounded body, which is a very important result from a practical point of view. The Stokes solution for the slow flow past a sphere has numerous applications in the natural sciences.

We could reverse the problem and consider a translational movement of an infinite rigid cylinder at a constant velocity $-\vec{U}$ in the fluid which is at the rest at infinity. Stokes' result implies that, in the course of time (as $t \to \infty$), all the fluid will start to move at the same velocity $-\vec{U}$ and the condition at infinity will be violated. The question is whether this result will change if we consider the complete Navier–Stokes equations. It is worth to notice that all the principal results on the Navier–Stokes equations were obtained, so to say, *in spite* of nonlinearity, by extending (while struggling against nonlinearity!) results that we can get more or less easily for linearized equations to the complete equations. As for the $2D$ flow problem, the desired result must be obtained *with the help* of nonlinearity. So far, this has been done only for low Reynolds numbers [6].

Similar questions for non-translational motions of a body and for motions periodic in time still stay without proper investigation.

Problem 2b. *Prove or disprove the global existence of stationary and periodic flows of a viscous incompressible fluid in the presence of interior sources and sinks.*

Consider the following steady-state boundary-value problem for the Navier–Stokes system. Let the flow domain D of $\mathbb{R}^3$ or $\mathbb{R}^2$ have boundary ∂D consisting of connected components $S_1, S_2, \ldots, S_k$, and let the velocity v be prescribed along the boundary:

$$v|_{\partial D} = q, \tag{5}$$

where q is a given vector field on the boundary. Then the incompressibility condition (2) imposes certain restriction on the vector field q, namely,

$$\sum_{l=1}^{k} \int_{S_l} q_n \, dS = 0 \tag{6}$$

(i. e., the total velocity flux through the boundary ∂D must be equal to zero). Meanwhile, in the classic work by J. Leray [7], the global existence theorem for the stationary flow was proved only under the more restrictive condition

$$\int_{S_1} q_n \, dS = \cdots = \int_{S_k} q_n \, dS = 0, \tag{7}$$

which coincides with the necessary condition (6) only in the case of a connected boundary (i. e., when $k = 1$). Condition (7) means that the fluid neither enters the flow domain D from the interior domains bounded by the surfaces $S_1, S_2, \ldots, S_k$ (we assume that S_k is the outer boundary of the flow domain), nor leaves the domain D through these surfaces. So there are no interior sources or sinks (more precisely, their sum is equal to zero).

It was necessary to impose the same boundary condition (7) on the boundary field $q(x, t)$ in order to prove the global existence theorem for periodic motions [8]. We face the following problem:

Prove (*or disprove by constructing a counterexample*) *a global theorem on existence of stationary and forced periodic motions of viscous incompressible fluids in the case when the domain D in $\mathbb{R}^2$ or $\mathbb{R}^3$ has boundary which consists of connected components $S_1, \ldots, S_k$, where $k > 1$, and only the necessary condition* (6) *holds.*

Here the boundary ∂D, the vector field q, and the external mass force $F(x)$ in the stationary case and $F(x, t)$ in the periodic case are assumed to be C^∞-smooth.

My feeling is that the solution is most likely negative. If this is true, then the necessary counter-examples can be constructed, probably even for the simplest case of the concentric circular ring.

The exterior problems with interior sources and sinks are also not without interest. Some unusual phenomena related to outer rotationally symmetric flows are considered in [9].

In addition, I would like to note that condition (7) makes it possible to prove the dissipativity of the non-stationary Navier–Stokes system [10]. When only the general condition (6) is valid, this result (except in the slow flow case) will probably fail as well.

It seems to be possible that, when the Reynolds number increases, the stationary regime can disappear (i. e., move to infinity in the corresponding function space) as $R \to R_*$, where the critical value of R_* is finite. However, before disappearing, this stationary regime becomes unstable and generates a self-oscillating periodic regime. Of course, there is a chance that at first the branching in the class of stationary regimes takes place. It would be interesting to examine the possibility of such a march of events, at least by a numerical experiment.

4. General stability theory for viscous fluid flows

Problem 3. *The existence of unstable stationary and periodic flows in an arbitrary domain.*

Let $a = a(x)$ be a velocity field of a stationary flow of a viscous incompressible fluid in a prescribed bounded domain D in $\mathbb{R}^2$ or $\mathbb{R}^3$. We assume a to be a solution to the boundary-value problem for the Navier–Stokes system with prescribed external forces and boundary velocity field. By linearizing the Navier–Stokes equation on this *basic flow* and searching for a solution in the form $e^{\sigma t}u(x)$, we get the following spectral problem:

$$\sigma u + (u, \nabla)a + (a, \nabla)u = -\nabla q + \nu\Delta u, \tag{8}$$

$$\nabla \cdot u = 0, \tag{9}$$

$$u|_{\partial D} = 0. \tag{10}$$

We define the *stability spectrum* of the main flow a (denoted by Σ_a) to be the set of the complex numbers σ for which the problem (8)–(10) has a nonzero solution. It is well-known that the stability spectrum of any flow is countable and the corresponding system of eigenvectors and adjoint vectors is complete [14]. (Note that, in the case of an unbounded domain, one should consider also the *continuous spectrum.*)

Let us try to imagine a great future hydrodynamic stability theory that has already solved all the fundamental problems and is able to entrust with computers the investigation of particular flows, their stability and transitions. Maybe, the following concepts will play an essential role in this theory.

Definition 1. The destabilizer $\mathcal{D} = \mathcal{D}(D)$ is the set of all smooth solenoidal ($\operatorname{div} v = 0$) vector fields a on the domain D such that the spectral problem (8)–(10)

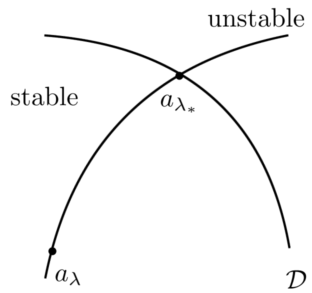

FIGURE 2.

has at least one eigenvalue σ_0 on the imaginary axis:

$$\mathcal{D} = \mathcal{D}(D) = \{a\colon \nabla\cdot a = 0,\ \exists\sigma_0 \in \Sigma_a\colon\ \operatorname{Re}\sigma_0 = 0\}. \tag{11}$$

Definition 2. The bifurcator $\mathcal{B} = \mathcal{B}(D)$ is the set of all solenoidal vector fields a on the domain D such that the stability spectrum Σ_a contains the point 0:

$$\mathcal{B} = \mathcal{B}(D) = \{a\colon \nabla\cdot a = 0,\ 0 \in \Sigma_a\}. \tag{12}$$

Definition 3. The oscillator $\mathcal{O} = \mathcal{O}(D)$ is the set of all smooth solenoidal vector fields a on the domain D such that the stability spectrum Σ_a contains at least one pair of complex conjugate numbers $\pm i\omega$, $\omega \neq 0$:

$$\mathcal{O} = \mathcal{O}(D) = \{a\colon \nabla\cdot a = 0,\ \exists\omega \in \mathbb{R},\ \omega \neq 0,\ i\omega \in \Sigma_a\}. \tag{13}$$

Now, let $a_\lambda = a_\lambda(x)$ be a solenoidal vector field depending on a real parameter λ. Assume that, if $\lambda = 0$, this flow $a_0 = a_0(x)$ is asymptotically stable, and its stability spectrum is situated in the left half-plane. Then the property of asymptotic stability is also preserved for small λ. Let us now gradually increase λ (without loss of generality, we can assume that $\lambda > 0$). It is possible that the flow a_λ is unstable for some λ. The critical values of λ_*, which correspond to transitions of the eigenvalues from the stable half-plane to the unstable one (in particular, those which separate the intervals of stability and instability), are determined by the condition that the spectrum Σ_a includes at least one point of the imaginary axis. In other words, the critical values λ_* are defined by the condition $a_{\lambda_*} \in \mathcal{D}(D)$ (Fig. 2). Of course, while we change the parameter λ, the curve $\{a_\lambda\}$ may cross the destabilizer $\mathcal{D}$ several times.

