Lecture notes on Partial Differential Equations

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Welcome to PDEs!

欢迎来到偏微分方程讲座课程!

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Class materials will be available at:
http://phys.nsu.ru/korobkov/Fudan_2022_PDEs/

Welcome to PDEs!

班组长叫什么名字？请在聊天中回答。请写下您的姓名和您的电子邮件地址.

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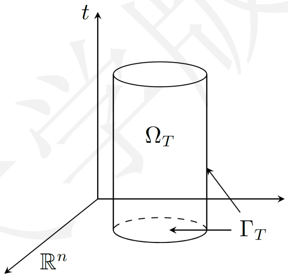
Roughly speaking: basic properties of elliptic equations hold for the parabolic equations in more subtle and sometimes more complicated forms.

WEAK MAXIMUM PRINCIPLE for the LAPLACE's EQUATION

MAXIMUM PRINCIPLE for the HEAT EQUATION



Let Ω — bounded open set in $\mathbb{R}^n$ and $T > 0$, then

$$\Omega_T = (0, T] \times \Omega, \qquad \Gamma_T = \partial' \Omega_T = \bar{\Omega}_T \setminus \Omega_T,$$

$$\Gamma_T = \Big(\{0\} \times \bar{\Omega} \Big) \cup \Big([0, T] \times \partial\Omega \Big).$$

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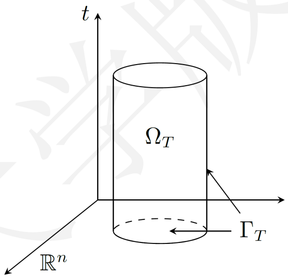

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$$u \in C^{1,2}(\Omega_T) \cap C(\bar{\Omega}_T), \qquad \partial_t u - \Delta u = 0 \ \Rightarrow \ \max_{(t,x) \in \bar{\Omega}_T} u = \max_{(t,x) \in \Gamma_T} u$$

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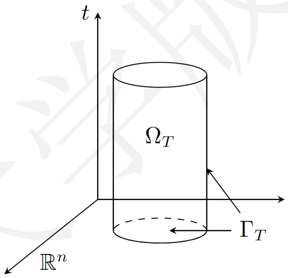

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$$\min_{(t,x) \in \bar{\Omega}_T} u = \min_{(t,x) \in \Gamma_T} u$$

HARNACK's INEQUALITY for HEAT EQUATION

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C. G. Axel von Harnack
(1851–1888)

HARNACK's INEQUALITY for HEAT EQUATION

Theorem 1. Let $u \geq 0$ be a smooth solution to the heat equation in $(0, T] \times B_{2R}$. Then for any $0 < t_1 < t_2 \leq T$, $x, y \in B_R$ the inequality

$$u(t_1, x) \leq u(t_2, y) \left(\frac{t_2}{t_1} \right)^n e^{\frac{|x-y|^2}{2(t_2-t_1)} + c\frac{t_2-t_1}{R^2}}$$

holds, where $c = c(n)$.

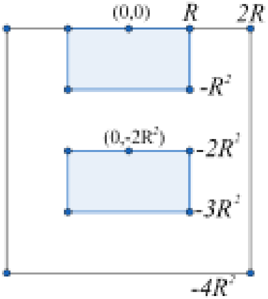
HARNACK's INEQUALITY for HEAT EQUATION

Denote $Q_R = (-R^2, 0] \times B_R$ — "parabolic ball".

Corollary 2. Let $u \geq 0$ be a smooth solution to the heat equation in Q_{2R}. Then

$$\max_{(t,x)\in(-3R^2,-2R^2)\times B_R} u \leq C_* \min_{(t,x)\in(-R^2,0)\times B_R} u$$

holds, where $C_* = C_*(n)$.



Proof:

Suppose $u > 0$ is continuous in $[0, T] \times B_{2R}$. Put $f(t, x) = \ln u(t, x)$. Then $\nabla f = \frac{\nabla u}{u}$. Suppose we can prove:

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Let $0 < t_1 < t_2 \le T$ and $x, y \in B_R$. Denote

$$L(s) = s(t_1, x) + (1-s)(t_2, y)$$

(the linear segment joining these two points). Then

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$$\int_0^1 \frac{d}{ds}(f(L(s)))ds = \int_0^1 \Big(\nabla f(L(s)) \cdot (x-y) + \partial_t f(L(s)) \cdot (t_1 - t_2)\Big) ds$$

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$$\overset{(*)}{\leq} \int_0^1 \Big(\nabla f(L(s)) \cdot (x-y) + (t_2 - t_1) \Big[-\frac{1}{2}|\nabla f|^2 (L(s)) + \frac{n}{st_1 + (1-s)t_2} + \frac{c}{R^2} \Big] \Big) ds.$$

We have

$$\ln\frac{u(t_1,x)}{u(t_2,y)}\leq\int_0^1\left(\nabla f(L(s))\cdot(x-y)+(t_2-t_1)\left[-\frac{1}{2}|\nabla f|^2(L(s))+\frac{n}{st_1+(1-s)t_2}+\frac{c}{R^2}\right]\right)ds$$

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By Cauchy inequality $ab\leq\frac{a^2}{2\varepsilon}+\frac{\varepsilon b^2}{2}$ we get

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Putting the last inequality into the first formula, we obtain

$$\ln\frac{u(t_1,x)}{u(t_2,y)} \le \frac{|x-y|^2}{2(t_2-t_1)} + n(t_2-t_1)\int_0^1 \frac{n}{st_1+(1-s)t_2}\,ds + c\frac{t_2-t_1}{R^2}$$

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So we have finished the proof! $\square$.