If it is already known that the flow a_λ loses its stability, the question arises about the nature of the corresponding transition. Generically, the answer depends mainly on the nature of the *neutral spectrum* (the intersection of the spectrum with the imaginary axis) of the critical flow a_{λ_*}. If $a_{\lambda_*} \in \mathcal{B}(D)$, one can expect branching of the stationary regimes. And if $a_{\lambda_*} \in \mathcal{O}(D)$, then (again generically) the Poincaré–Andronov–Hopf bifurcation of branching off the cycle (self-oscillatory periodic regime) takes place.

If we imagine that the sets $\mathcal{D}$, $\mathcal{B}$, and $\mathcal{O}$ are stored in a computer memory, then, in each particular case, we need only to track up when the family $\{a_\lambda\}$ hits them.

Let us state the problem in the following form.

Prove that, for any domain D in $\mathbb{R}^3$ or $\mathbb{R}^2$, the sets $\mathcal{D}(D)$, $\mathcal{B}(D)$, and $\mathcal{O}(D)$ are nonempty.

So far, this result is known only for rotationally symmetric domains in $\mathbb{R}^3$ [11]–[13]. Certainly, when it is proved that the sets $\mathcal{D}(D)$, $\mathcal{B}(D)$, and $\mathcal{O}(D)$ are nonempty (which is the main property of any set), the questions on the structures of these sets will arise. Each of them are likely to be stratified with respect to the codimensions of the bifurcations arising when the family $\{a_\lambda\}$ intersects them.

It would be natural to raise quite similar questions for periodic regimes (for instance, of fixed period p). In addition to the sets $\mathcal{D}_p$, $\mathcal{B}_p$, and $\mathcal{O}_p$, which naturally generalize the sets $\mathcal{D}$, $\mathcal{B}$, and $\mathcal{O}$, we need to include here the *duplicator* $\mathcal{D}b_p$, i. e., the set of all time-dependent solenoidal vector fields in the domain D such that they are p-periodic in t and the corresponding monodromy operator has multiplier -1. Nonlinear perturbations of these critical situations lead generically to the *period doubling bifurcation.*

Finally, we notice that, in the finite-dimensional case, the definitions given above work fairly well. For example, it turns out to be possible to construct sets similar to $\mathcal{D}$, $\mathcal{B}$, and $\mathcal{O}$ for the Galerkin approximations to the Navier–Stokes equations (Yudovich V. I., *On the bifurcators, oscillators and destabilizers of the Navier–Stokes system* (in preparation)).

Problem 4. *The completeness of the Floquet solution system in the stability problem for periodic flows of viscous fluids.*

Linearization of the Navier–Stokes equations on a known T-periodic flow $a(x, t)$ in a (bounded) domain D with a rigid boundary produces a system of equations with T-periodic coefficients. Searching for solutions of the form $e^{\sigma t}u(x, t)$, where the vector-function u is T-periodic, we obtain a spectral problem with complex parameter σ:

$$\frac{\partial u}{\partial t} + \sigma u + (u, \nabla)a + (a, \nabla)u = -\nabla q + \nu\Delta u, \tag{14}$$

$$\nabla \cdot u = 0, \tag{15}$$

$$u|_{\partial D} = 0. \tag{16}$$

The set of complex numbers σ for which this problem has a nonzero solution is called *the stability spectrum*, or *Floquet spectrum*, of the flow $a(x, t)$. Note that, if this spectrum contains a point σ, then it also contains a countable number of points $\sigma + in\omega$, where $\omega = 2\pi/T$ and $n \in Z$. Along with $e^{\sigma t}u(x, t)$, the solutions of the form $e^{\sigma t}\sum_{k=0}^{n} t^k u_m^k(x, t)$, where $m = 0, 1, \ldots, r$, are also referred to as Floquet solutions. Here u_m^k are T-periodic vector-functions, which are called adjoint Floquet solutions, or generalized Floquet solutions (which is extremely ambiguous).

Let us define the Hilbert space $S_2(D)$ as the closure of the set of all C^∞-smooth and compactly supported solenoidal vector fields on the domain D with $L_2(D)$-norm. We formulate the following problem:

Prove that, for any T-periodic solution a, the system of Floquet solutions is complete, i. e., their values at $t = 0$ form a complete system in $S_2(D)$.

Let us introduce the *monodromy operator* U_T of the linearized system, which is obtained from (14)–(16) when $\sigma = 0$. By definition, for any solution $u(x, t)$ of this system, we have $U_T u_0 = u(\cdot, T)$, where u_0 is the initial value: $u_0(x) = u(x, 0)$. An equivalent statement of the problem is:

Prove that the monodromy operator U_T has a complete system of eigen- and adjoint vectors.

In the case of a stationary flow, the completeness of the normal modes system was proved long ago by S. G. Krein (see [14]). The application of the well-known Keldysh theorem plays the crucial role in this proof. The idea is that for $a = 0$, we have a self-adjoint spectral problem, for which the completeness of the eigenvector system can be derived in a standard way, with the help of the Hilbert–Schmidt theorem. The terms that contain the flow a form a weak in a sense, if not small, perturbation of the basic self-adjoint positive-definite operator. This is what makes it possible to apply the Keldysh theorem.

It is rather surprising that, in the periodic case, the Keldysh theorem is non-applicable. Some results on the completeness of Floquet solutions were obtained in the 1970s by my post-graduate student A. I. Miloslavsky [15]. He instead applied the Dunford–Schwartz theorem [16], which also tells us that the completeness of the root vector system is preserved under perturbations, though it is based on principles quite different from those used by the Keldysh theorem. The conditions of this theorem prohibit the eigenvalues of the unperturbed operator from coming arbitrarily close to each other. This kind of behavior is typical for spectral boundary-value problems on the interval or on a plane domain. However, for the Laplace operator (and for the Stokes operator as well) in a bounded domain D in $\mathbb{R}^m$, the eigenvalues λ_n grow like $n^{2/m}$ as $n \to \infty$ and approach each other for $m \geq 3$. It is due to this restriction that the desired conclusion follows from Miloslavsky's general theorems only in the cases of total separation of variables, when the problem is reduced to a second-order parabolic equation (or a system of such equations) with coefficients periodic in t and only one spatial variable. This class certainly includes a lot of interesting flows, such as parallel flows in a circular pipe or in a channel, time-periodic symmetric flows between two coaxial cylinders, etc. But the general problem turned out to be very complicated, and the difficulties here are of a fundamental nature.

Concentrating our attention on the essential properties of the linearized Navier–Stokes system which we are really able to deal with, we come to the following abstract statement of the problem.

Consider the following ordinary differential equation in the Hilbert space H:

$$\frac{du}{dt} + Au = B(t)u, \tag{17}$$

where A is a (constant) self-adjoint operator (similar to the Laplace operator $-\Delta$ or the Stokes operator $-\Pi\Delta$) such that its inverse operator $A^{-1} = G$ (Green operator) is completely continuous. The operator-function $B(t)$ is t-periodic with period T; for any t, it is subordinate to the operator A in the strong sense which means that the operator-functions $G^{1/2}B(t)$ and $B(t)G^{1/2}$ are bounded and continuous in t with respect to the uniform operator topology (generated by the operator norm on H). More precisely, the operator $B(t)$ can be unbounded and not defined everywhere, but its domain $(\mathrm{dom}(B(t))$ should be everywhere dense and contain the image of the operator $G^{1/2}$. At the same time, the above-mentioned operator-functions must be continuously extendable up to bounded and continuous operator-functions.

In the case where B is constant, the completeness can be derived from the Keldysh theorem. Seemingly, this result can be extended to operators $B(t)$ periodic in t. However, Miloslavsky constructed an example of an equation of form (17) with bounded coefficient $B(t)$ for which the monodromy operator is quasi-nilpotent! This equation does not have Floquet solutions at all. Note that the importance of the completeness property of the normal modes or of the Floquet solutions in the stability/instability problem is strongly exaggerated by many authors. The fact that the equation (17) does not have Floquet solutions indicates its *overstability*: each of its solutions decays faster than any exponential as $t \to +\infty$, at least like $e^{-kt \ln t}$ for some $k > 0$, or even like $e^{-kt^{\alpha}}$ for some $\alpha > 1$.

The example of Miloslavsky is of a rather abstract nature. It would be very interesting to determine if this kind of overstability is possible for parabolic partial differential equations and for the linearized Navier–Stokes system. My guess is that this is not possible; it is more likely that overstability exists on some invariant subspace. The reason is that any pair of differential operators with variable coefficients, say, of orders m and n, "almost commute", their commutator "loses the order". This differential operator is of order less than $m + n$. On the other hand, for commuting (in the natural sense) operators A and $B(t)$, the conclusion on the completeness certainly holds.

I would also like to refer to the paper [18], where an example of a parabolic equation of the form $\frac{\partial u}{\partial t} - \Delta u = q(x,\, t)u$ on the torus T^3 with coefficients bounded with respect to x and t is constructed. In this example, the equation has some solutions decaying like e^{-ct^2} with $c > 0$ as $t \to +\infty$. However, the issue remains open, whether or not such a fast damping is possible when the function q is periodic in t.

There exists a class of equations (17) with self-adjoint and strictly positive monodromy operators. These are equations for which the following condition ([12], [13]) is satisfied:

$$B^*(-t) = B(t). \tag{18}$$

In this case, the monodromy operator of equation (17) certainly has an orthonormal eigenbasis. Unfortunately, only a few rotational periodic fluid flows and periodic convective flows of a stratified fluid lead to equations of form (17) with the condition (18) satisfied.

5. Stability of ideal fluid flows

Problem 5. *Justify the validity of linearization in the problem on the instability of a stationary flow of an ideal incompressible fluid with respect to weak norms.*

Instability must be understood as a lack of Lyapunov stability. The definition of Lyapunov stability uses the norm on the function space of solenoidal vector fields tangent to the boundary of the flow domain. The answer to the stability question depends crucially on the choice of this norm [14], [19]–[22]. There exist strong reasons to believe that all the flows of ideal incompressible fluid are unstable with respect to "strong" norms, such as $\max_x |\nabla \times v(x, t)| + \cdots$ in the $3D$-case and $\max_x |\nabla(\nabla \times v(x, t))| + \cdots$ in the $2D$-case. Here the dots stand for weaker norms, for instance the L_2-norm. Although this statement in its general form still remains a hypothesis, a large number of examples and theorems, related to various types of flow, strongly confirms its correctness. Apparently, even a solid rotation of a fluid is not an exception.

Everything convinces us that even L_p-norms of a curl in the $3D$-case and its C^λ-norms in the $2D$-case also grow infinitely as $t \to \infty$ for very wide classes of flows. These classes are likely to be so wide that no stationary flow that is Lyapunov stable with respect to these norms exists.

In the $2D$-case, the known global existence theorem suggests a natural choice of the norm. It is the norm in the space V of solenoidal vector fields in the domain $D \subset \mathbb{R}^2$ with a bounded vortex:

$$\|v\|_V = \operatorname*{ess\,max}_{x \in D} |\nabla \operatorname{curl} v(x)| + \cdots . \tag{19}$$

Here dots again denote a minor norm. The norm of the vector field $v(x, t)$ in the space V is estimated uniformly in $t \in R$ [26]. This is the strongest norm that is still uniformly bounded for all $t \in R$.

In any case, it is obvious that the stability and instability definitions are of interest only when stable flows exist. It is easy to check that flows with constant vortex are Lyapunov stable in the space V.

In the $3D$-case, the "right" choice of a norm is obscure, at least because we do not know of any global existence theorem for the initial boundary-value problem. At the same time, in the case of Lyapunov stability, according to the definition, the motion in the presence of small initial perturbations must be defined for any $t > 0$. In principle, this does not prevent us from treating the collapse (going of the motion to infinity for a finite time) as a special case of *instability.* Maybe, we have to soften the definition of the Lyapunov stability, admitting only smooth perturbations from some set which should be everywhere dense in the chosen function space. Somehow or other, at the moment, there are only two reasonable candidates for the role of the "right" norm, namely, the C-norm and L_2-norm. In Problem 5, another choice is, of course, possible; however, the norm must be weaker than $\max |\operatorname{curl} v|$ in the case of $3D$ flows.

At present, only one general result on the justification of linearization in the stability problem for stationary flows of ideal incompressible fluid is known [23]. However, in [23], in the case when the stability spectrum contains a point of the right half-plane, the instability was proved only for norms which were much too strong. The other weakness of this article is the presence of an additional restriction on the spectrum. This is the requirement that the spectrum must contain a spectral set which lies entirely in the right half-plane (in fact, a stronger requirement may be needed). I suppose, this disadvantage can be easily removed with the use of Krein's [24] approach connected with the so-called "almost eigenvectors".

Problem 6. *Justification of Arnold's method in the stability problem for an ideal fluid flow.*

In spite of the significant progress achieved by applying Arnold's method, beginning from his pioneering works of the mid-1960s (see [25]), many fundamental questions of the theory remained in a shadow and are still unclear.

Problem 6a. *Prove the Lyapunov stability in the space V in the case when the stationary flow satisfies Arnold's criterion.*

Let me remind the reader that, in the case of a stationary flow with stream function ψ satisfying the equation

$$\psi = F(\Delta\psi), \tag{20}$$

this criterion requires that the quadratic form

$$\mathcal{H} = \mathcal{H}[\varphi] = \frac{1}{2}\int_D \left[(\nabla\varphi)^2 + \frac{\nabla\psi}{\nabla\Delta\psi}(\Delta\varphi)^2\right] dx\, dy, \tag{21}$$

$$\varphi\big|_{\partial D} = 0. \tag{22}$$

be positive-definite or negative-definite.

Under natural restrictions, Arnold proved an *a priori* estimate for L_2-norm of the vorticity perturbation $\|\Delta\varphi(\cdot,\, t)\|_{L_2(D)}$. However, for such initial data, although we know the global existence theorem [26], there is no uniqueness theorem. To prove uniqueness, it is sufficient to assume that the initial velocity belongs to V, i. e., $\Delta\varphi \in L_\infty(D)$ (see also [27], where the uniqueness is proved for some class of flows with unbounded vorticity). So the stability in this space is proved only in some impaired sense: even if there are many perturbed flows (corresponding to the same initial data), all of them are very close to the basic stationary flow. In a natural way, the following problem arises.

Problem 6b. *Prove* (*or disprove*) *the uniqueness of the solution to the basic initial boundary value problem for the Euler equations in a bounded domain D in the case when the initial vorticity belongs to $L_p(D)$ for some $p > 1$.*

So far, we have to acknowledge that the Lyapunov stability in the space V is proved completely only for the flows with constant vorticity. By the way, for such flows, the form (21) is not defined, and the result about stability is obtained directly. It is time to note that by no means all stationary flows satisfy an equation of form (20) with a univalent and smooth function F. Generally, a stationary flow is defined by the equation

$$\frac{\mathcal{D}(\psi,\, \Delta\psi)}{\mathcal{D}(x,\, y)} = 0, \tag{23}$$

i. e., the requirement that ψ and $\Delta\psi$ are functionally dependent. For instance, the starting hypotheses of Arnold are broken for the stream function defined uniquely by the boundary-value problem

$$\begin{gathered} -\Delta\psi = -\psi^3 + 1, \\ \psi\big|_{\partial D} = 0. \end{gathered} \tag{24}$$

In this case, $\psi = \sqrt[3]{\Delta\psi + 1}$, so the function F exists, but it is not smooth. Another example in which there does not generally exist any univalent function F: the function ψ satisfies the equation

$$\psi^2 + (\Delta\psi)^2 = 1 \tag{25}$$

and the boundary condition is $\psi\big|_{\partial D} = 0$. The stability problem for such flows remains completely uninvestigated.

Problem 6c. *Investigate the stability of the flows* (24), (25) and similar.

It is most likely that all flows which do not admit a univalent function F are unstable. Thereupon, I would like to draw attention to the works [28], [29], where it was proved that the solution of the problem about maximum of the kinetic energy on the set of isovortical vector fields leads to flows with univalent dependence between ψ and $\Delta\psi$.

Further, it is important in principle to develop the Arnold approach, which is a special form of the direct Lyapunov method, as applied to the instability problem.

Problem 6d. *Prove the instability of a stationary flow in the case when the Arnold criterion is roughly violated.*

Seemingly, everything confirms the opinion of the famous author that his criterion is "close to necessary" (see, for instance, [30]). However, as a matter of fact, in hydrodynamics, instability is very rarely established by using the direct Lyapunov method. A kind of exception is given by the results of Vladimirov [31] obtained with the use of virials in problems related to the motion of a body in a fluid.