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So we have finished the proof! Not yet,really:)

It remains us to prove the used inequality

$$-\partial_t f \leq -\frac{1}{2}|\nabla f|^2 + \frac{n}{t} + \frac{c}{R^2} \qquad \text{in } (0, T] \times B_R \qquad (*)$$

if $u > 0$ is a solution to the heat equation in $[0, T] \times B_{2R}$, $f = \ln u$.

Step 1.

Put $F = t\big(|\nabla f|^2 - 2\partial_t f\big)$. We would like to estimate $\partial_t F - \Delta F$ and then to estimate maximum of F. This is a very close to the "maximum principle for the heat equations", but this is a rather long way.

Step 1.

Put $F = t\big(|\nabla f|^2 - 2\partial_t f\big)$. From $f = \ln u$, $\partial_t u = \Delta u$ we have

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Step 2.

For any scalar function h one has:

$$\frac{1}{2}\Delta h^2 = \frac{1}{2}\operatorname{div}(\nabla h^2) = \operatorname{div}(h\nabla h) = |\nabla h|^2 + h\Delta h.$$

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Applying it for $h = |\nabla g|^2$, one has

$$\frac{1}{2}\Delta|\nabla g|^2 = |\nabla^2 g|^2 + \langle \nabla g, \nabla\Delta g\rangle \geq \frac{|\Delta g|^2}{n} + \langle \nabla g, \nabla\Delta g\rangle.$$

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$$\Delta F \geq \frac{F^2}{2nt} + \frac{1}{n}|\nabla f|^2 F - 2\langle \nabla f, \nabla F\rangle - \frac{F}{t} + \partial_t F.$$

Step 4.

$$\Delta F \geq \frac{F^2}{2nt} + \frac{1}{n}|\nabla f|^2 F - 2\langle \nabla f, \nabla F \rangle - \frac{F}{t} + \partial_t F.$$

Made the scaling coordinate transformation:

$$(t, x) \mapsto (R^2 t, Rx),$$

then

$$(t, x) \in \big(0, \frac{T}{R^2}\big] \times B_1.$$

All previous estimates are invariant under this coordinate transformation.

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In other words, we can assume without loss of generality, that $R = 1$.

Step 5.

$$\Delta F \geq \frac{F^2}{2nt} + \frac{1}{n}|\nabla f|^2 F - 2\langle \nabla f, \nabla F \rangle - \frac{F}{t} + \partial_t F. \quad (**$$

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Now we have a good estimate for F and for $\partial_t F - \Delta F$. We can try to apply maximum principle for F. The problem is, that the maximum can be attained on the boundary $\partial B_{2R} = \partial B_2$, where F is not under our control. We have to find a way somehow to exclude the boundary.

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Let $0 \leq \varphi \leq 1$ be a C^∞-smooth functions such that

$$\varphi(x) = \begin{cases} 1, & x \quad \in B_1, \\ 0, & |x| \quad \notin B_2. \end{cases}$$

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Put $\eta = \varphi^2$.

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$$\begin{aligned}(\partial_t - \Delta)(\eta F) &= -\Delta\eta \cdot F - 2\langle \nabla\eta, \nabla F\rangle + \eta(\partial_t - \Delta)F \\ &\overset{(**)}{\leq} cF - 2\langle \nabla\eta, \nabla F\rangle - \eta\left(\frac{F^2}{2nt} + \frac{1}{n}|\nabla f|^2 F - 2\langle \nabla f, \nabla F\rangle - \frac{F}{t}\right).\end{aligned}$$

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Step 6.

$$(\partial_t - \Delta)(\eta F) \leq cF - 2\langle \nabla\eta, \nabla F\rangle - \eta\left(\frac{F^2}{2nt} + \frac{1}{n}|\nabla f|^2 F - 2\langle \nabla f, \nabla F\rangle - \frac{F}{t}\right). \qquad (***$$

Let $F_0 = (\eta F)(t_0, x_0) = \max\limits_{(t,x)\in[0,T]\times B_2} F(t,x)$. If $F_0 < 0$, nothing to estimate. Suppose $F_0 > 0$. Then $t_0 > 0$ and $|x_0| < 2$, i.e., x_0 is an interior point of B_2. In particular,

$$\partial_t F \geq 0, \qquad \nabla(\eta F) = 0, \qquad \Delta(\eta F) \leq 0 \qquad \text{at the point } (t_0, x_0).$$

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Therefore,

$$0 \leq (\partial_t - \Delta)(\eta F)(t_0, x_0) \leq c_1 - \frac{\eta F^2}{2nt_0} - \frac{1}{n}|\nabla f|^2 \cdot \eta F - 2\langle \nabla f, \nabla\eta\rangle F + \frac{\eta F}{t_0}.$$

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Therefore,

$$0 \le (\partial_t - \Delta)(\eta F)(t_0, x_0) \le c_1 - \frac{\eta F^2}{2nt_0} - \frac{1}{n}|\nabla f|^2 \cdot \eta F - 2\langle \nabla f, \nabla\eta\rangle F + \frac{\eta F}{t_0}. \qquad (5)$$

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In other words, $(\eta F)(t_0, x_0) \le 2n + c_3 t_0$.

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The homework problems:

1. In the beginning of our proof we assume that $u > 0$ is a smooth solution to the heat equation in $[0, T] \times B_{2R}$. Prove the same estimate under original assumption, namely, that $u \geq 0$ is a smooth solution to the heat equation in the domain $(0, T) \times B_{2R}$ continuous on $(0, T] \times B_{2R}$.

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2. For any constant $\alpha > 1$ prove the similar estimate

$$|\nabla f|^2 - \alpha \partial_t f \leq \frac{n\alpha^2}{2t} + \frac{c\alpha^2}{R^2}\left(1 + \frac{\alpha^2}{\alpha - 1}\right) \qquad \text{in } (0, T] \times B_R$$

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Hint: We have proved this estimate for $\alpha = 2$.

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3. Taking $R \to +\infty$ and $\alpha \to 1$, prove that

$$|\nabla f|^2 - \partial_t f \leq \frac{n}{2t} \qquad \text{in } (0, T] \times \mathbb{R}^n$$

if $u \geq 0$ is a solution to the heat equation in the strip $(0, T] \times \mathbb{R}^n$ and $f = \ln u$.

4. Using

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prove

Theorem 2. Let $u \geq 0$ be a smooth solution to the heat equation in the strip $(0,T] \times \mathbb{R}^n$. Then for any $0 < t_1 < t_2 \leq T$, $x, y \in \mathbb{R}^n$ the inequality

$$u(t_1,x) \leq u(t_2,y) \left(\frac{t_2}{t_1}\right)^{\frac{n}{2}} e^{\frac{|x-y|^2}{4(t_2-t_1)}}$$

holds.

Theorem 3. Let $T > 0$ and let $\Omega \subset \mathbb{R}^n$ be an open set. Then for any compact subdomain $\Omega' \Subset \Omega$ and for any pair $0 < t_1 < t_2 \leq T$ there exists a constant $C = C(\Omega', \Omega, t_1, t_2)$ such that

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$$r = \min_{x \in \bar{\Omega}', y \in \partial\Omega} |x - y|.$$

The Harnack inequality for general domains

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$$r = \frac{1}{2} \min_{x \in \bar{\Omega}', y \in \partial\Omega} |x - y|.$$

By our assumptions, the closure $\bar{\Omega}'$ is a compact subset of Ω, so $r > 0$.

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Proof.

Take $r = \frac{1}{2} \min\limits_{x \in \bar{\Omega}', y \in \partial\Omega} |x - y| > 0$. Fix $0 < t_1 < t_2 \le T$ and $x_1, x_2 \in \bar{\Omega}'$. Since the set $\bar{\Omega}'$ is connected and compact, there exists a finite r-net $y_0, y_1, \ldots, y_N \in \bar{\Omega}'$ such that $y_0 = x_1$, $y_N = x_2$,

$$\bar{\Omega}' \subset \bigcup_{i=0}^{N} B_r(y_i), \qquad |y_i - y_{i+1}| < r, \qquad i = 0, 1, \ldots, N-1.$$

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So we have finished the proof! $\square$

The **strict** maximum principle for the heat equation

Theorem 4. Let $T > 0$, $\Omega \subset \mathbb{R}^n$ be an open set, and let u be a solution to the heat equation in $\Omega_T = (0, T] \times \Omega$.

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Proof. Fix $\varepsilon > 0$, $t_1 < t_0$, and take arbitrary compact subdomain $\Omega' \Subset \Omega$. Put

$$v_\varepsilon(t, x) = u(t_0, x_0) - u(t, x) + \varepsilon.$$

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Proof. Fix $\varepsilon > 0$, $t_1 < t_0$, and take arbitrary compact subdomain $\Omega' \Subset \Omega$. Put

$$v_\varepsilon(t, x) = u(t_0, x_0) - u(t, x) + \varepsilon.$$

Then by construction $v_\varepsilon \geq \varepsilon > 0 \quad$ in Ω_T and v_ε is a solution to the heat equations as well.

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The proof of strict maximum principle is finished. $\square$