It is a pity that the beautiful criterion for the stability of a three-dimensional stationary flow obtained by Arnold, as it was expected by its author, turned to be inapplicable to any flow, except, maybe, to the rigid rotation.

Problem 6e. *Does a stable three-dimensional stationary flow of an ideal incompressible fluid exist?*

It is likely that even a rigid rotation is unstable with respect to strong norms, for instance, with respect to the vorticity norm $\max|\operatorname{curl} v| + \cdots$. Thus, if the stable flow does exist, we still need to explain in what sense (in what function space, etc.) it is stable.

6. Stability of the simplest laminar flows and first transition

Problem 7. *Prove that the Hagen–Poiseuille flow in a circular pipe and the Couette flow in a channel are absolutely stable* (*i. e., stable at any Reynolds number*)*.*

This time, we are dealing with rigorously formulated spectral boundary-value problems for ordinary differential equations (see, for instance, [32]). In the case of the Poiseuille flow, confining ourselves to axisymmetric perturbations, we should prove that all eigenvalues σ of the following spectral problem are located in the left half-plane $\operatorname{Re}\sigma < 0$:

$$\{(L-\alpha^2) - [\sigma + i\alpha R(1-r^2)]\}(L-\alpha^2)\psi = 0, \tag{26}$$

$$\psi = \psi' = 0, \quad r = 1, \tag{27}$$

where α is the wave number of the perturbation, R is the Reynolds number, σ is a complex parameter, and the function $\psi = \psi(r)$ is defined on the segment $[0, 1]$. The second-order differential operator L is defined by the equality

$$L = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} = \frac{d}{dr}\left(\frac{d}{dr} + \frac{1}{r}\right). \tag{28}$$

There is no boundary condition at $r = 0$, and it is easy to understand why: the points of the axis of the pipe are interior for the initial problem in the cylinder, and there are no singularities at them. Instead of a boundary condition, we state the "boundedness condition" coming from the requirement that the rate of energy dissipation is finite (or that the velocity field belongs to the class $W_2^{(1)}$). This condition has the form

$$\int_0^1 (|(L-\alpha^2)\psi|^2 + |\psi|^2) r\,dr < \infty. \tag{29}$$

In fact, we must also handle the corresponding spectral boundary-value problems for non-axisymmetric perturbations. They are well-known (see [32]), and let me omit them here.

It is necessary to prove that, for arbitrary real α and R, all possible eigenvalues σ of the spectral problem (26)–(28) are situated in the left half-plane $\operatorname{Re}\sigma < 0$.

For $R = 0$ (and an arbitrary α), we have the self-adjoint boundary-value problem and all eigenvalues σ are negative (the equilibrium state is, of course, asymptotically stable). Being guided by the results of perturbation theory and simple estimates of the σ-spectrum, we can easily prove that the eigenvalues cannot go to infinity for finite values of the Reynolds number R. Hence the absolute stability property is equivalent to the non-existence of a critical value of the Reynolds number.

By a critical value, we mean a value $R = R_*$ such that there exists at least one eigenvalue σ_0 on the imaginary axis. It is convenient to express it in the form $\sigma_0 = -i\alpha c R$, where c is an unknown real constant (the phase velocity of the neutral perturbation). Thus, the absolute stability problem can be formulated in the following form.

Prove that, for any real α and R, the differential equation

$$(L-\alpha^2)^2\psi - \lambda g(r)(L-\alpha^2)\psi = 0 \tag{30}$$

with conditions (27), (28) has only zero solution. Here we put

$$g(r) = 1 - r^2 - c; \qquad \lambda = i\alpha R. \tag{31}$$

Let us emphasize that only pure imaginary eigenvalues have physical sense. If, instead of the boundary conditions (27), we take the conditions

$$\psi = L\psi = 0, \qquad r = 1, \tag{32}$$

then we obtain the self-adjoint Sturm–Liouville boundary-value problem for the function $\omega = (L-\alpha^2)\psi$. In this case, all eigenvalues of the parameter λ are real. This proves the absolute stability in the case of "soft" boundary conditions (32).

The idea appears to watch the change of eigenvalues λ resulting from the change of the boundary conditions (27) into the conditions (32) and prove that they remain real. If this were the case, then the problem would be resolved. Alas, this is true only in the case $c \notin (0, 1)$. However, for $c \in (0, 1)$, the boundary-value problem (26), (28), (32) admits complex (unreal) eigenvalues together with the set of real eigenvalues [33]. The fact that the phase velocity c should lie in the interval of values of the velocity of the Poiseuille flow can be obtained more directly from integral estimates [33].

In the case of the Couette flow in a channel, we come by the same way to the following problem.

Prove the absolute stability of the plane Couette flow in a channel by establishing that the boundary-value problem

$$(D^2 - \alpha^2)^2 u = i\alpha R(y - c)(D^2 - \alpha^2)u, \qquad D = \frac{d}{dy}, \tag{33}$$

$$u = Du = 0 \quad (y = \mp 1) \tag{34}$$

on the segment $[0, 1]$ *has only the zero solution for all real* α *and* R.

It should be noted that, beyond a shadow of a doubt, the Poiseuille flow in a pipe and the Couette flow in a channel are absolutely stable. This statement is supported by repeated and very bulky calculations (that is true, time and again, "no doubt" statements turn out to be wrong). Moreover, for the Couette flow, an "almost analytical" proof was constructed [34]. The numerical part of this work was reduced to the checking of some, not so complicated, inequality for the Bessel function.

However, I would like to believe that it is possible to construct some beautiful algebraic-analytical proof. I imagine a general theorem which imply without embarrassment the absolute stability of both flows. This theorem cannot be very general, because it should be based on rather deep and special properties of the linearized Navier–Stokes equations. The point is that, for parallel flows in non-circular tubes, which are quite similar to the Poiseuille flow, stability most likely can be lost already for finite values of the Reynolds number. It would be desirable (and I hope not so difficult) to prove this rigorously for tubes with elongated rectangular and elliptic cross-sections.

It is also interesting to note that, for the Poiseuille–Couette flow in a channel with the profile $\mathcal{U}(y) = ay + b(1 - y^2)$, according to calculations of several authors, absolute stability takes place not only for $b = 0$ (pure Couette) but also for sufficiently small values of the parameter $k = |b/a|$, say for $k < k_*$; for $a = 0$, we have the Poiseuille flow in a channel, which is unstable for large R. A really good theory should also predict the value k_* separating absolutely stable and unstable flows. The role of a computer should be reduced to the calculation of the concrete values of the parameters $k_* = k_*(\alpha)$ and $R_* = R_*(\alpha)$ for $k > k_*(\alpha)$.

Several other absolutely stable flows are known. Such is, for example, the Couette flow in the case when only the outer cylinder is rotating. Another example is given by the Kolmogorov spatially periodic flow with sinusoidal profile in the case of a short longitudinal period. For these particular flows the absolute stability is proved, and for the former, even global (nonlinear) stability takes place [36]. However, the problems on global nonlinear stability and the development of turbulence still remain topical. We discuss these problems below, in this and next sections.

Problem 8. Exchange stabilities principle. When a parameter on which the basic regime depends arrives at its critical value, generically, the following two basic cases can occur: either a pair of complex conjugate eigenvalues appear on the imaginary axis or the eigenvalue σ_0 is 0. The term *oscillatory instability* is attributed to the former case, while with the latter we connect the term *monotonous instability*.

In the second case, we also say that the monotonicity principle, or the *exchange of stabilities principle*, takes place. The former term has remained from that (short) time when researchers believed that, in viscous fluid dynamics, instability is always monotonic.

In several cases, it turn out to be possible to prove the monotonicity principle rigorously. I mention the free convection problem, in which the result was achieved by reduction to the spectral problem for a self-adjoint operator. For the spatially periodic Kolmogorov flow with velocity profile $\mathcal{U} = \sin y$, the monotonicity principle was proved in [35] with the help of explicit analytical considerations based on the possibility to express the characteristic equation by means of continued fractions. For several special steady rotational flows periodic in t, the monotonicity principle was established in [12], [13]. Sometimes, in order to justify the monotonicity principle, we can apply the theorem on the positive leading eigenvalue of a positive linear operator (Perron–Frobenius–Ientch–Rutman–M. G. Krein).

However, in the most interesting hydrodynamical case of the Couette–Taylor flow between *co-rotating* rigid cylinders, the principle is still not justified. The following mathematical problem is not resolved.

Prove that, for an arbitrary $\alpha \in \mathbb{R}$, the minimal critical Reynolds number R of the spectral boundary value problem

$$(L - \alpha^2)^2 u - \sigma(L - \alpha^2)u = 2\alpha^2 R\omega(r)v, \tag{35}$$

$$(L - \alpha^2)v - \sigma v = -\lambda g(r)u, \tag{36}$$

$$u = u' = v = 0 \quad (r = r_1,\, r_2) \tag{37}$$

corresponds to the eigenvalue $\sigma = 0$.

Here r_1, r_2 are the radii of the cylinders, $0 < r_1 < r_2$, α is the axial wave number, and the functions ω and g are expressed through the basic Couette profile

$$v_0 = v_0(r) = Ar + \frac{B}{r} \tag{38}$$

by the equalities

$$\omega(r) = \frac{v_0(r)}{r}; \quad g(r) = -\left(\frac{dv_0}{dr} + \frac{v_0}{r}\right). \tag{39}$$

The constants A and B are defined by the boundary conditions $v_0(r_1) = \Omega_1 r_1$, $v_0(r_2) = \Omega_2 r_2$, where Ω_1 and Ω_2 are the angular velocities of the cylinders. At that we should assume that $\omega(r) \geq 0$ for $r \in [r_1, r_2]$) and the Synge condition for instability at $A < 0$ holds (for $A \geq 0$, stability takes place for all R; see [32]).

Repeated many times, calculations and natural experiments show us, beyond a shadow of a doubt, that the monotonicity principle is true. In some particular cases (narrow gap $r_2 - r_1$, close angular velocities Ω_1 and Ω_2), it was actually proved. However, a rigorous proof for the general case is still absent. Let me repeat, the problem here is not the mathematical rigor *per se*, but it is necessary to understand the reasons of the phenomenon. Why self-oscillations do not appear at the first transition? When the desirable result is achieved, no doubt, it will enable us to decide, together with this particular case, many other problems, and first of all for the rotational flows with profiles different from the Couette one (38).

Note that, after the spectral problem (35)–(37), we must consider also the spectral problems for non-rotationally symmetric modes depending on the polar angle through the multiplier $e^{im\theta}$.

The existence of an infinite sequence of critical values $R_1(\alpha) < R_2(\alpha) < \dots$ going to infinity at $\sigma = 0$ (and even for $\sigma > 0$) was proved by reduction to the integral equation with an *oscillatory* (in the Gantmakher–Krein sense) kernel [36]. Thus we will have the right to consider the problem as completely resolved when it is rigorously proved that, at $R < R_1$, the entire σ-spectrum (including its part corresponding to the non-symmetric modes, for $m \neq 0$) is located in the left half-plane.

In the case of counter-rotating cylinders, the existence of monotonous instability critical values corresponding to the creation of Taylor vortices was proved in [37]. In this case, however, the monotonicity principle is not always valid: for large angular velocities of the outer cylinder, the first transition may be connected with the appearance of an unstable oscillatory mode, which is not symmetric.

In the Soviet times, scientists often had to answer the question about the economical effect of their results. They had to spin answers out of thin air. However, in the case of the monotonicity principle, this effect can be really evaluated in roubles or pounds. The point is that researchers time and again find themselves in a typical position described by Confucius. They have to catch a black cat in a dark room, and, in addition, the cat is absent. A huge work is necessary to get the result that oscillatory instability is impossible in this or that situation. And as the only result, the melancholic sentence "Oscillatory instability is not found" appears at the end of the paper. Moreover, as it is impossible to get the absolute certainty, the succeeding authors re-examine the result again and again. Only a rigorous proof of the monotonicity principle preserves us at one go from this Sisyphean toil and saves a lot of human and computer time.

Problem 9. *Instability "in the large" of the Poiseuille flow in a pipe and the Couette flow in a channel* (*asymptotic bifurcation theory*).

How can we achieve an agreement between the result on the absolute stability of the Poiseuille flow in a circular pipe and the Couette flow in a channel and, on the other hand, the results of experimentalists who, beginning from Reynolds, report regularly about the observed instability and development of turbulent regimes? In general, the frame of the answer is discernible, though we are still far from the complete clarity. Most likely, these flows, staying stable "in the small" for all Reynolds numbers, are globally stable only for sufficiently small R. However, beginning from some value of the Reynolds number R, these flows become *unstable in the large*. The reason for this is that, as $R \to \infty$, the domain of attraction contracts (at least in one direction) to the point corresponding to the basic flow itself. This happens,

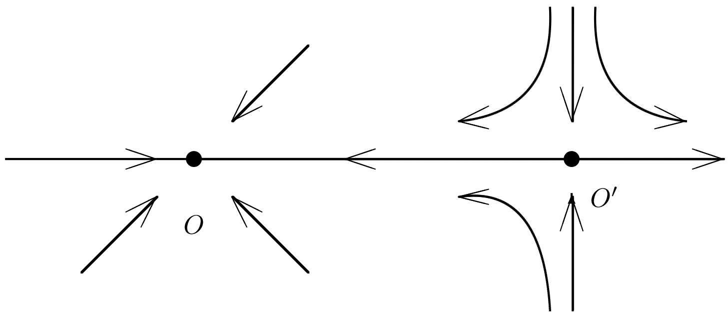

Figure 3.

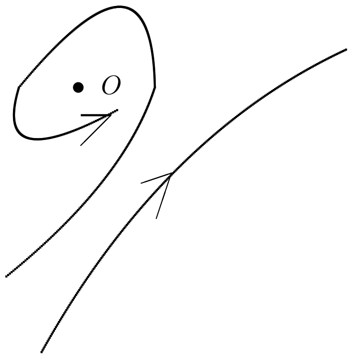

Figure 4.

for instance, when some unstable equilibrium O' is coming closer and closer to a given asymptotically stable equilibrium O (see Fig. 3), and in the limit they merge together. Of course, it may happen that this is an unstable limit cycle or an even more complicated invariant set that approaches the equilibrium O. There is also another variant for which the equilibrium O is stuck in the limit into a separatrix trajectory (Fig. 4). The results of computer experiments seemingly testify to the first variant (Fig. 3), though the situation of Fig. 4 is more difficult to disclose and, maybe, it will be also found out.

Problem 9a. *Prove that the Poiseuille flow in a pipe and the Couette flow in a channel are unstable in the large.*

It is very important to clarify the nature of this instability. Indeed, if the situation of Fig. 3 is realized, then the question is what the nature of the unstable regime O' is? The authors of the work [38] got over the huge computational difficulties and calculated the two-dimensional and three-dimensional soliton-like stationary regimes near the Couette flow in a channel (see also the references in this paper to the results of other authors on the existence nearby the Couette flow of space-periodic regimes even at lower Reynolds numbers). The results on the intermittency of "turbulent plugs" and zones of the laminar Poiseuille flow in the transition interval of Reynolds numbers also can serve as some evidence on the existence of soliton-like solutions to the Navier–Stokes equations.

Problem 9b. *Prove that, for sufficiently large Reynolds numbers, there exist stationary (plane and periodic in the transversal direction) solutions to the Navier–Stokes equations tending to the Couette flow as* $|x| \to \infty$*.*

Problem 9c. *Prove that there exist solutions of the Navier–Stokes system in a circular pipe of the travelling soliton type tending to the Poiseuille flow as* $z - ct \to \infty$ *(*z *is the axial variable,* c *is the phase velocity).*

Problem 9d. *Prove the existence of travelling waves that are space-periodic and soliton-like or stationary and periodic in time tend, as* $R \to \infty$*, to the Poiseuille flow in a pipe or to the Couette flow in a plane channel, respectively.*

It is most likely that all these problems, 9a–d, will be resolved along with the construction of an *asymptotic bifurcation theory.* I mean the case when the critical Reynolds number is infinitely large: $R_* = \infty$. The fact is, of course, that $R = \infty$ is an essentially singular point for the Navier–Stokes system. Therefore, such a theory should include the construction of boundary layer (or, maybe, other?) asymptotics of the secondary regimes merging with the basic one as $R \to \infty$.

When this theory is built (no doubt it will happen!), many other applications will be found in hydrodynamics as well as in other branches of mathematical physics.

7. Transitions and chaotic regimes

Problem 10. *Find and rigorously justify the existence of strange attractors in the Navier–Stokes system and its nearest relatives* (convection problem, multi-component fluids, magnetic hydrodynamics, etc.).

In a series of hydrodynamical problems, with the use of natural and computer experiments, we have already settled chains of transitions leading from an asymptotically stable equilibrium (stationary motion) through secondary equilibria, limit cycles, and/or invariant two-dimensional tori to complicated chaotic regimes. The treatment of experimental data was obtained in the framework of ideas and images of the general bifurcations theory. However, even in the case of ordinary differential equations, for the most interesting cases, this work was not carried through up to the rigorous checking of the conditions of general theorems. During the past decades, several prominent mathematicians even began to declare that, in such problems, the abilities of rigorous mathematical analysis are exhausted and, henceforward, we have to pin our hopes only on computer calculations. And even the realization of rigorous proofs has been entrusted with computers. It is impossible to accept this point of view.

No doubt, direct numerical calculations always played a significant role in the development of mathematics. Archimedes, however, not even for a minute, thought it was sufficient to determine the volumes of a ball and a cone by weighing. He followed up on developing his methods of calculating volumes up to the triumphant conclusion. Euler, Gauss, and Ramanujan got many of their discoveries, especially in number theory, as a result of extensive calculations and observations. However their discoveries became the property of mathematics only after the development of the corresponding rigorous theories. The application of computers widened the scope of numerical experiments utterly and now plays the decisive role in the investigation of processes described by differential equations.

Perhaps, mathematicians never thought that literally everything in the world must be justified with complete rigorousness. If we consider mathematics as a tool for investigating nature (for me, this is only one but the main of its sides), then its characteristic feature is the aspiration to get absolutely reliable results. Meanwhile, very often it is sufficient to obtain results with probability 0.99 or maybe even 0.6. High reliability is very expensive — for instance, a rigorous proof of error estimates ("demonstrative calculations" according to K. I. Babenko) requires much more computer time than even the calculation itself.

It is not out of place to note that rigorous mathematical proofs give us results with complete reliability, but... in the limit $t \to \infty$ only. So many times it occurred that the falseness of proofs of important results (or even of the results themselves) has been detected only years or even decades later. I heard that about 30% of the theorems published in journals like "Comptes Rendus" or "Doklady Akademii Nauk SSSR" turn out to be wrong.

It seems obvious that the most fundamental results supporting a great deal in mathematics should be justified absolutely rigorously. Otherwise, very quickly, the house-of-cards effect will convert our reasoning into taking shots in the dark.

I admit that, in future, the rigorous justifications of results within computers such that their verification is also accessible for computers only will play an essential role, for example, in structural calculations in extremely important situations when human lives depend crucially on the work of the device. However, mathematics has its "human side". It is notable that, recently, this was a sufficient reason for increasing the NSF grants for mathematicians.

Going back to our problem, note that the highest chances to its resolution are connected with asymptotic methods and methods of bifurcation theory. In particular, when investigating intersections of bifurcations in the Couette–Taylor problem (viscous flow between rigid rotating cylinders), the Navier–Stokes system is reduced to *amplitude systems* on the central manifold ([39], [40]). I performed extensive computer experiments with these amplitude systems and found homoclinic bifurcations, doubling cascades of limit cycles, resonance breaking up of tori, and, probably, other transitions leading to the creation of various chaotic regimes. The original problem for the Navier–Stokes equations is reduced to the investigation of a system of ordinary differential equations of comparatively small order (the sixth or eighth, and, after the further reduction, even of fourth and, respectively, fifth order).

Now, it is just the moment to recall that there are fairly big gaps even in the theory of ordinary differential equations. For instance, Smale [1] states the problem which, in a somewhat free exposition, sounds as follows:

"Prove that there exists the Lorenz attractor in the Lorenz system."

In fact, up to now the experimental observations of the Lorenz attractor and the general existence theorems are not united: nobody has examined the fulfilment of the general conditions for the special Lorenz system. I expect that the further development of asymptotic methods will enable us to resolve both this Smale's problem and our Problem 10, say, for the Couette–Taylor problem. That is true, in the former case, the analysis of the reduced system should be supplemented with a proof that the found attractors sustain the addition of (initially removed) higher order terms of the power series. This technical problem seems quite surmountable, though we should be ready to see that some of the observed attractors will disappear.

It is not so difficult to find other situations in fluid dynamics displaying different degenerated bifurcations and predisposed to the appearance of strange attractors and chaotic regimes.

Of course, various limiting cases, first of all the case of vanishing viscosity, suggest us different ways of searching for chaotic flow regimes.

8. Asymptotics of vanishing viscosity and turbulence

It cannot be even doubted that the problem of fluid motion at a very low viscosity (i. e., at very large Reynolds numbers) is the central one in hydrodynamics. All the problems discussed above are its more or less essential parts.

Problem 11a. *Prove (or disprove) that, as $\nu \to 0$, a solution to the Navier–Stokes system in a bounded domain $D \subset \mathbb{R}^m$ with fixed rigid boundary ($v|_{\partial D} = 0$) and assigned initial velocity field ($v|_{t=0} = v_0(x)$) approaches the solution to the Euler equations with the same initial condition and boundary condition ($v_n|_{\partial D} = 0$).*

Assume that the data of the problem, namely, ∂D and v_0, and the exterior mass force (if it is present) are C^∞-smooth.

Of course, it should be specified what kind of passage to the limit is in use here. Is this the uniform convergence on an arbitrary interior subdomain? Or the convergence in mean, in the norm $L_p(D)$? Or, maybe, in some measure?.. So far it is hard to state any reasonable conjecture regarding this matter.

The main difficulty here is related to the presence of a rigid wall. Even in the two-dimensional case, where we have strong existence theorems for the Navier–Stokes equations as well as for the Euler equations, the situation is completely unclear. If there is no boundary, say, in the case of spatially-periodic flows (equations on the torus T^2), or if the boundary conditions are "soft" ($v_n|_{\partial D} = 0$, $\nabla \times v|_{\partial D} = 0$), then everything is fine [41]: the passage to the limit as $\nu \to 0$ is justified by using the integral estimates of vorticity.

The other complicated case is when the initial velocity field has discontinuities on some curves. The case of a *weak discontinuity*, when the velocity and hence the pressure are continuous and only a vorticity jump takes place, can be analyzed fairly well. The point is that, in a fluid without a boundary, when the initial conditions are smooth, there are no boundary layers at all, and in the case of a weak discontinuity or "soft" boundary conditions, the boundary layer equations are linear and admit an explicit solution. It is also very important that we know in advance where this boundary layer is formed, along the whole boundary (in the case of soft boundary conditions) or along the whole weak discontinuity curve. On curves of strong discontinuity and on rigid walls, the boundary layers are described by the Prandtl equations, which are nonlinear and cannot describe flow near the whole rigid boundary or the whole curve of strong discontinuity. The fundamental obstacle here is the *separation of a boundary layer*.

Let me notice that, on a fluid free boundary the boundary layer is also weak and linear. (See [43], [44] on the results of V. A. Batishchev, L. S. Srubshchik, and V. V. Pukhnachev.) However, we are still far from a rigorous theory because of the fundamental difficulties related to the existence theorems.

Problem 11b. *Determine the limit of a stationary solution to the Navier–Stokes system as* $\nu \to 0$. *In particular, find the asymptotics of the stationary flow of a viscous fluid past a rigid body.*

In addition to the difficulties related to separation of the boundary layer, in stationary (and periodic in t) problems we encounter another serious difficulty related to the determination of the limiting flow regime. The stationary Euler equations have infinitely many stationary solutions, and the question is which of them is the limiting one for the prescribed initial conditions. Things get even more complicated since we do not know beforehand the smoothness degree of this limiting stationary regime. If it is discontinuous, the location and nature of discontinuities also have to be determined.

Furthermore, the question on the stability of these stationary regimes naturally arises. They are most likely unstable at large Reynolds numbers. However, fluid-dynamicists used to assume optimistically that the asymptotics is still valid for those moderate Reynolds number, at which the stability is preserved. One can say in addition that the self-oscillatory regimes arising at slightly supercritical Reynolds numbers remain so close to the stationary flow that the integral characteristics, say the resistance force, differ very little from their stationary values.

Problem 11c. *Determine the average velocity field in the turbulence developed as* $\nu \to 0$ (*Reynolds number* $R \to \infty$) *and calculate the correlations*

$$\langle v(x',\, t) \otimes v(x'',\, t)\rangle,$$

i. e., the average values of all possible products of the velocity field components at the prescribed time t *in the points* x' *and* x'' *of the flow domain. Determine also the correlation functions corresponding to distinct times* t' *and* t'',

$$\langle v(x',\, t') \otimes v(x'',\, t'')\rangle.$$

A huge amount of literature is devoted to this problem [46] (see also the recent survey [47]). However, all existing turbulence theories without exception are based on hydrodynamics equations only to some, rather small, extent. In any case, all of them include some hypotheses that are not derived from the Navier–Stokes equations and may even contradict them. It is worth noticing that some of these theories predict the average velocity field quite well. For instance, it is hardly an accidental coincidence that the average profile of turbulent Couette flow in a channel calculated in [49] is the same as the one obtained experimentally. However, I suppose, no theory manages to predict more complicated flow characteristics, beginning with second-order correlation functions, not to mention the higher orders.

I do not have a sufficiently clear answer to the inevitably arising question on the exact type of averaging that we should bear in mind while stating Problem 11c. The most common ones are averaging with respect to time and with respect to an invariant measure on the phase space of the system. In the case of ergodicity, these two types of averaging are the same. While pressing for theoretical results to coincide with experimental data, we shall probably need to take into account that a measuring instrument makes some kind of spatial averaging over a small region.

However, against all the odds, I believe that the consecutive theory of developed turbulence, which describes flows with very low viscosity, can and will be built. It is not inconceivable that this theory will bifurcate into several branches. Probably, we will have to describe differently turbulent flows in pipes, turbulent convective flows, Couette–Taylor flows, turbulent flows past bodies, etc. That is true, the lack of a general theory leads to much more branching. Almost every flow needs to be considered separately, which requires introducing new *ad hoc* hypotheses every time. So far, the problem is to construct the asymptotics for at least one particular case.

Sometimes experiments provide us with so beautiful and clear results that it is a shame on theorists that they cannot interpret them. For instance, for the Couette–Taylor flow between two cylinders (when the exterior cylinder is fixed), Taylor obtained (1923) the expression $v_\theta = \frac{c}{r}$ for the azimuthal velocity, which is valid everywhere except narrow boundary layers (see [48]). It is quite remarkable that Taylor's vortices, which have lost their stability long ago, come alive once again for very large Reynolds numbers. Apparently, they survive (stay stable) for arbitrary large Reynolds numbers. An excellent survey on turbulent Taylor vortices is presented in [45]. A large number of simple and explicit relations has been also discovered in experiments on Bénard's convection in a horizontal fluid layer. I believe that the problem of rigorous mathematical interpretation of these phenomena and patterns is not hopeless.

There is no reason to expect uniqueness in Problems 11b and 11c. It is quite possible that there exist several stationary regimes having different asymptotical behavior at vanishing viscosity. In fact, this is the case for the Couette flow between two rigid spheres, as well as for Karman's flow between rotating planes. There exist some other turbulent regimes along with Taylor's turbulent vortices; moreover, the former themselves are not uniquely determined under the given boundary conditions (their axial wave numbers vary).

In conclusion of this section, I would like to emphasize that it was the developed turbulence, at the limit of vanishing viscosity, that we discussed here. Ideally, the transition theory for moderate Reynolds numbers must describe all possible types of flow regimes and the conditions of their births and deaths while moving along parameters, and it should also provide us with methods for computing them. It seems impossible to predict without particular calculations what sequence of transitions will be realized in a prescribed situation. Rather, the theory must teach researchers the rules according to which flow regimes change and methods for the calculation of various regimes. Of course, it is very important to learn how to formulate right questions and to point out the quantities that need to be calculated first of all. One would say that this theory will more resemble traffic regulations than a train schedule. Although, I suppose, we have good chances that asymptotic methods will enable us to predict even transition sequences in various limiting cases.

Acknowledgements. I am sincerely grateful to Natalya Popova and Marc Burton for their invaluable help in translating of this paper into English.

I am greatful to the reviewer for the suggestion to include the interesting modern books [50]–[52] into the list of references. Meanwhile one should remember that nothing can substitute for reading of pioneers — such authors as Lyapunov, Leray, and Hopf.

Let me conclude this paper with some small edification for a young researcher who intends to start solving one or another of the problems discussed here. All fore-quoted authors (and the present author is, of course, not an exception) have never resolved these fundamental problems. Therefore one should study their works with care not to go a hopeless way.

References

[1] S. Smale, *Mathematical problems for the next century*, Math. Intelligencer **20** (1998), no. 2, 7–15. MR **99h:**01033

[2] G. A. Martynov, *Phase transitions problem in statistical mechanics*, Usp. Phys. Nauk **169** (1999), no. 6, 595–624 (Russian).

[3] V. M. Fridkin (ed.), *Ferroelectrics–semi-conductors*, Nauka, Moscow, 1976 (Russian).

[4] V. I. Yudovich, *Global regularity vs collapse in dynamics of incompressible fluids*, Mathematical events of XX century (V. B. Filippov, ed.), PHASIS, to appear (Russian).

[5] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Mathematics and its Applications, Vol. 2, Gordon and Breach Science Publishers, New York, 1969. MR 40 #7610

[6] R. Finn and D. R. Smith, *On the stationary solutions of the Navier–Stokes equations in two dimensions*, Arch. Rational Mech. Anal. **25** (1967), 26–39. MR 35 #3238

[7] J. Leray, *Etude de diverses équations integrales non lineaires et de quelques problèmes que pose l'hydrodynamique.*, J. Math. Pures Appl. Ser 9 **12** (1933), 1–82.

[8] V. I. Yudovich, *Periodic motions of a viscous incompressible fluid*, Sov. Math. Dokl. **1** (1960), 168–172.

[9] V. I. Yudovich, *Rotationally-symmetric flows of incompressible fluid through a circular ring. part* 1, 2, Preprint VINITI, 2001.

[10] E. Hopf, *Ein allgemeiner Endlichkeitssatz der Hydrodynamik*, Math. Ann. **117** (1941), 764–775. MR 3,92a

[11] V. I. Yudovich, *An example of the loss of stability and the generation of a secondary flow of a fluid in a closed container*, Mat. Sb. (N.S.) **74 (116)** (1967), 565–579 (Russian). MR 36 #4867. English translation in: Math. USSR-Sb. **3** (1967), 519-533.

[12] V. I. Yudovich, *On periodic differential equations with a selfadjoint monodromy operator*, Dokl. Akad. Nauk **368** (1999), no. 3, 338–341 (Russian). MR **2000k:**34092. English translation in: Dokl. Phys. **44** (1999), no. 9, 648–651.

[13] V. I. Yudovich, *Periodic differential equations with a selfadjoint monodromy operator*, Mat. Sb. **192** (2001), no. 3, 137–160 (Russian). MR **2002i:**34104. English translation in: Sb. Math. **192** (2001), no. 3–4, 455–478.

[14] V. I. Yudovich, *The linearization method in hydrodynamical stability theory*, Translations of Mathematical Monographs, vol. 74, American Mathematical Society, Providence, RI, 1989. MR **90h:**76001

[15] A. I. Miloslavsky, a) *The Floquet theory for abstract parabolic equations*, Ph.D. thesis, Rostov-on-don, 1976; b) *On the Floquet theory for parabolic equations*, Funktsonal. Anal. i Prilozhen. **10** (1976), no. 2, 80–81. MR 57 #13057; c) *The Floquet theory for abstract parabolic equations. Basis properties of the generalized eigen-spaces of the monodromy operator*, Preprint

VINITI, no. 3073-75; d) *Floquet solutions, and the reducibility of a certain class of differential equations in a Banach space*, Izv. Severo-Kavkaz. Nauchn. Centra Vyssh. Shkoly Ser. Estestv. Nauk. (1975), no. 4, 82–87, 118. MR 57 #10149

[16] N. Dunford and J. T. Schwartz, *Linear operators. Part III: Spectral operators*, Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1971. MR 54 #1009

[17] E. Hopf, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr. **4** (1951), 213–231. MR 14,327b

[18] V. Z. Meshkov, *On the possible rate of decrease at infinity of the solutions of second-order partial differential equations*, Mat. Sb. **182** (1991), no. 3, 364–383 (Russian). MR **92d:**35032. English translation in: Math. USSR-Sb. **72** (1992), no. 2, 343–361.

[19] V. I. Yudovich, *The loss of smoothness of the solutions of Euler equations with time*, Dinamika Sploshnoy Sredy (1974), Vyp. 16, Nestacionarnye Problemy Gidrodinamiki, 71–78, 121 (Russian). MR 56 #12670

[20] V. I. Yudovich, *On the gradual loss of smoothness and instability that are intrinsic to flows of an ideal fluid*, Dokl. Akad. Nauk **370** (2000), no. 6, 760–763 (Russian). MR **2001c:**76053. English translation in: Dokl. Phys. **45** (2000), no. 2, 88–91.

[21] V. I. Yudovich, *On the loss of smoothness of the solutions of the Euler equations and the inherent instability of flows of an ideal fluid*, Chaos **10** (2000), no. 3, 705–719. MR **2002i:**76010

[22] V. I. Yudovich, *On the unbounded growth of vorticity and velocity circulation of flows of a stratified and a homogeneous fluid*, Mat. Zametki **68** (2000), no. 4, 627–636 (Russian). MR **2001j:**76117. English translation in: Math. Notes **68** (2000), no. 3-4, 533–540.

[23] S. Friedlander, W. Strauss, and M. Vishik, *Nonlinear instability in an ideal fluid*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), no. 2, 187–209. MR **99a:**76057

[24] Y. L. Daletsky and M. G. Krein, *Stability of solutions of differential equations in Banach space*, Nauka, Moscow, 1970 (Russian). MR 50 #5125

[25] V. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier (Grenoble) **16** (1966), no. fasc. 1, 319–361. MR 34 #1956

[26] V. I. Yudovich, *Plane unsteady motion of an ideal incompressible fluid*, Soviet Phys. Dokl. **6** (1961), no. 3, 18–20.

[27] V. I. Yudovich, *Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid*, Math. Res. Lett. **2** (1995), no. 1, 27–38. MR **95k:**35168

[28] A. I. Shnirelman, *Lattice theory and flows of ideal incompressible fluid*, Russian J. Math. Phys. **1** (1993), no. 1, 105–114. MR **95d:**58040

[29] G. R. Burton, *Rearrangements of functions, maximization of convex functionals, and vortex rings*, Math. Ann. **276** (1987), no. 2, 225–253. MR **88d:**49020; *Variational problems on classes of rearrangements and multiple configurations for steady vortices*, Ann. Inst. H. Poincaré Anal. Non Linéaire **6** (1989), no. 4, 295–319. MR **90h:**58017

[30] L. Belenkaya, S. Friedlander, and V. Yudovich, *The unstable spectrum of oscillating shear flows*, SIAM J. Appl. Math. **59** (1999), no. 5, 1701–1715 (electronic). MR **2000e:**76051

[31] V. A. Vladimirov, *On the instability of equilibrium in fluids*, Zh. Prikl. Mekh. i Tekhn. Fiz. (1989), no. 2, 108–116 (Russian). MR **90g:**76068. English translation in: J. Appl. Mech. Tech. Phys. **30** (1989), no. 2, 269–276.; *Lyapunov's direct method in problems of fluid equilibrium instability*, Arch. Mech. (Arch. Mech. Stos.) **42** (1990), no. 4–5, 595–607 (1991). MR **92h:**76047

[32] C. C. Lin, *The theory of hydrodynamic stability*, Cambridge, at the University Press, 1955. MR 17,1022b

[33] V. I. Yudovich, *On some spectral problems with differential weight (towards the absolute stability problem for the Poiseuille flow)*, Preprint VINITI, no. 2168-V88, 1988.

[34] V. A. Romanov, *Stability of plane-parallel Couette flow*, Funktsional. Anal. i Prilozhen. **7** (1973), no. 2, 62–73. MR 48 #4536

[35] L. D. Meshalkin and J. G. Sinai, *Investigation of the stability of a stationary solution of a system of equations for the plane movement of an incompressible viscous liquid*, Prikl. Mat.

Mekh. **25** (1961), 1140–1143 (Russian). MR 25 #871. English translation in: J. Appl. Math. Mech. **25** (1961), 1700–1705.

[36] V. I. Yudovich, *Secondary flows and fluid instability between rotating cylinders*, Prikl. Mat. Mekh. **30** (1966), 688–698 (Russian). MR 36 #4868 English translation in: J. Appl. Math. Mech. **30** (1966), 822–833.

[37] Y. S. Barkovsky and V. I. Yudovich, *Spectral properties of a class of boundary value problems*, Mat. Sb. (N.S.) **114(156)** (1981), no. 3, 438–450, 480 (Russian). MR **82e:**34019. English translation in: Math. USSR-Sb. **42** (1982), no. 3, 387–398.

[38] A. Cherhabili and U. Ehrenstein, *Finite-amplitude equilibrium states in plane Couette flow*, J. Fluid Mech. **342** (1997), 159–177. MR **98d:**76069

[39] P. Chossat and G. Iooss, *The Couette–Taylor problem*, Applied Mathematical Sciences, vol. 102, Springer-Verlag, New York, 1994. MR **95d:**76049

[40] V. V. Kolesov and V. I. Yudovich, *Calculation of oscillatory regimes in Couette flow in the neighborhood of the point of intersection of bifurcations initiating Taylor vortices and azimuthal waves*, Fluid Dynam. **33** (1998), no. 4, 532–542 (1999). MR **2001j:**76045

[41] V. I. Yudovich, *Non-stationary flows of an ideal incompressible fluid*, Zh. Vychisl. Mat. i Mat. Fiz. **3** (1963), 1032–1066 (Russian). MR 28 #1415. English translation in: U.S.S.R. Comput. Math. and Math. Phys. **3** (1963), 1407–1456.

[42] L. I. Srubshchik and V. I. Yudovich, *Asymptotic behavior of first-order discontinuities of fluid flows with vanishing viscosity*, Soviet Phys. Dokl. **16** (1971), 530–533.

[43] V. A. Batishchev and L. I. Srubshchik, *Diffusion of a spherical Hill vortex with vanishing viscosity*, Soviet Phys. Dokl. **16** (1971), 286–288.

[44] V. V. Pukhnachëv, *Nonclassical problems of the theory of a boundary layer*, Novosibirsk. Gos. Univ., Novosibirsk, 1979. MR **83g:**76002

[45] E. L. Koschmieder, *Bénard cells and Taylor vortices*, Cambridge Monographs on Mechanics and Applied Mathematics, Cambridge University Press, New York, 1993. MR **94h:**76031

[46] A. S. Monin and A. M. Yaglom, *Statistical fluid mechanics. The mechanics of turbulence, Vol. 1*, MIT Press, Cambridge, Mass., 1971; *Vol. I, Chap. 2*, CTR Monograph, Center for Turbulence Research, Stanford, 1997, preprint.

[47] A. M. Yaglom, *New trends in turbulence*, Proc. of Summer School, Les Houches (to be published).

[48] A. A. Townsend, *The structure of turbulent shear flow*, Cambridge University Press, New York, 1956. MR 17,1249b

[49] V. L. Berdichevsky, A. A. Fridlyand, and V. G. Sutyrin, *Prediction of turbulent velocity profile in Couette and Poiseuille flows from the first principles*, Phys. Rev. Lett. **76** (1996), no. 31, 3967–3970.

[50] P. Constantin and C. Foias, *Navier–Stokes equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988. MR **90b:**35190

[51] G. Gallavotti, *Foundations of fluid dynamics*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 2002. MR **2003e:**76002

[52] J.-Y. Chemin, *Perfect incompressible fluids*, Oxford Lecture Series in Mathematics and its Applications, vol. 14, The Clarendon Press Oxford University Press, New York, 1998. MR **2000a:**76030

Math. Dept., Rostov State University, 5, Zorge st., Rostov-Don 344090 Russia
E-mail address: yudovich@ns.math.rsu.ru